Simulation of thermoelastic damping effect(TED) in plates

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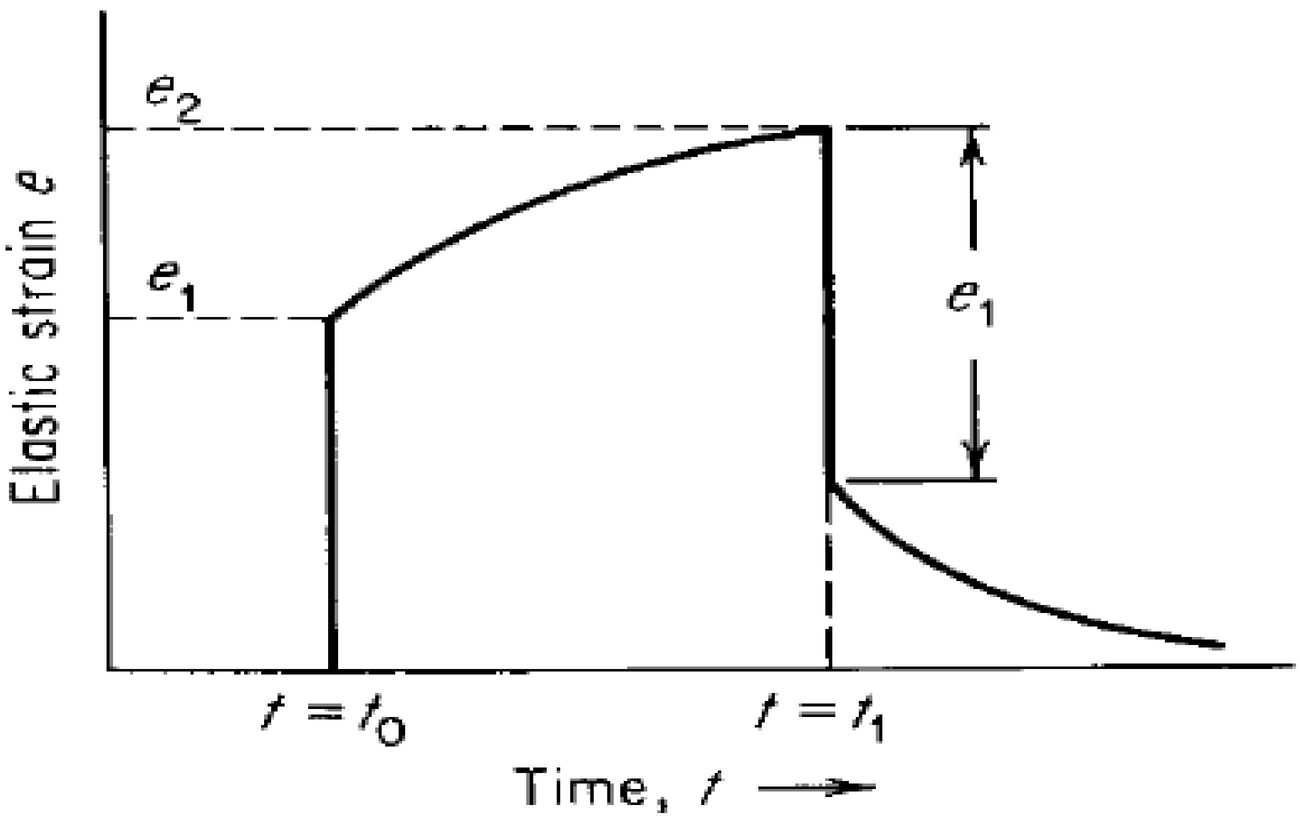
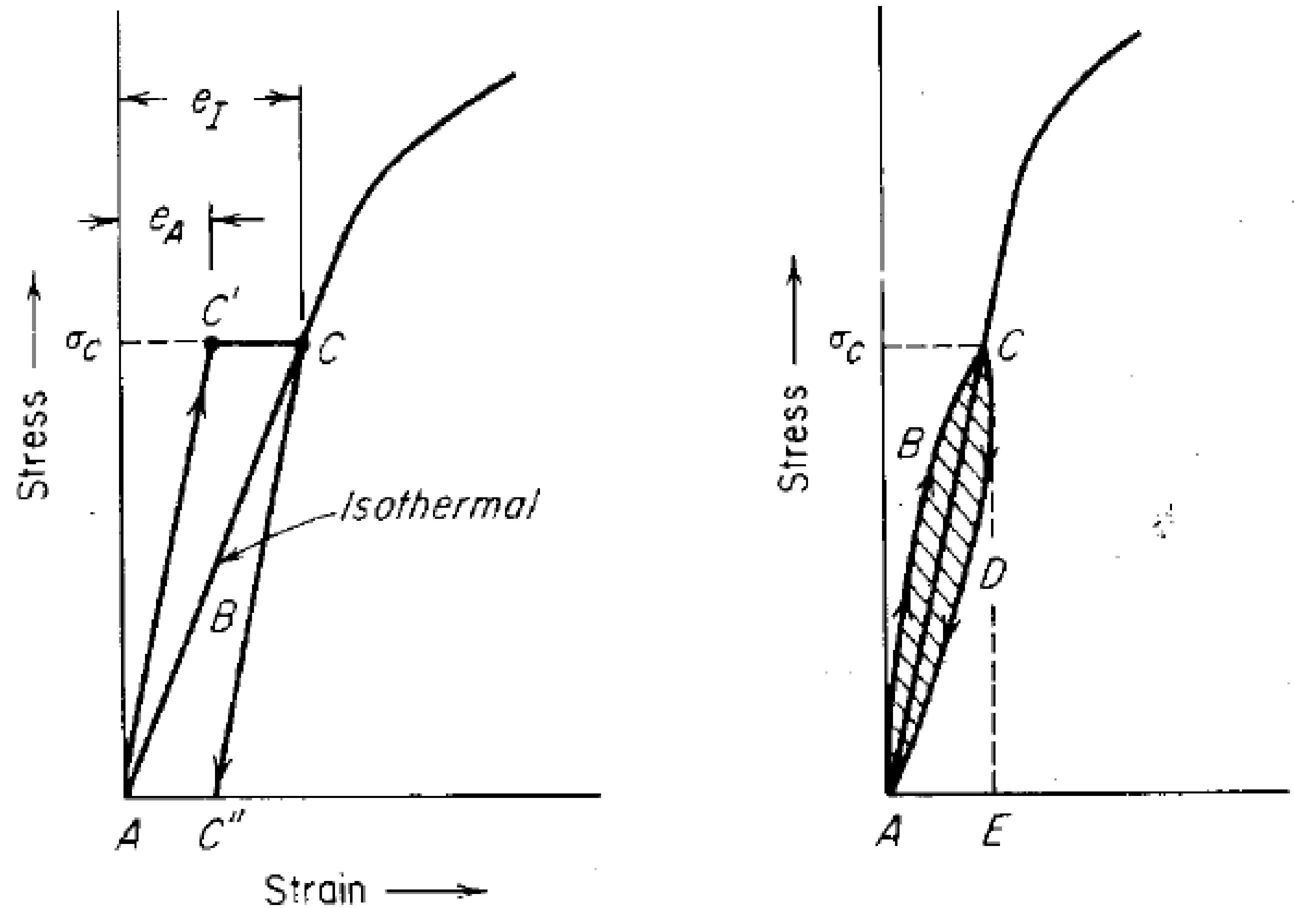
Thermoelastic damping effect(TED)

