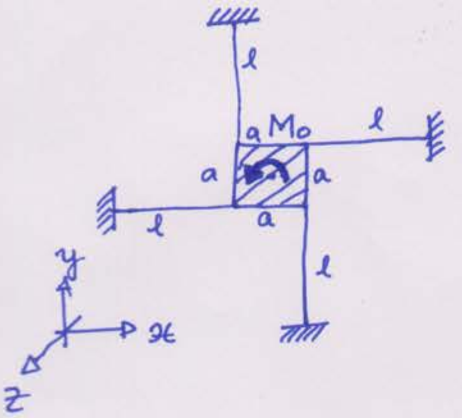
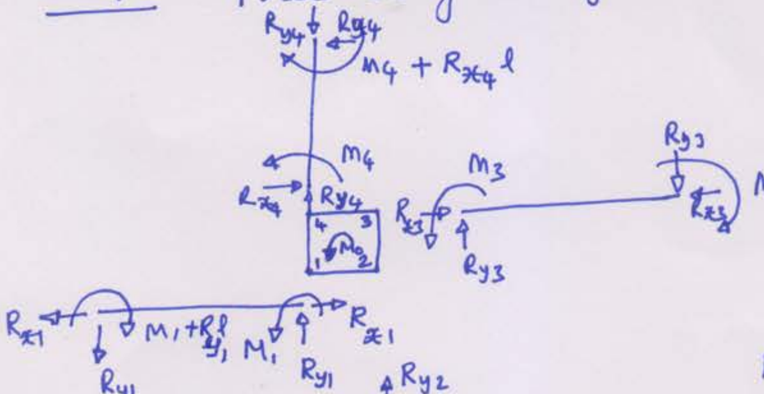
Rotational stiffness of a pin-wheel suspension

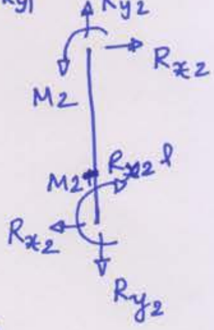


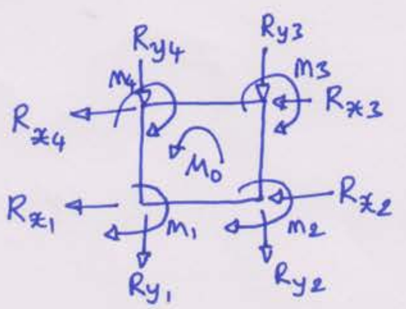
A square-shaped plate is attached to four beams to make a pin-wheel suspension. All beams are of length l. The side of the plate is a. The thickness of all beams and the plate is t. The in-plane width of the beams $= w$. Young's modulus is Y.

If M_0 is applied on the plate as a moment about the z-axis, how much does the plate rotate? (This is equivalent to asking the z-axis rotational stiffness.)

Step 1 Free-body diagrams and reaction forces







Because of the way we have written the reaction forces all four beams are already in static equilibrium.

Because of rotational symmetry we can write:

$$R_{x1} = R_{y2} = -R_{x3} = -R_{y4} \overset{\text{def}}{=} R_x$$

$$R_{y1} = -R_{x2} = -R_{y3} = R_{x4} = R_y$$

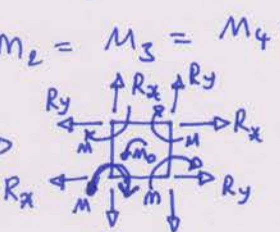
$$M_1 = M_2 = M_3 = M_4 = M$$

Now, we have only three unknowns: R_x, R_y, M.

But only one equation: $\Sigma M_z = 0$

$$-4R_x \frac{a}{2} + 4R_y \frac{a}{2} - 4M + M_0 = 0$$

for this



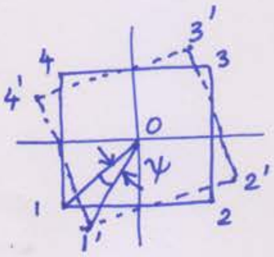
Note: $\Sigma F_x = 0$ $\Sigma F_y = 0$ Here.

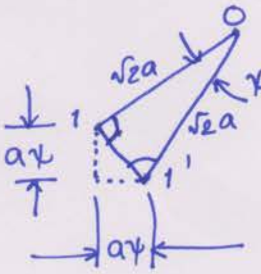
Since we have only one equation ($M_0 - 2aR_x + 2aR_y - 4M = 0$) in three unknowns (R_x , R_y , and M), it is a statically inderminate system.

Step 2

We need two more equations. For that, let us consider kinematics. How does the plate move? It undergoes a pure rotation about the z-axis. Let the angle of rotation be ψ.

(ψ is a small angle as we are concerned with linear-small-displacement stiffness.)



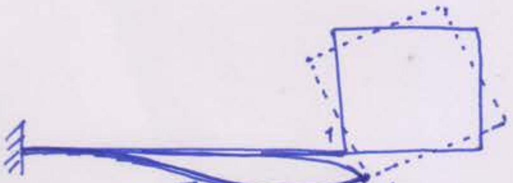


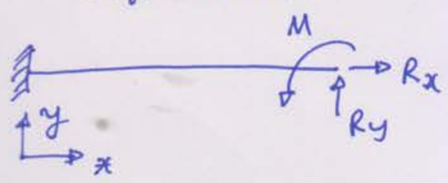
$\cos\psi \approx 1$
$\sin\psi \approx \psi$
$\Uparrow$
for small ψ

Note the coordinates of 1 and 1':

$1 \rightarrow (-a, -a)$ if o is the origin.

$1' \rightarrow \left(-\sqrt{2}a\cos\left(\frac{\pi}{4}+\psi\right),\ -\sqrt{2}a\sin\left(\frac{\pi}{4}+\psi\right)\right) \approx \left(-a(1-\psi),\ -a(1+\psi)\right)$





So, the displacements and slope (i.e., rotation) of the tip of the cantilever at 1 due to R_x, R_y, and M are:

$$\left.\begin{aligned} u &= a\psi \\ v &= -a\psi \\ \theta &= \psi \end{aligned}\right\}$$

$$\left.\begin{aligned} u &= a\psi = \frac{R_x l}{Y t w} \\ v &= -a\psi = \frac{R_y l^3}{3Y(tw^3/12)} + \frac{Ml^2}{2Y(tw^3/12)} \\ \theta &= \psi = \frac{Ml}{Y(tw^3/12)} + \frac{R_y l^2}{2Y(tw^3/12)} \end{aligned}\right\}$$

We got three equations and an extra unknown (namely ψ).

Step 3

Equations:

$$M_0 - 2aR_x + 2aR_y - 4M = 0 \quad - (1)$$

$$\frac{R_x l}{Y t w} = a\psi \quad - (2)$$

$$\frac{Ml^2}{2Y(tw^3/12)} + \frac{R_y l^3}{3Y(tw^3/12)} = -a\psi \quad - (3)$$

$$\frac{R_y l^2}{2Y(tw^3/12)} + \frac{Ml}{Y(tw^3/12)} = \psi \quad - (4)$$

Unknowns: R_x, R_y, M, and ψ.