Mass-Spring-Damper or LCR Oscillator

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By referring to Figure A1, we can write an equation to describe the dynamic behavior of the one degree-of-freedom mass-spring-damper system by summing all the forces acting on it:

$$m\ddot{u} + b\dot{u} + ku = F(t) \tag{A.1}$$

where $m\ddot{u}$ is the inertia force, $b\dot{u}$ the damping force, ku the spring force, and $F(t)$ the applied force. The displacement and its first and second time-derivatives are indicated with u, $\dot{u}$, and $\ddot{u}$ respectively.

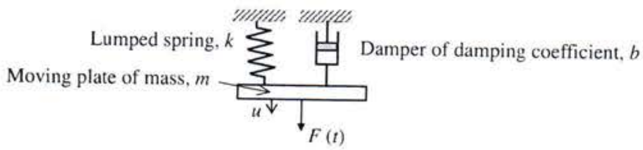

Figure A.1 A mass-spring-damper model.

The dynamics of an electrical LCR (inductor-capacitor-resistor) circuit (see Figure A2) is also governed by the same equation but in terms of different variables.

$$L\frac{dI}{dt} + RI + \frac{\int I\,dt}{C} = V(t) \Rightarrow L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = V(t) \tag{A.2}$$

where Q is charge, I is current, L is inductance, R is resistance, C is capacitance, and $V(t)$ is the applied voltage.

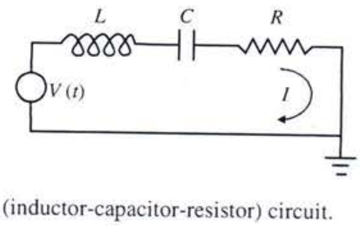

Figure A.2 An electrical LCR (inductor-capacitor-resistor) circuit.

Equations (A.1) and (A.2) are ordinary differential equations (ODEs) of second order because they involve up to the second derivative of the variable to be computed, u in the case of mass-spring-damper and Q in the case of the LCR circuit. In the subsequent discussion, we consider only Eq. (A.1) but note that all of what is presented here applies to the LCR circuit as well.

A.1. FREE VIBRATIONS

Let us first consider Eq. (A.1) with force (i.e., the right hand side term) and damping (i.e., b) equal to zero.

$$m\ddot{u} + ku = 0 \tag{A.3a}$$

$$\Rightarrow u(t) = A\sin(\omega_n t) + B\cos(\omega_n t) \tag{A.3b}$$

where

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \text{(unit : rad/s; divide by } 2\pi \text{ to get in Hz or cycles/s)} \tag{A.3c}$$

It is easy to verify this solution by differentiating u and substituting into the equation. The constants A and B can be found from the initial conditions: $u = 0$ and $\dot{u} = 0$ at $t = 0$.

$$B = u(0)$$
$$A = \frac{\dot{u}(0)}{\omega_n} \tag{A.4}$$

We can readily see that Eq. (A.3b) represents sinusoidal response with frequency, ω_n. This is called the *natural frequency*. It is an important dynamic characteristic of an elastic system with inertia. This gives us the frequency at which the system would naturally oscillate if it were to be disturbed from its state of rest and then let to oscillate without any external influence.

A.2 DAMPED FREE VIBRATIONS

As the first external influence, let us introduce the damping term (i.e., $b\dot{u}$) and write the new solution.

$$m\ddot{u} + b\dot{u} + ku = 0 \tag{A.5a}$$

$$\Rightarrow u(t) = e^{-(b/2m)t}\left\{Ae^{\left(\sqrt{(b/2m)^2-(k/m)}\right)t} + Be^{-\left(\sqrt{(b/2m)^2-(k/m)}\right)t}\right\} \tag{A.5b}$$

where the constants A and B are to be determined from the initial conditions. The quantity under the square root in Eq. (A.5b) may be positive, negative, or zero depending on the values of m, b, and k. Let us keep m and k constant and vary b. The value of b that makes the quantity under the square root sign in Eq. (A.5b) zero is called the *critical damping* and is denoted by b_c.

$$(b/2m)^2 - (k/m) = 0 \Rightarrow b_c = 2\sqrt{km} \tag{A.6}$$

We use this critical damping to define a non-dimensional quantity called the *damping ratio*, ζ.

