Chapter 2

The Principle of Minimum Potential Energy

The objective of this chapter is to explain the principle of minimum potential energy and its application in the elastic analysis of structures. Two fundamental notions of the finite element method viz. discretization and numerical approximation of the exact solution are also explained.

2.1 The principle of Minimum Potential Energy (MPE)

Deformation and stress analysis of structural systems can be accomplished using the principle of Minimum Potential Energy (*MPE*), which states that

> *For conservative structural systems, of all the kinematically admissible deformations, those corresponding to the equilibrium state extremize (i.e., minimize or maximize) the total potential energy. If the extremum is a minimum, the equilibrium state is stable.*

Let us first understand what each term in the above statement means and then explain how this principle is useful to us.

A constrained structural system, i.e., a structure that is fixed at some portions, will deform when forces are applied on it. Deformation of a structural system refers to the incremental change to the new deformed state from the original undeformed state. The deformation is the principal unknown in structural analysis as the strains depend upon the deformation, and the stresses are in turn dependent on the strains. Therefore, our sole objective is to determine the deformation. The deformed state a structure attains upon the application of forces is the equilibrium state of a structural system. The Potential energy (PE) of a structural system is defined as the sum of the strain energy (*SE*) and the work potential (*WP*).

$$PE = SE + WP \tag{1}$$

The strain energy is the elastic energy stored in deformed structure. It is computed by integrating the strain energy density (i.e., strain energy per unit volume) over the entire volume of the structure.

$$SE = \int_V (strain\, energy\, density)\, dV \tag{2}$$

The strain energy density is given by

$$Strain\, energy\, density = \frac{1}{2}(stress)(strain) \tag{2a}$$

The work potential *WP*, is the negative of the work done by the external forces acting on the structure. Work done by the external forces is simply the forces multiplied by the displacements at the points of application of forces. Thus, given a deformation of a structure, if we can write down the strains and stresses, we can obtain *SE*, *WP*, and finally *PE*. For a structure, many deformations are possible. For instance, consider the pinned-pinned beam shown in Figure 1a. It can attain many deformed states as shown in Figure 1b. But, for a given force it will only attain a unique deformation to achieve equilibrium as shown in Figure 1c. What the principle of *MPE* implies is that this unique deformation corresponds to the extremum value of the MPE. In other words, in order to determine the equilibrium deformation, we have to extremize the *PE*. The extremum can be either a minimum or a maximum. When it is a minimum, the equilibrium state is said to be stable. The other two cases are shown in Figure 2 with the help of the classic example of a rolling ball on a surface.

