Analytical Solutions for Squeeze Film Effect Model

Presented by

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Introduction

Squeeze film effect naturally occurs in dynamic MEMS structures because most of these systems have parallel plates or beams that trap a thin film of gas between the plate and the fixed substrate.

Introduction

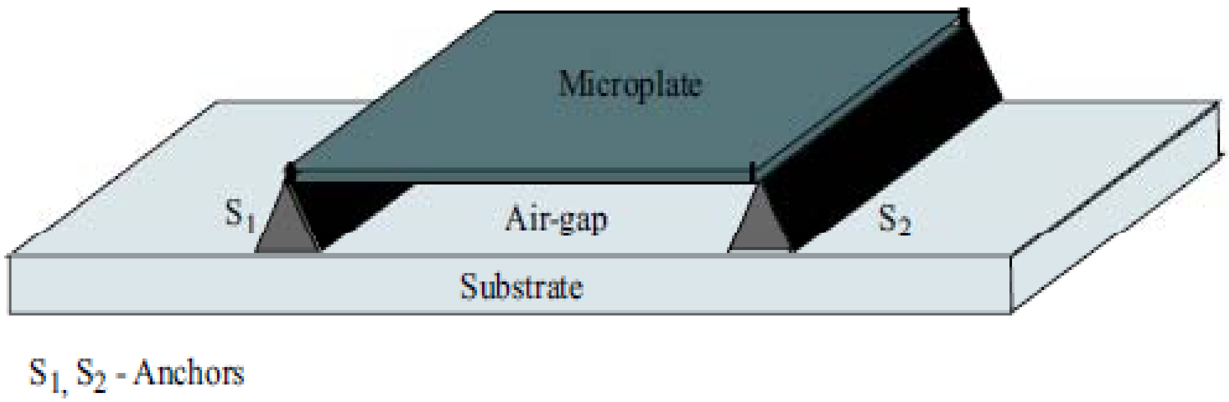


Figure taken from [1]