$$\zeta = \frac{b}{b_c} \tag{A.7}$$

With the help of Eqs. (A.3c, A.6) and (A.7), we can write Eq. (A.5b) as

$$u(t) = e^{-\zeta\omega_n t}\left\{Ae^{i\left(\sqrt{1-\zeta^2}\right)\omega_n t} + Be^{-i\left(\sqrt{1-\zeta^2}\right)\omega_n t}\right\} \tag{A.8}$$

When $\zeta < 1$, Eq. (A.8) gives *under-damped* oscillatory motion that can be re-written as

$$u(t) = e^{-\zeta\omega_n t}\left\{C\sin\left(\omega_n t\sqrt{1-\zeta^2}\right) + D\cos\left(\omega_n t\sqrt{1-\zeta^2}\right)\right\} \tag{A.9}$$

so that there are two constants that are to be determined by initial conditions after Eq. (A.8) is expanded[1] and the imaginary terms are cancelled. The reader may verify that this solution satisfies the ODE, $m\ddot{u} + b\dot{u} + ku = 0$ for $\zeta < 1$. By examining Eqs. (A.3b) and (A.9), we can see that we can define *damped natural frequency*, ω_d, as

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{A.10}$$

Thus, the under-damped response gives an oscillatory motion of a slightly different frequency from the natural frequency. Furthermore, because of $e^{-\zeta\omega_n t}$ term in Eq. (A.9), the amplitude of this oscillation decreases exponentially with time. The solid-line curve in Figure A.3a shows this behavior schematically.

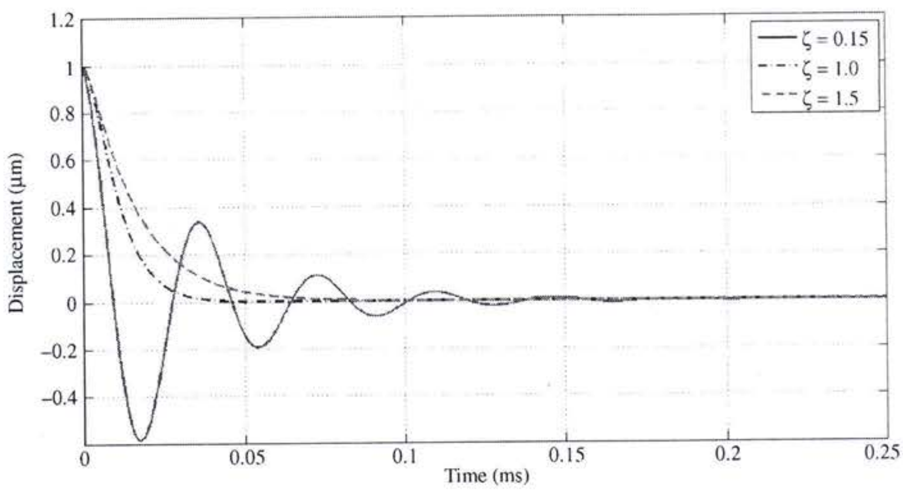

Figure A.3 Under, critical, and over damped cases of a mass-spring-damper system.

[1] Use the identity: $e^{i\theta} = \cos\theta + i\sin\theta$.

When $\zeta = 1$ (called the *critically damped* case), Eq. (A.8) becomes $u(t) = e^{-\zeta\omega_n t}(A + B)$. Since $(A + B)$ can be written together as one constant, we now fall short of one equation to satisfy two initial conditions: $u(0) = u_0$ and $\dot{u}(t) = \dot{u}_0$. So, the solution for $\zeta = 1$ should be re-written as

$$u(t) = e^{-\omega_n t}\{A + Bt\} \tag{A.11}$$

because it not only satisfies the ODE but also gives two constants to satisfy the initial conditions. The dash-dot-line curve in Figure A.3 shows this behavior schematically.

When $\zeta > 1$, Eq. (A.8) becomes directly applicable because $i = \sqrt{-1}$ in $e^{i\left(\sqrt{1-\zeta^2}\right)\omega_n t}$ disappears because $\sqrt{1-\zeta^2}$ can be written as $i\sqrt{\zeta^2 - 1}$ yielding

$$u(t) = Ae^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} + B^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} \tag{A.12}$$

The behavior of this *over-damped* case is illustrated by the dashed-line curve in Figure A.3.