- It is a consequence of internal friction which is a general term for various energy dissipating effects in solids.
- Internal friction arises from the anelastic behavior of materials.
- And anelastic behavior arises when strain in any material isn't only a single valued function of stress but also of time.

A sudden stress is applied and after sometime suddenly released.[1]



TED in stress-strain diagram[1]



- The following is an equation of heat transfer involving thermoelastic damping. it can be derived by calculating the rate of entropy change with time while considering the work done by stress[2],[3]

$$k\nabla^2 T = \frac{\partial T}{\partial t} + \frac{E\alpha T_0}{(1-2v)C_v}\frac{\partial \bar{e}}{\partial t},$$

$$\bar{e} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz},$$

- the first hero of this field.[4]

“Clarence Melvin Zener”

There are other scientific phenomena on

His name-

-Zener breakdown

-zener pinning

-zener-holloman parameter

-zener ratio

- BELL LAB named a diode on his name

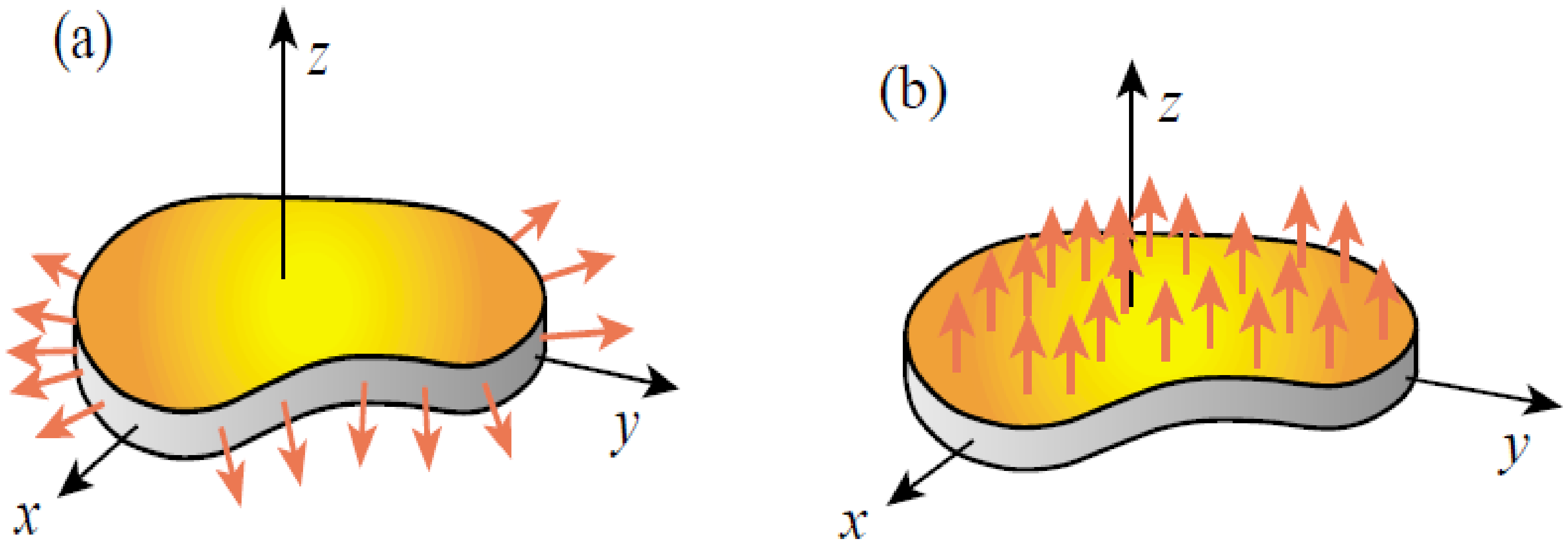
‘zener diode’

- Another hero in this field appear around 2000, named Lifshitz.
- Well,,,,more appropriate is to say that every scientist is a hero.

- to deal with the TED problem in plate, we need to solve the equation of motion of plates simultaneously with the last equation.
- There are various approach to solve plate deformation problem;
- Here are a few;[5]

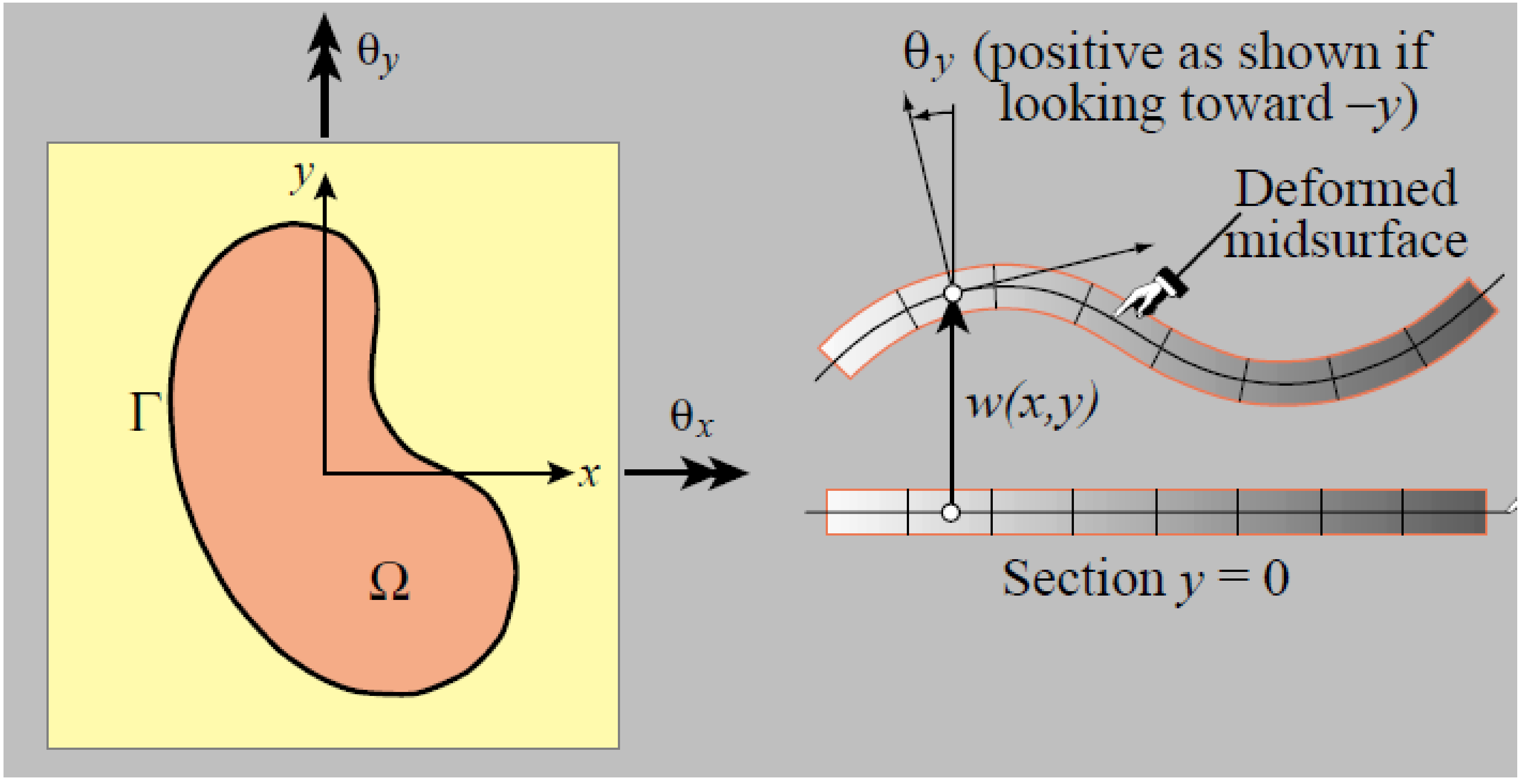
- Membrane shell model
- Von karman model
- Kirchhoff model
- Reissner-mindlin model
- Higher order composite model
- Exact model

