Analytical Verification of Equivalence of Maxwell Stress Tensor and Electrostatic Surface Force

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Outlines

- Conductors and Dielectrics
- Ponderomotive force
- Equivalent Stress Tensor
- Maxwell Stress Tensor
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- Traction on Surface charge of Conductor
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Conductors and Dielectrics

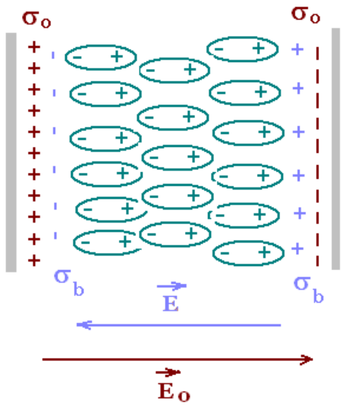

Photo: Scientific sentence

- Materials can be conductors, insulators, semiconductors.
- Any charge introduced into conductor distributes itself over its surface. This displacement, though microscopic, is also found in dielectrics.
- This displacement is characterized by 'electric displacement' ***D*** and defined as,

$$\boldsymbol{D} \stackrel{\text{def}}{=} \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \stackrel{\text{def}}{=} \varepsilon \boldsymbol{E}$$

P is electric polarization vector, ε_0 is permittivity of free space, ε is permittivity of dielectric.

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Ponderomotive force

- The expression of Ponderomotive body force of an electric field is expressed as

$$\boldsymbol{f} = \Psi_v \boldsymbol{E} - \frac{1}{2}\boldsymbol{E}.\boldsymbol{E}\ \boldsymbol{\nabla}\varepsilon + \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{E}.\boldsymbol{E}\ \frac{\partial \varepsilon}{\partial \rho_m}\rho_m)$$

ρ_m is the mass density of dielectric.

- Total force applied on the body

$$\boldsymbol{F} = \int_{\boldsymbol{V}} \boldsymbol{f}\, dV$$

Assumptions :

1. Value of ε changes smoothly in the interfaces of different media.
2. Surface free charge is absent.
3. V is total electric field.

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Equivalent Stress Tensor

- Let $\boldsymbol{\tau_e}$ denote stress tensor equivalent to body force. So,

$$\boldsymbol{F} = \int_{\boldsymbol{V}} \boldsymbol{f}\, dV = \oint_{S} \boldsymbol{\tau_e}\, \boldsymbol{n}\, dS$$

$$\boldsymbol{N} = \int_{\boldsymbol{V}} \boldsymbol{R}\, X\, \boldsymbol{f}\, dV = \oint_{S} \boldsymbol{R}\, X(\boldsymbol{\tau_e}\, \boldsymbol{n})\, dS$$

where ***F*** and ***N*** denotes total force and total moment.

- Total moment equality makes $\boldsymbol{\tau_e}$ to be symmetric.
- $\boldsymbol{f}$ is decomposed into two components

$$\boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2 \text{, where } \boldsymbol{f}_1 = \Psi_v \boldsymbol{E} - \frac{1}{2} \boldsymbol{E}.\boldsymbol{E}\, \boldsymbol{\nabla}\varepsilon$$

$$\text{and} \quad \boldsymbol{f}_2 = \frac{1}{2} \boldsymbol{\nabla}(\boldsymbol{E}.\boldsymbol{E}\, \frac{\partial \varepsilon}{\partial \rho_m} \rho_m)$$

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Maxwell Stress Tensor

- Now, $f_{1x} = (\boldsymbol{\nabla}.\boldsymbol{D})E_x - \frac{1}{2}\boldsymbol{E}.\boldsymbol{E}\ \frac{\partial \varepsilon}{\partial x}$

$$(\boldsymbol{\nabla}.\boldsymbol{D})E_x = \boldsymbol{\nabla}.(E_x\boldsymbol{D}) - \boldsymbol{D}\boldsymbol{\nabla}.E_x$$

and $\boldsymbol{D}\boldsymbol{\nabla}.E_x = \varepsilon\boldsymbol{E}\boldsymbol{\nabla}.E_x = \frac{\varepsilon}{2}\frac{\partial \boldsymbol{E}.\boldsymbol{E}}{\partial x}$

- Hence, $f_{1x} = \frac{\partial(E_x D_x - \frac{\varepsilon}{2}\boldsymbol{E}.\boldsymbol{E})}{\partial x} + \frac{\partial E_x D_y}{\partial y} + \frac{\partial E_x D_z}{\partial z}$
- Similarly ,$f_{1y} = \frac{\partial E_y D_x}{\partial x} + \frac{\partial(E_y D_y - \frac{\varepsilon}{2}\boldsymbol{E}.\boldsymbol{E})}{\partial y} + \frac{\partial E_y D_z}{\partial z}$

$$f_{1z} = \frac{\partial E_z D_x}{\partial x} + \frac{\partial E_z D_y}{\partial y} + \frac{\partial(E_z D_z - \frac{\varepsilon}{2}\boldsymbol{E}.\boldsymbol{E})}{\partial z}$$

- Tensor $\boldsymbol{\tau_{em}}$corresponding to $\boldsymbol{f}_1$ is termed as Maxwell stress tensor.

$$\boldsymbol{\tau_{em}} = \ \varepsilon(\boldsymbol{E} \otimes \boldsymbol{E} - \frac{1}{2}(\boldsymbol{E}.\boldsymbol{E})\boldsymbol{I})$$