It is important to get an intuitive understanding of the effect of damping on the dynamics of the system by studying Figure A.3 carefully. This is how a system would move from one state to another state when it is disturbed. The term 'state' here represents both the position and the velocity. High damping ($\zeta > 1$) means that it returns to original state quickly and low damping means that ($\zeta < 1$) it will oscillate around the original state for some time before settling. The time constant, τ, which is the reciprocal of the damped natural frequency, indicates how fast the system responds to sudden changes in the state of the system or the external forces.

$$\tau = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{A.13}$$

As can be seen in the curve corresponding to the under-damped case in Figures A.3, a few oscillations elapse before the displacement reaches the steady state. This means that the response time for the system is a multiple of τ.

Example A.1

If $k = 3.0$ N/m and $m = 0.1$E-9 kg, obtain the displacement, $u(t)$, for $\zeta = 0.15$, 1.0, and 1.5 with the initial conditions: $u(0) = 1$E-6 m, and $\dot{u}(0) = 0.0$ m/s.

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3.0}{0.1E-9}} = 0.1732E6 \text{ rad/s} = 27.5664 \text{ kHz}$$

$$b_c = 2\sqrt{km} = 2\sqrt{3 \times 0.1E-9} = 34.6410\text{E-6N/(m/s)}$$

Case (i): $\zeta = 0.15$ (under-damped case)

$$b = \zeta b_c = 0.15 \times 34.6410E-6 = 5.1962\text{E-6 N/(m/s)}$$

From Eq. (A.9) and the given initial conditions, we get

$$u(t) = u(0)\ e^{-\zeta\omega_n t}\cos\left(\omega_n t\sqrt{1-\zeta^2}\right) = 1E-6\left(e^{-25.9087t}\right)\cos(0.1712E6\ \ t)$$

This is plotted as a solid-line curve in Figure A.3.

Case (ii): $\zeta = 1.0$ (critically-damped case)

$$b = \zeta b_c = 1.0 \times 34.6410E - 6 = 34.6410E\text{-}6 \text{ N/(m/s)}$$

From Eq. (A.11) and the given initial conditions, we get

$$u(t) = e^{-\omega_n t}\{A + Bt\} = 1E - 6(e^{-0.1732E6\ t})(1 + 0.1732E6\ t)$$

This is plotted as a dash-dot-line curve in Fig. A.3.

Case (iii): $\zeta = 1.5$ (over-damped case)

$$b = \zeta b_c = 1.5 \times 34.6410E - 6 = 51.9615\text{ E} - 6 \text{ N/(m/s)}$$

From Eq. (A.12) and the given initial conditions, we get

$$u(t) = Ae^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} + B^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} = -0.1708E - 6(e^{-0.4534E6\,t}) + 1.1708E - 6(e^{-0.0662E6\,t})$$

This is plotted as a dashed-line curve in Fig. A.3.

A.3 DAMPED VIBRATIONS WITH A PERIODIC FORCE

For a periodic force, $F_0 \sin(\omega t)$, we have

$$m\ddot{u} + b\dot{u} + ku = F_0 \sin(\omega t) \tag{A.14}$$

whose solution is given by

$$u = U\sin(\omega t - \varphi) \tag{A.15a}$$

where

$$U = \frac{(F_0/k)}{\sqrt{\left(1 - (\omega^2/\omega_n^2)\right)^2 + \left(2\zeta(\omega/\omega_n)\right)^2}} \tag{A.15b}$$

and

$$\tan\varphi = \frac{2\zeta(\omega/\omega_n)}{\left(1 - (\omega^2/\omega_n^2)\right)} \tag{A.15c}$$

It can be readily seen from Eqs. (A.15b-c) that when $\omega = \omega_n$, the amplitude U equals $(F_0/2\zeta k)$ and the phase, φ, becomes 90°. This condition is referred to as *resonance*. If damping is very low, Eq. (A.15b) tells us that U becomes very large. When there is no damping at all, it reaches infinity. That is, an elastic system vibrates with large amplitude when the frequency of the periodic force equals the natural frequency of the system. Figure A.4 shows the amplitude variation with applied frequency for the numerical data of Example A.1. Such a plot is called the *frequency response* of the system and it plays a very important role in characterizing the dynamics of a system. Figure A.4 shows U in Eq. (A.15b) on a semi-log plot where the applied frequency (i.e., ω) is plotted on a log scale. On the other hand, in Figure A.5, both axes are in the log scale.