- We'd work with kirchhoff model.[5]



- basic assumption of kirchhoff's plate model-[5]

"material normal to the original surface remain straight and normal to the deformed surface"



- These are the following strain equation one can come up with[5]

$$
\begin{aligned}
e_{xx} &= \frac{\partial u_x}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} = -z\,\kappa_{xx}, \\
e_{yy} &= \frac{\partial u_y}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} = -z\,\kappa_{yy}, \\
e_{zz} &= \frac{\partial u_z}{\partial z} = -z\frac{\partial^2 w}{\partial z^2} = 0, \\
2e_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} = -2z\,\kappa_{xy}, \\
2e_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0, \\
2e_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = -\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} = 0.
\end{aligned}
$$

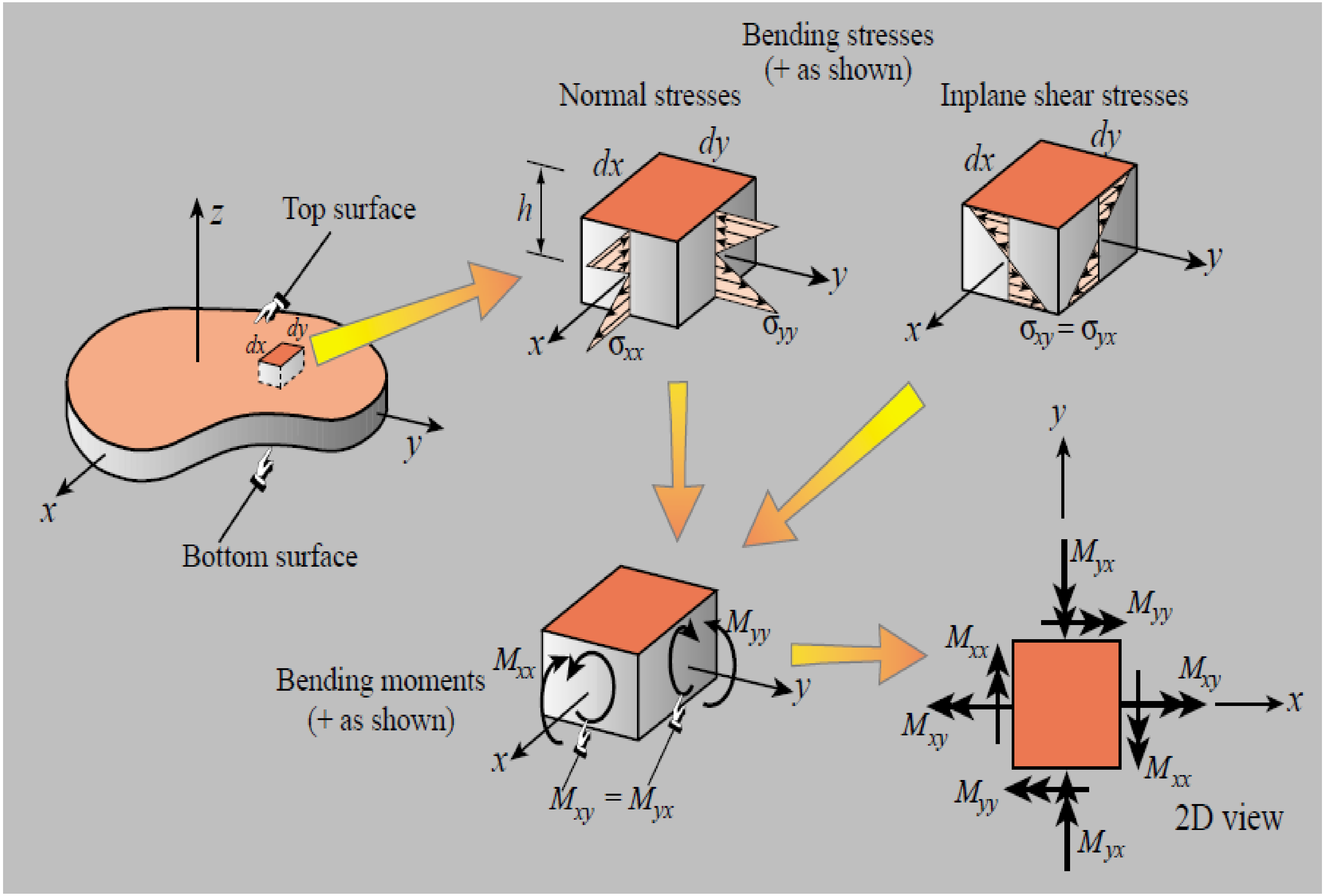
- We get shear and longitudinal strain in z direction 0,,evidently because of the assumption…w(x,y) is only the function of x and y,not of z..

Now here comes the paradox;

- the shear strain along z direction is zero,,so no transverse shear force but we can’t derive lagrange’s equation without assuming shear forces

- Also the plate is thin and bending in this model is assumed to happen because of in plane stress acting above and below the mid surface. So it’s kind of plane stress problem but having longitudinal strain along with all the shear strain in z direction as zero also implies it to be a plain strain condition.

- Plane strain and plane stress condition can occur simultaneously only when poisson’s ratio is zero.


Bending stresses
(+ as shown)
Normal stresses
Inplane shear stresses
Top surface
Bottom surface
z
x
y
dx
dy
h
σxx
σyy
σxy = σyx
Bending moments
(+ as shown)
Mxx
Myy
Mxy = Myx
Mxx
Myy
Mxy
Myx
2D view

- According to hook’s law for plane stress,(so no shear forces)[5]

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = -z \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix}.$$