Squeeze Film Damping Mechanism

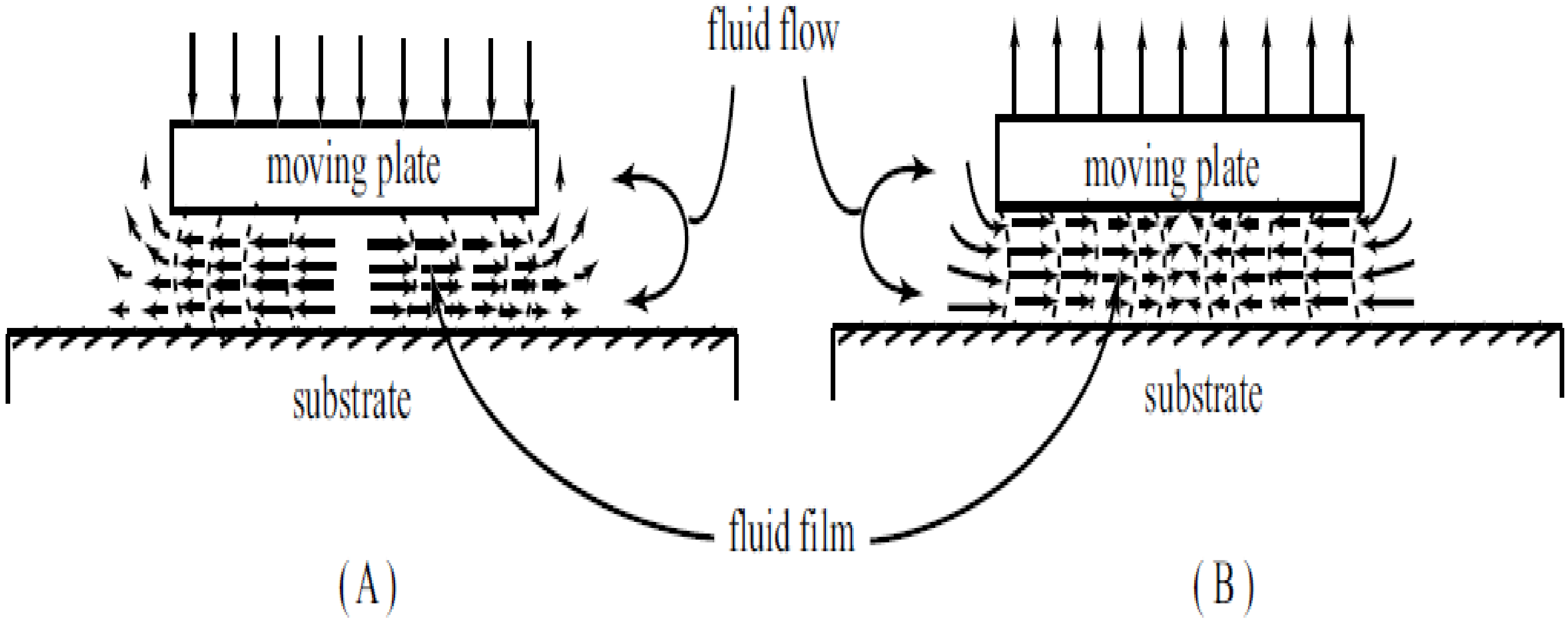


Figure taken from [1]

Squeeze Film Modeling

Continuity Equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0; \quad (1)$$

Navier – Strokes equation:

$$\rho \left\{ \frac{\partial \mathbf{u}}{\partial t} + (\nabla \mathbf{u})\mathbf{u} \right\} = F - \nabla p + (\mu^* + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla \cdot (\nabla \mathbf{u}); \quad (2)$$

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z} \right) \quad \text{and} \quad \rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z} \right) \quad (3)$$

Squeeze Film Modeling

Since the gap is much smaller than the surface dimensions and due to small v & u convective inertia terms are ignored. → $u\frac{\partial v}{\partial x}$, $v\frac{\partial v}{\partial y}$, $u\frac{\partial u}{\partial x}$ and $v\frac{\partial u}{\partial y}$

$$\rho\left(\frac{\partial u}{\partial t}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) \quad \text{and} \quad \rho\left(\frac{\partial v}{\partial t}\right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) \qquad (4)$$

Squeeze Film Modeling

After neglecting the unsteady inertia term

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) \quad \text{and} \quad \frac{\partial p}{\partial y} = \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right) \qquad (5)$$

Using (9) and the no slip B.C we get

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(z^2 - \frac{h^2}{4}\right) \quad \text{and} \quad v = \frac{1}{2\mu}\frac{\partial p}{\partial y}\left(z^2 - \frac{h^2}{4}\right) \qquad (6)$$

Squeeze Film Modeling

We now integrate (1) across –h/2 to h/2 using (6)

$$h\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}\left[\rho\int_{-h/2}^{h/2}\mathrm{udz}\right]+\frac{\partial}{\partial y}\left[\rho\int_{-h/2}^{h/2}v\mathrm{dz}\right]+\rho\frac{\partial h}{\partial t}=0. \qquad (7)$$

$$w=\frac{\partial h}{\partial t}$$

and assuming isothermal flow conditions

$$\frac{\partial}{\partial x}\left(\frac{ph^3}{\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{ph^3}{\mu}\frac{\partial p}{\partial y}\right)=12\frac{\partial (ph)}{\partial t} \qquad (8)$$

Squeeze Film Modeling

Using perturbation parameters $p = P_a + \hat{p}$ and $h = (h_a + \hat{h})$

We get the linearized compressible Reynolds equation

$$\left[\frac{\partial^2 \hat{p}}{\partial x^2} + \frac{\partial^2 \hat{p}}{\partial y^2}\right] = \frac{12\mu}{P_a h_a^3}\left[h_a \frac{\partial \hat{p}}{\partial t} + P_a \frac{\partial \hat{h}}{\partial t}\right] \quad (9)$$

Eq (9) is now non-dimensionalized using

$$\Phi = \frac{\hat{p}}{P_a},\ \epsilon = \frac{\hat{h}}{h_a},\ \tau = \omega t,\ X = x/L\ ,\ Y = y/L$$

$$\left[\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2}\right] = \sigma\left[\frac{\partial \Phi}{\partial \tau} + \frac{\partial \epsilon}{\partial \tau}\right] \quad (10)$$

where $\sigma = \dfrac{12\mu L^2 \omega}{P_a h_a^2}$

Solution by separation of variables

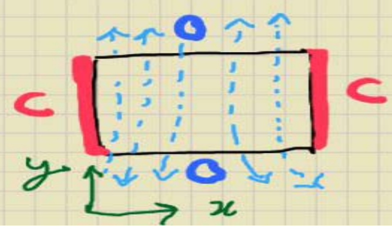
$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\sigma\left[\frac{\partial\phi}{\partial t}+\frac{\partial\epsilon}{\partial t}\right]$$

$$\Downarrow$$

$$\nabla^2 P-\sigma\frac{\partial P}{\partial t}=\sigma\frac{\partial H}{\partial t} \qquad (11)$$

$$\left\{\begin{array}{l} P=\phi \\ \epsilon = H \\ \sigma = \alpha^2 \end{array}\right\}$$