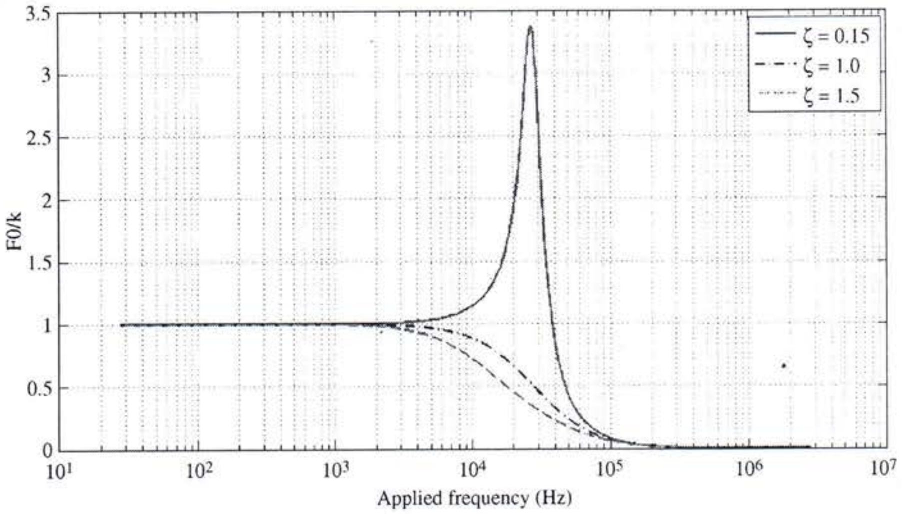

Figure A.4 Frequency response on the semi-log scale for under, critical, and over damped cases.

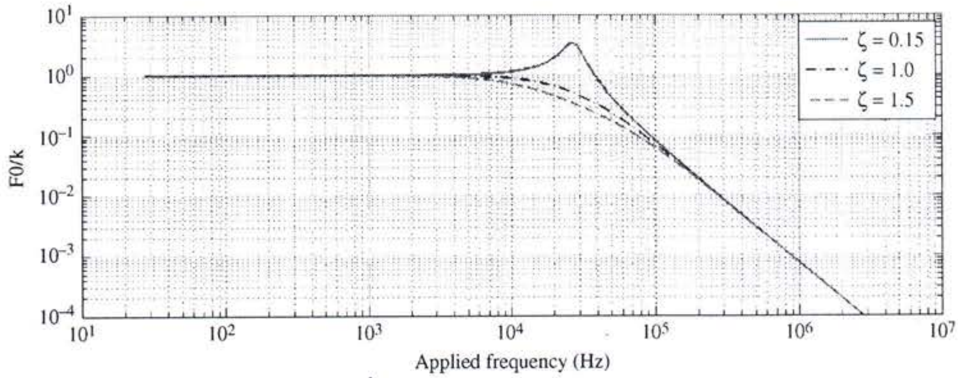

Figure A.5 Frequency response on the ~~semi~~ log-log scale for under, critical, and over damped cases.

It can be seen that the response, i.e., (F_0/k), stays constant only for a certain band of frequency from zero to about one fifth to one third of ω_n. This is called the *bandwidth* of the system wherein the output in the frequency response is independent of the applied frequency. Thus, the output of a sensor or an actuator is reliable only within the system's bandwidth. This is because, outside this range there will be ambiguity if the same (F_0/k) arises due to different value s of F_0 and ω. In other words, $\{F_{01}, \omega_1\}$ and $\{F_{02}, \omega_2\}$ may give the same (F_0/k). To avoid such ambiguities, in practice, it is customary to take one third the natural frequency to be the bandwidth of the system.

One more feature worth noticing in Figures A.4-5 is how sharp the peak is at the resonance frequency. For many applications, we want this peak to very sharp so that the response is high only over a very narrow band of frequency around the resonance frequency. The sharpness of the resonance is quantified by what is known as the *quality factor*, Q. It is given by

$$Q = \frac{1}{2\zeta} \tag{A.16}$$

The range 2ζ around the natural frequency has the property that (F_0/k) is more than $(0.707/2\zeta)$. From Eqs. (A.6-7), we see that

$$Q = \frac{2\sqrt{km}}{b} \tag{A.17}$$

Therefore, high quality factor is obtained when damping is very low or mass and stiffness are very high.