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Striction Stress

- Tensor $\boldsymbol{\tau_{es}}$ corresponding to $\boldsymbol{f}_2$ is striction tensor

$$\boldsymbol{\tau_{es}} = \frac{1}{2}(\boldsymbol{E}.\boldsymbol{E}\ \frac{\partial \varepsilon}{\partial \rho_m}\rho_m)\boldsymbol{I}$$

- Equivalent stress tensor, $\boldsymbol{\tau_e} = \boldsymbol{\tau_{es}} + \boldsymbol{\tau_{em}}$

- If we are not interested in the distribution of Ponderomotive forces over the volume of an arbitrary body, but only their resultant, it is sufficient to consider only Maxwell's force and stress.
  - Condition: The body is surrounded by a liquid dielectric in mechanical equilibrium or vacuum.

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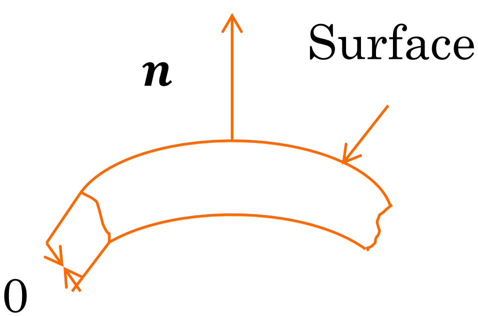
Traction

- The traction on a surface element with unit normal $\boldsymbol{n}$ is

$$\boldsymbol{t} = \boldsymbol{\tau_e}\boldsymbol{n} = \varepsilon E_n \boldsymbol{E} - \frac{1}{2}\boldsymbol{E}.\boldsymbol{E}\,(\varepsilon - \frac{\partial \varepsilon}{\partial \rho_m}\rho_m)\boldsymbol{n}$$

$$\boldsymbol{t}_m = \boldsymbol{\tau_{em}}\boldsymbol{n} = \varepsilon E_n \boldsymbol{E} - \frac{\varepsilon}{2}\boldsymbol{E}.\boldsymbol{E}\ \boldsymbol{n}$$

- Surface charge is a discontinuity.
- The traction on this discontinuity can be derived from above traction by considering charged layer of finite thickness, then passing over in limit to an infinitely thin layer.



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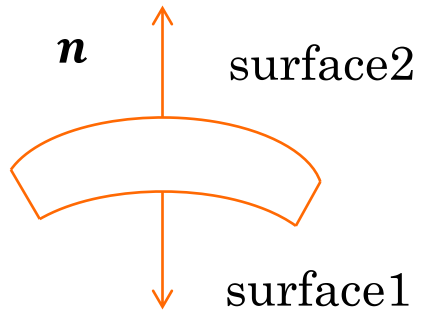
Traction on Surface charge

- $t_{1n} dS = -\varepsilon_1 (E_{1n} \boldsymbol{E}_1 - \frac{1}{2} \boldsymbol{E}_1 . \boldsymbol{E}_1 \, \boldsymbol{n}) dS$
- $t_{2n} dS = \varepsilon_2 (E_{2n} \boldsymbol{E}_2 - \frac{1}{2} \boldsymbol{E}_2 . \boldsymbol{E}_2 \, \boldsymbol{n}) dS$
- Net traction on this thick layer,

$$\boldsymbol{t} = \boldsymbol{t}_{2n} - \boldsymbol{t}_{1n}$$
$$= \varepsilon_2 \left(E_{2n} \boldsymbol{E}_2 - \frac{1}{2} \boldsymbol{E}_2 . \boldsymbol{E}_2 \, \boldsymbol{n} \right) + \varepsilon_1 \left(E_{1n} \boldsymbol{E}_1 - \frac{1}{2} \boldsymbol{E}_1 . \boldsymbol{E}_1 \, \boldsymbol{n} \right)$$



- This relation also holds in the limit.

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Traction on Surface charge of Conductor

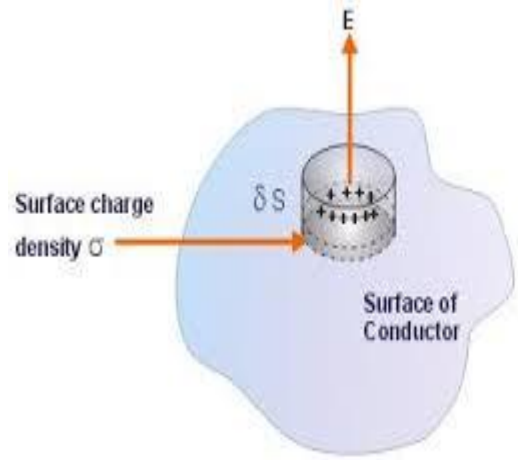
- For Metal $\boldsymbol{E_1} = \mathbf{0}$, $\boldsymbol{E_2} = E_2\boldsymbol{n}$

$$\boldsymbol{t} = \varepsilon_2\left(\frac{1}{2}E_2E_2\right)\boldsymbol{n}$$

- Taking a gaussian surface on the surface of conductor,

$$\varepsilon_2 E_2 = \Psi_s$$

$$\boldsymbol{t} = \varepsilon_2\left(\frac{1}{2}E_2E_2\right)\boldsymbol{n} = \frac{1}{2}\frac{{\Psi_s}^2}{\varepsilon_2}\boldsymbol{n}$$



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