Here we will solve for the "OCOC" B.C's:



Solution by separation of variables

First we will consider the homogenous part of (11) with B.C's:

$$\nabla^2 P - \alpha^2 \frac{\partial P}{\partial t} = 0 \quad (12)$$

B.C's:

$$P_x(0,y,t) = P_x(L,y,t) = 0$$

$$P(x,0,t) = P(x,W,t) = 0$$

We assume

$$P(x,y,t) = \sum_{m,n} f_{mn}(x,y)\, T(t) \quad (13)$$

$$\& \quad f_{mn}(x,y) = a_{mn} \Psi_{mn}(x,y)$$

Solution by separation of variables

putting (13) into (12) we get

$$\frac{\nabla^2 f}{f} = \frac{\alpha^2 \frac{\partial T}{\partial t}}{T} = -k_{mn}^2 \quad (14)$$

Depends only on (x, y) Depends only on t

"Hence they must be constant"

Solution by separation of variables

From (14) we get the Eigen value equation

$$\nabla^2 f + k_{mn}^2 f = 0 \quad (15)$$

Now since we have

$$f_{m,n}(x,y) = a_{mn}\,\psi_{mn}(x,y)$$

we assume

$$\psi_{mn} = X_m(x)\,Y_n(y) \quad (16)$$

putting (16) in (15) we get

$$\nabla^2 \psi_{mn} + k_{mn}^2 \psi_{mn} = 0$$

Solution by separation of variables

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] X_m Y_n + k_{mn}^2 \left[X_m Y_m\right] = 0$$

$$X_m'' Y_n + Y_n'' X_m + k_{mn}^2 \left[X_m Y_n\right] = 0$$

$$\left[k_{mn}^2 = k_n^2 + k_m^2\right]$$

$$\frac{X_m''}{X_m} + \frac{Y_n''}{Y_n} + k_n^2 + k_m^2 = 0$$

From this we get two eigen value eqn's:

$$X_m'' + k_m^2 X_m = 0 \qquad (17)$$

$$Y_n'' + k_n^2 Y_n = 0 \qquad (18)$$

Solution by separation of variables

Now sol^n for eq (17) which is a II order ODE is

$$X_m = A\sin(k_m x) + B\cos(k_m x) \quad (18)$$

Applying B.C's :

$$X_m'(0) = X_m'(L) = 0$$

and using eq (18) we get

$$A = 0 \quad \text{and} \quad k_m = \frac{m\pi}{L}$$

$$\& \quad X_m = \cos(k_m x) \quad -(19)$$

Similarly we get

$$Y_n = \sin(k_n y) \quad -(20)$$

Solution by separation of variables

Hence from (19) & (20)

$$\psi_{mn}(x,y) = \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{W}\right) \quad (21)$$

Now

$$P = \sum_{mn} f_{mn}(x,y)\, T(t)$$

$$= \sum_{mn} a_{mn}\, \psi_{mn}(x,y)\, T(t) \quad (22)$$

Assuming a harmonic source

$$H = \delta\, \phi\, e^{i\omega t} \quad (23)$$

and choosing

$$T(t) = e^{i\omega t} \quad (24)$$

Solution by separation of variables

we put (22), (23) & (24) in the Reynold's eqn:

$$\nabla^2 P - \alpha^2 \frac{\partial P}{\partial t} = \alpha^2 \frac{\partial H}{\partial t}$$

$$\Rightarrow \sum_{m,n}\left[\nabla^2(a_{mn}\psi_{mn}T) - \alpha^2\frac{\partial}{\partial t}(a_{mn}\psi_{mn}T)\right] = \delta i\phi\omega e^{i\omega t}$$

since $\nabla^2\psi + k_{mn}^2\psi = 0 \Rightarrow \nabla^2\psi = -k^2\psi$

using this relation we get

$$\sum -a_{mn}k_{mn}^2\psi e^{i\omega t} - \alpha^2 a_{mn}\psi_{mn} i\omega e^{i\omega t} = i\omega\delta\phi e^{i\omega t} \qquad (25)$$

Solution by separation of variables

Multiplying both sides of eq^n (25) by ψ_{pq} & using the property :

$$\iint \psi_{mn} \psi_{pq} \, dx\, dy = \delta_{mn} \delta_{pq}$$

we get

$$a_{mn} = \delta \frac{\int_0^L \int_0^W i\omega\alpha^2 \phi \psi_{mn} \, dy\, dx}{\int_0^L \int_0^W \left[-k_{mn}^2 - i\omega\alpha^2\right] \psi_{mn}^2 \, dy\, dx} \qquad (26)$$

Here $\phi(x,y) = \sin^2\left(\frac{\pi x}{L}\right) \sin^2\left(\frac{\pi y}{W}\right)$ is the assumed 1^{st} mode slope of the plate.

