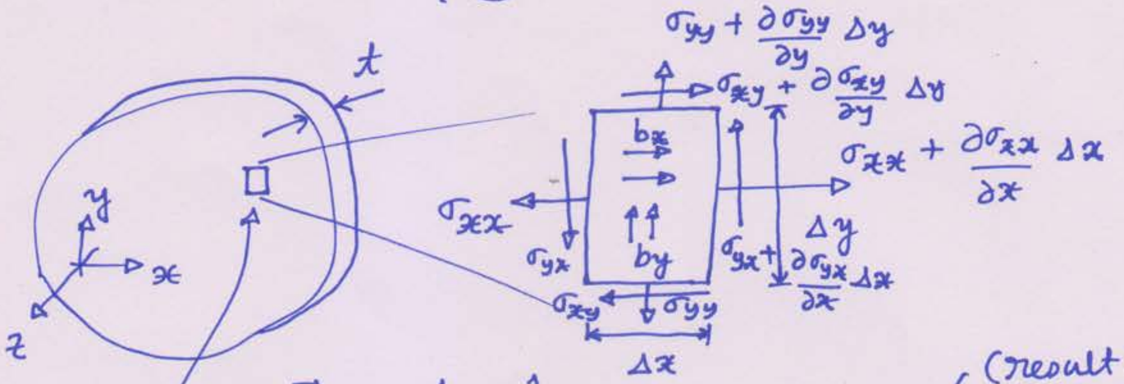
Derivation of the governing differential equations for static equilibrium of 2D elastic continuum.



$\vec{b} = b_x \hat{i} + b_y \hat{j}$
(body forces, N/m^3)

An infinitesimal element that is as small as one can imagine

σ_{xx}, σ_{yy} } normal stresses (result of stretch or contraction)

σ_{xy}, σ_{yx} } shear stresses (result of rotations)

σ_{ab}: a — direction of the force; b — direction of the normal to the surface over which the force is acting.

We now write static equilibrium equations for the infinitesimal element by treating it as a free-body:

$$\Sigma F_x = 0 \Rightarrow -\sigma_{xx}\cancel{t\,\Delta y} + \left(\cancel{\sigma_{xx}} + \frac{\partial \sigma_{xx}}{\partial x}\Delta x\right) t\,\Delta y$$
$$- \cancel{\sigma_{xy}\,t\,\Delta x} + \left(\cancel{\sigma_{xy}} + \frac{\partial \sigma_{xy}}{\partial y}\Delta y\right) t\,\Delta x + b_x (t\,\Delta x\,\Delta y) = 0$$

$$\Rightarrow \boxed{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0}$$

Similarly, $\Sigma F_y = 0 \Rightarrow$ $\boxed{\dfrac{\partial \sigma_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0}$

$$\Sigma M_z = 0 \Rightarrow \sigma_{yx}\,t\,\Delta y\,\frac{\Delta x}{2} + \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial x}\Delta x\right) t\,\Delta y\,\frac{\Delta x}{2}$$
$$- \sigma_{xy}\,t\,\Delta x\,\frac{\Delta y}{2} - \left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial y}\Delta y\right) t\,\Delta x\,\frac{\Delta y}{2} = 0$$

$\Rightarrow$ $\boxed{\sigma_{yx} = \sigma_{xy}}$ (neglecting moment contributions of $\frac{\partial \sigma_{yx}}{\partial x}\Delta x$ and $\frac{\partial \sigma_{xy}}{\partial y}\Delta y$, because they are small.)

Writing the equations for a 3D elastic continuum is similar. So, we get:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

and $\sigma_{xy} = \sigma_{yx}$; $\sigma_{yz} = \sigma_{zy}$; and $\sigma_{xz} = \sigma_{zx}$.

Strain (linear) definition:

Normal strains:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

Shear strains:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \varepsilon_{yx}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \varepsilon_{zy}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \varepsilon_{zx}$$

$\vec{u} = (u_x, u_y, u_z)$ = displacements along x, y, and z axes at a point.

Generalized Hooke's law = Hooke's law + Poisson effect

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{Y} - \frac{\nu\sigma_{yy}}{Y} - \frac{\nu\sigma_{zz}}{Y}$$

$$\varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{Y} + \frac{\sigma_{yy}}{Y} - \frac{\nu\sigma_{zz}}{Y}$$

$$\varepsilon_{zz} = -\frac{\nu\sigma_{xx}}{Y} - \frac{\nu\sigma_{yy}}{Y} + \frac{\sigma_{zz}}{Y}$$

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{2G} \; ; \quad \varepsilon_{xz} = \frac{\sigma_{xy}}{2G}$$

$$\varepsilon_{yz} = \frac{\sigma_{yz}}{2G}$$

By inversion ...

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{Y}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}$$

ν = Poisson's ratio