M_{XX}, M_{YY} and M_{XY} are moment per unit Length

$$M_{xx} = -\int_{-h/2}^{h/2} \sigma_{xx}\, z\, dz,$$

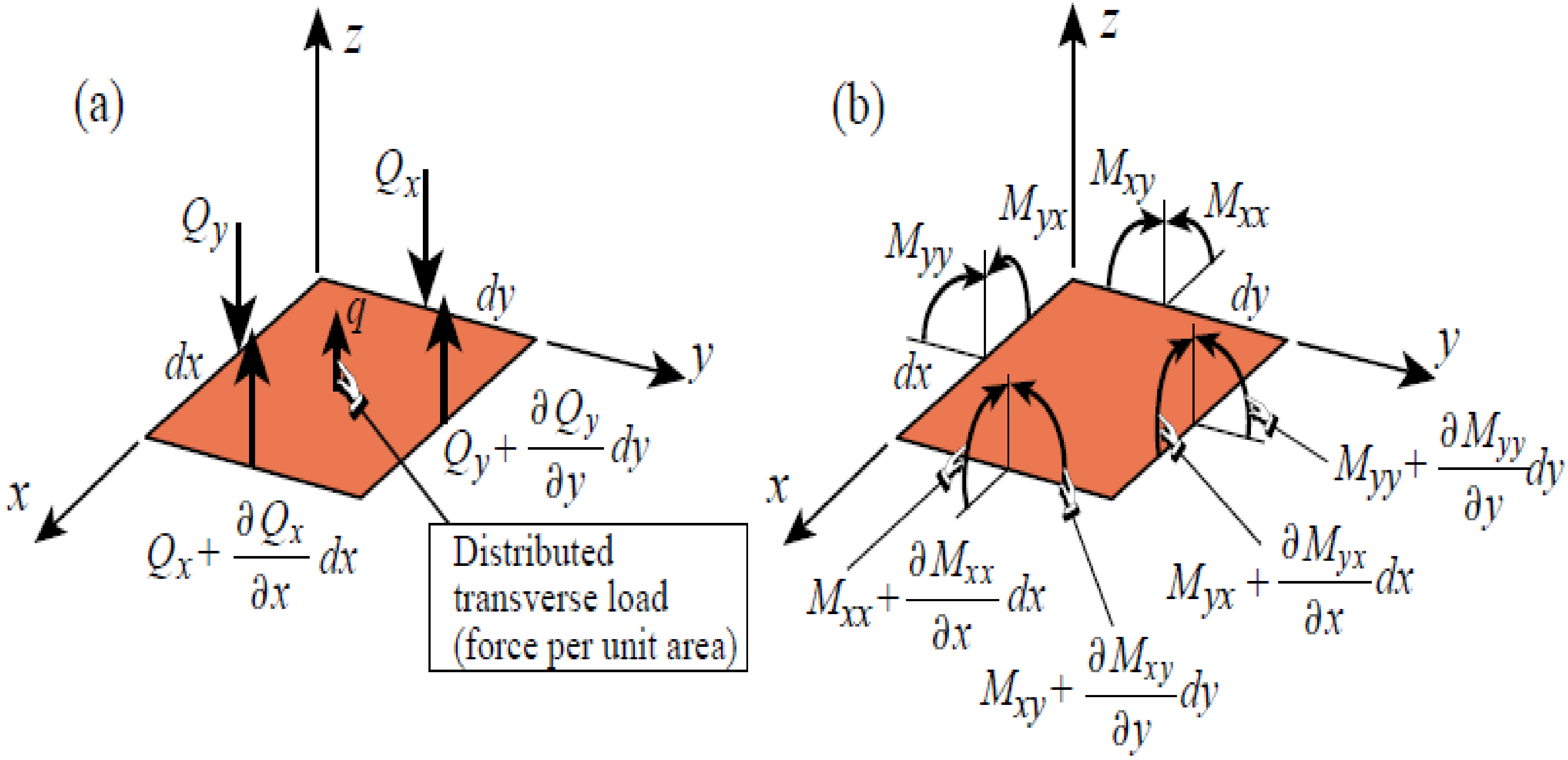
$$M_{yy} = -\int_{-h/2}^{h/2} \sigma_{yy}\, z\, dz,$$

$$M_{xy} = -\int_{-h/2}^{h/2} \sigma_{xy}\, z\, dz,$$

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1+\nu) \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix}$$

$D = \frac{1}{12} E h^3 / (1 - \nu^2)$ D is isotropic plate rigidity coefficient and above equation Is for isotropic plate

- On considering shear forces-[5]



- So on considering shear forces, here are the equations we get
- $$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q\,,$$ force balance along z-direction,
- Moment equilibrium about x and y axis,

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -Q_x\,, \qquad \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} = -Q_y$$

- Replacing the values of Q and Q from above moment equation,

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = q\,.$$

- Writing above equation in terms of w(x,y);[as moment is related to curvature which is given by the double derivative of w with respect to x and y, following is what we get[5]

$$D\,\nabla^4 w = D\,\nabla^2\nabla^2 w = q\,,$$

- Previous equation was derived for any irregular shaped thin plates with uniform thickness, now the vibration of plates while considering the inertia term and thermal moment can be expressed with following equation,[6]

$$D\nabla^2\nabla^2 w + D(1+v)\alpha_T \nabla^2 M_T + \rho h \frac{\partial^2 w}{\partial t^2} = 0,$$

$$M_T = \frac{12}{h^3}\int_{-h/2}^{h/2} \vartheta z \, \mathrm{d}z$$

- the heat transfer equation involving thermoelatic damping

$$\kappa \nabla^2 \vartheta + \kappa \frac{\partial^2 \vartheta}{\partial z^2} = \rho c_v \frac{\partial \vartheta}{\partial t} + \beta T_0 \frac{\partial e}{\partial t},$$

$$\beta = E\alpha_T/(1-2v)$$

- A little modification to the equations lead us to,

$$D\nabla^2\nabla^2 w + D(1+\nu)\alpha_T \nabla^2 M_T + \rho h \frac{\partial^2 w}{\partial t^2} = 0, \quad \text{..........(1)}$$

$$\kappa \frac{\partial^2 \vartheta}{\partial z^2} = \rho c_v \frac{\partial \vartheta}{\partial t} - \beta T_0 z \frac{\partial}{\partial t}(\nabla^2 w). \quad \text{...........(2)}$$

$$e = -z\nabla^2 w.$$

- That equation 2 has been obtained by placing the value of e and ignoring the temperature gradient over the surface of the plate.
- Now we need to solve equation 1 and 2 simultaneously to solve the problem