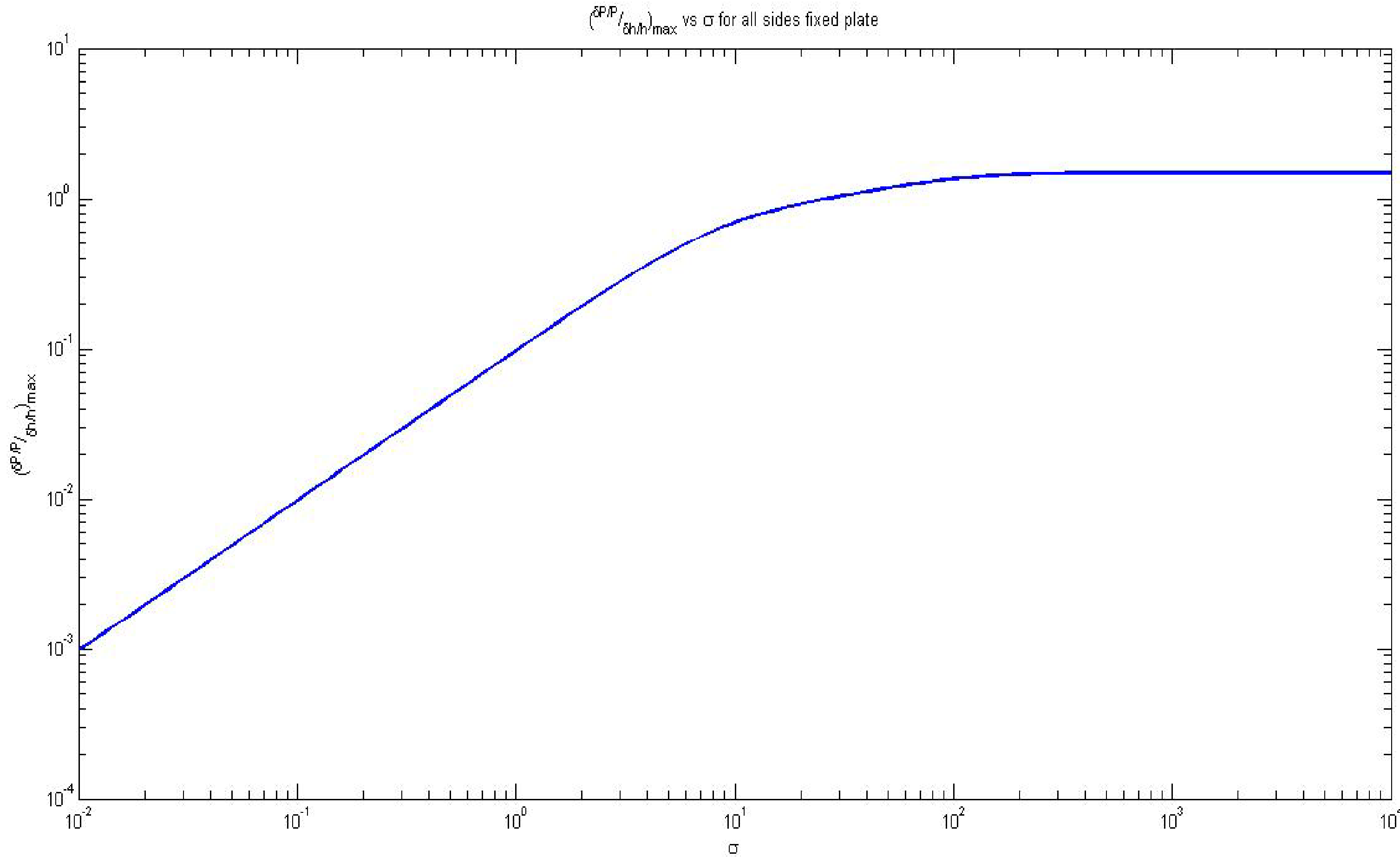
Solution by separation of variables

Hence we get

$$p = \sum_{m,n} a_{mn} \psi_{mn}(x,y) T(t)$$

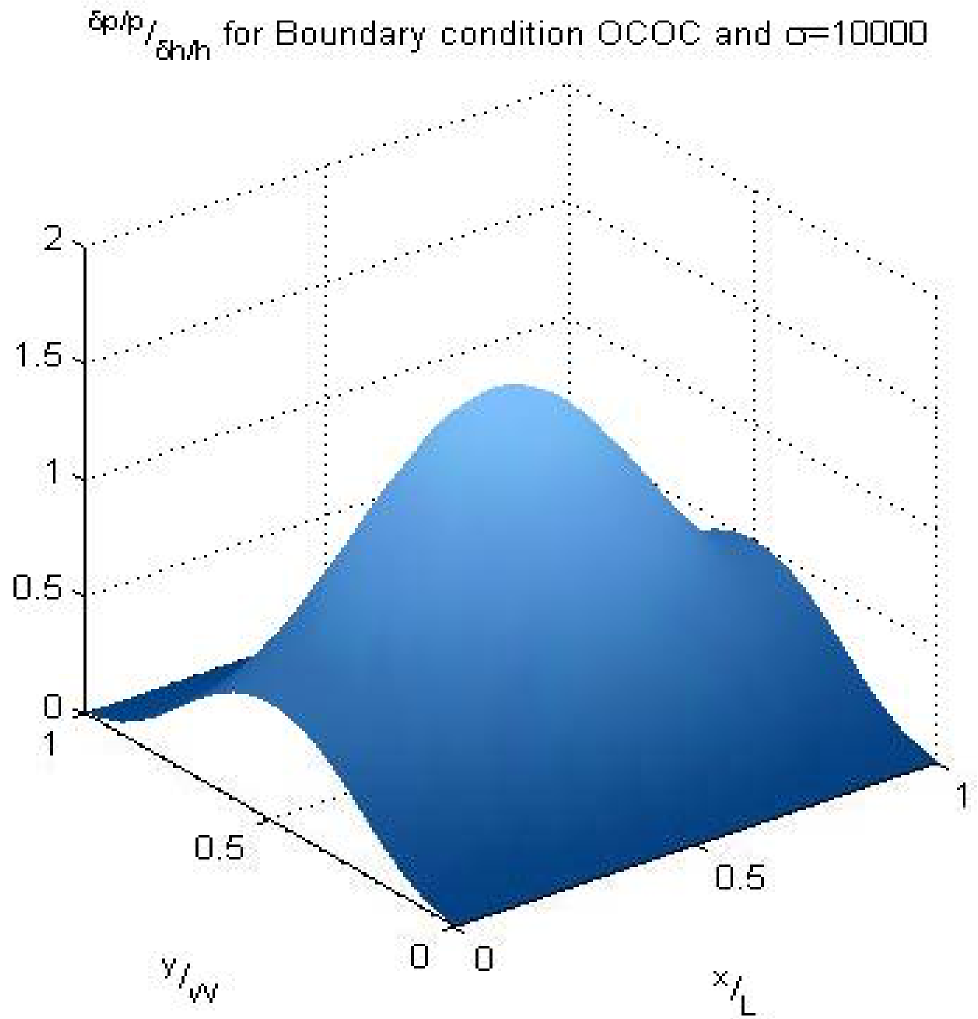
$$F_{tot} = \int_0^L \int_0^W p_a \hat{p}(x,y)\, dx\, dy$$

Results



$(^{\delta P/P}/_{\delta h/h})_{max}$ vs σ for all sides fixed plate
$(^{\delta P/P}/_{\delta h/h})_{max}$
σ
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}

Results


δp/p/δh/h for Boundary condition OCOC and σ=10000
2
1.5
1
0.5
0
1
0.5
0
y/W
0
0.5
1
x/L

References

[1] Rud.ra Pratap, Suhas Mohite and Ashok Kumar Pandey "Squeeze film effect in MEMS devices"

[2] H. C. Nathanson, W. E. Newell, R. A. Wickstrom, and J. R. Davis, \The resonant gate transistor," IEEE Transaction on Electron Devices ED-14, No. 3, pp. 117{133, 1967.

[3] [online], \Available:http://www.analogdevices.com," Analog devices Inc .

[4] T. Juneau and A. P. Pisano, \Micromachined dual axis input axis angular rate sensor," Tech. Dig. Proc. Solid-State Sensor Actuator Workshop , pp. 299{302, 1996.