- We solve the problem by assuming two harmonic functions separately for displacement and temperature,[6]

$$w(r,t) = w_0(r)\mathrm{e}^{\mathrm{i}\omega t}, \quad \vartheta(r,z,t) = \vartheta_0(r,z)\mathrm{e}^{\mathrm{i}\omega t},$$

- We substitute these values in equation 1 and 2 and then try to solve,,here w_0 is mode shape which is determined by applying boundary condition and we attempt to find an expression relating the vibration frequency and mode shape.
- vibrational frequency is expected to have complex value where real and imaginary part imply new eigenfrequency and attenuation effect respectively.
- TED effect is shown as quality factor. Quality factor is the ratio of the total vibrational energy to the loss of vibrational energy per cycle.

- So putting those values in equation 1 and 2 we have,

$$D\nabla^2\nabla^2 w_0 + D(1+v)\alpha_T \nabla^2 M_{T0} - \rho h \omega^2 w_0 = 0, \quad \text{..........(3)}$$

- $$\kappa \frac{\partial^2 \vartheta_0}{\partial z^2} = \mathrm{i}\omega\rho c_v \vartheta_0 - \mathrm{i}\omega\beta T_0 z \nabla^2 w_0, \quad \text{...............(4)}$$

Solution of eq 4 is as follow,

$$\vartheta_0 - \frac{\beta T_0}{\rho c_v} z \nabla^2 w_0 = A \sin(mz) + B \cos(mz),$$

$$m = \sqrt{-\frac{\mathrm{i}\omega\rho c_v}{\kappa}} = (1-\mathrm{i})\sqrt{\frac{\omega\rho c_v}{2\kappa}},$$

As there is no heat flow across upper and lower surface, $\partial\vartheta_0/\partial z = 0$

$$A = -\frac{\beta T_0}{\rho c_v} \frac{1}{m \cos(mh/2)} \nabla^2 w_0, \qquad \vartheta_0(r,z) = \frac{\beta T_0}{\rho c_v} \nabla^2 w_0 \left(z - \frac{\sin(mz)}{m \cos(mh/2)} \right).$$

$$B = 0.$$

- As thermal moment is given as follow,

$$M_{T0} = \frac{12}{h^3} \int_{-h/2}^{h/2} \vartheta_0 z \, \mathrm{d}z.$$

- Putting the value of tem profile obtained in last slide in the above equation and subsequently putting the thermal moment value in equation in eq 3,

$$D(1 + (1 + v)\alpha_T \frac{\beta T_0}{\rho c_v} (1 + \frac{24}{m^3 h^3} \left(\frac{mh}{2} - \tan\left(\frac{mh}{2} \right) \right) \nabla^2 \nabla^2 w_0 - \rho h \omega^2 w_0 = 0,$$

- Solution of above eq is of the following form,

$$w_0(r) = A_0 J_0(pr) + B_0 Y_0(pr) + C_0 I_0(pr) + D_0 K_0(pr),$$

$$p^4 = \rho h \omega^2 / D_\omega,$$

- After a quite cumbersome analysis, real and imaginary part of vibrational frequency is as

$$\mathrm{Re}(\omega) = \omega_0 \left[1 + \frac{\Delta_D}{2} \left(1 - \frac{6}{\xi^3} \frac{\sinh \xi - \sin\xi}{\cosh \xi + \cos\xi} \right) \right],$$

$$\mathrm{Im}(\omega) = \omega_0 \frac{\Delta_D}{2} \left(\frac{6}{\xi^3} \frac{\sinh \xi + \sin \xi}{\cosh \xi + \cos \xi} - \frac{6}{\xi^2} \right).$$

- Here,

$$\Delta_D = \frac{(1+\nu)\alpha_T \beta T_0}{\rho c_v}. \qquad \xi = h\sqrt{\frac{\omega_0 \rho c_v}{2\kappa}},$$

$$\omega_0 = \frac{q_n}{a^2}\sqrt{\frac{D}{\rho h}}.$$

- q_n is determined by the boundary conditions, whether the plate is clamped on both sides or simply supported at one end etc. it can’t have all values but discrete values which satisfy the equations derived by applying boundary conditions.

- one more important relation relating eigenfrequency with other variable is as [6]

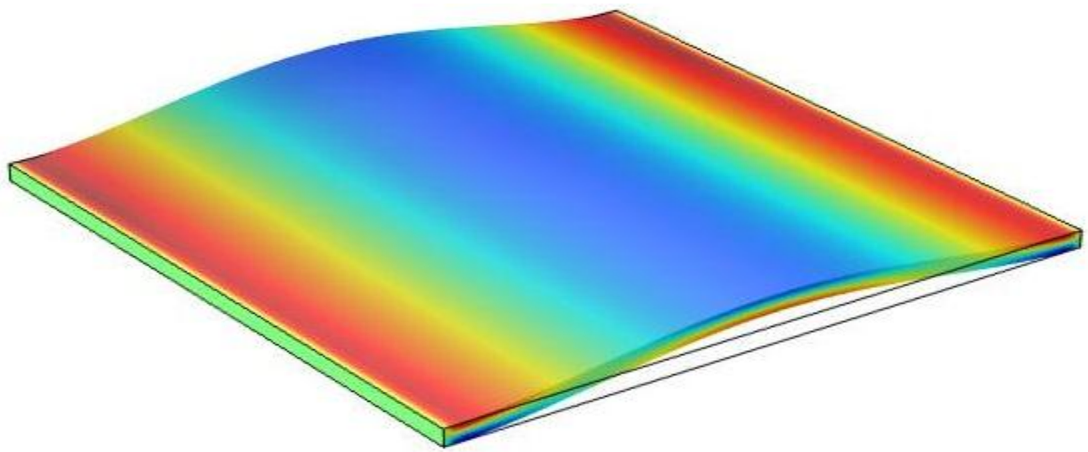
$$\omega = \omega_0 \left[1 + \frac{\Delta_D}{2} (1 + f(\omega_0)) \right].$$

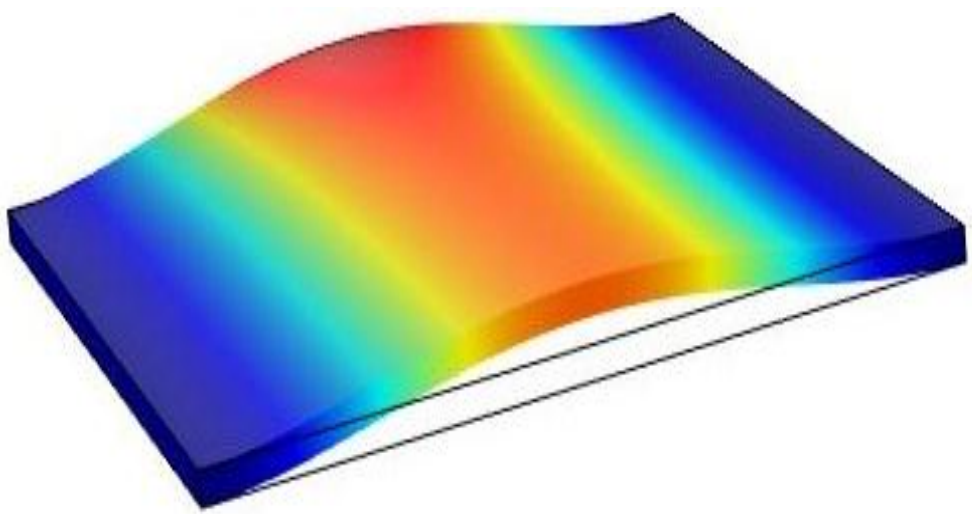
- Hence the quality factor is as follow,

$$Q^{-1} = 2 \left| \frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)} \right| = \Delta_D \left(\frac{6}{\xi^2} - \frac{6}{\xi^3} \frac{\sinh \xi + \sin \xi}{\cosh \xi + \cos \xi} \right)$$

- Simulation on comsol- a square plate of dimension 400 micron and thickness 12 micron at a frequency 0.65 mega hertz.

Temp profile-





-mode shape

• References;-

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[3] comsol mems module model

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[5]AFEM-chapter 20-kirchhoff plates-field equations

[6]thermoelastic damping of axis-symmetric vibration of circular plate resonator, science direct, journal of sound and vibration, Yuxin Sun & Hironori Tohmyoh, department of nanomechanics, Tohoku University, japan.