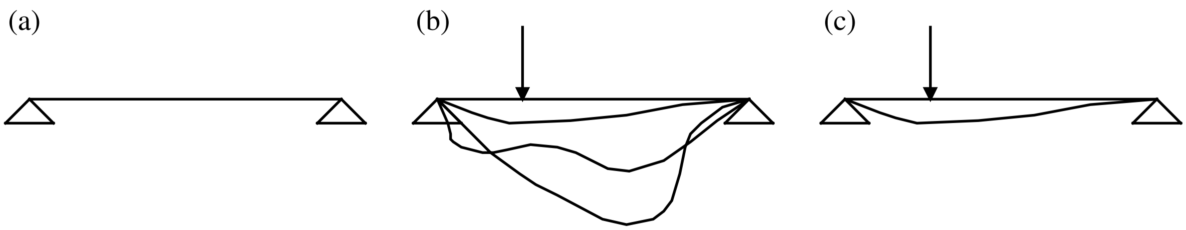


Figure 1 The notion of equilibrium deformed state of a pinned-pinned beam

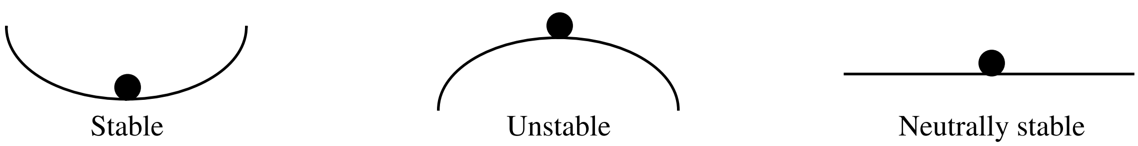


Figure 2 Three equilibrium states of a rolling ball

There are two more new terms in the statement of the principle of *MPE* that we have not touched upon. They are conservative system and the kinematically admissible deformations. Conservative systems are those in which *WP* is independent of the path taken from the original state to the deformed state. Kinematically admissible deformations are those deformations that satisfy the geometric (kinematic) boundary conditions on the structure. In the beam example above (see Figure 1), the boundary conditions include zero displacement at either end of the beam. Now that we have defined all the terms in the statement, it is a good time to read it again to make more sense out of it before we apply it.

2.2 Application of MPE principle to lumped-parameter uniaxial structural systems

Consider the simplest model of an elastic structure viz. a mass suspended by a linear spring shown in Figure 3. We would like to find the static equilibrium position of the mass when a force F is applied. We will first use the familiar force-balance method, which gives

$$F = spring\ force = kx \qquad \text{at equilibrium (} k \text{ is the spring constant)}$$

$$\therefore \qquad x_{equilibrium} = \delta = \frac{F}{k} \tag{3}$$

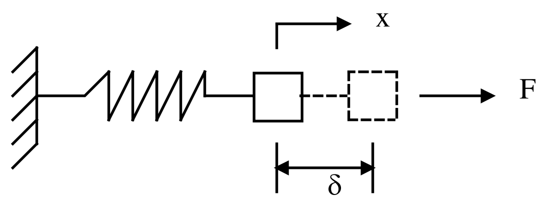


Figure 3 Simplest model of an elastic structural system

We can arrive at the same result by using the MPE principle instead of the force-balance method. Let us first write the *PE* for this system.

$$PE = (SE) + (WP) = \left(\frac{1}{2}kx^2\right) + \left(-Fx\right) = \frac{1}{2}kx^2 - Fx \tag{4}$$

As per the *MPE* principle, we have to find the value of x that extremizes *PE*. The condition for extremizing *PE* is that the first derivative of *PE* with respect to x is zero.

$$\frac{d(PE)}{dx} = 0 \Rightarrow kx - F = 0 \Rightarrow x_{equilibrium} = \delta = \frac{F}{k} \tag{5}$$

We got the same result as in Equation (3). Further, verify that the second derivative of *PE* with respect to x is positive in this case. This means that the extremum is a minimum and therefore the equilibrium is stable.

Figure 4 pictorially illustrates the *MPE* principle: of all possible deformations (i.e., the values of x here), the stable equilibrium state corresponds to that x which minimizes *PE*. For the assumed values of k = 5, and F = 10, equilibrium deflection is 2 which is consistent with Figure 4. As illustrated in Figure 3, the *MPE* principle is an alternative way to write the equilibrium equations for elastic systems. It is, as we will see, more efficient than the force-balance method. Let us now consider a second example of a spring-mass system with three degrees of freedom viz. q_1, q_2, and q_3. *The number of degrees of freedom of a system refers to the minimum number of independent scalar quantities required to completely specify the system*. It is easy to see that the system shown in Figure 5 has three degrees of freedom because we can independently move the three masses to describe this completely.

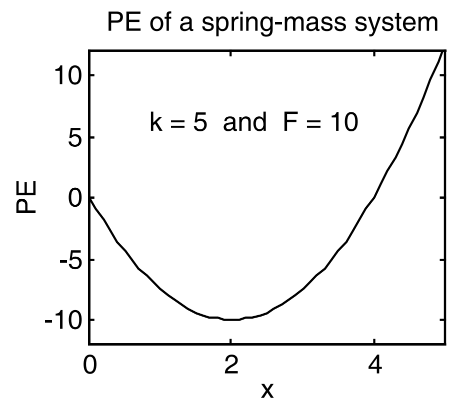


Figure 4 PE of a spring-mass system

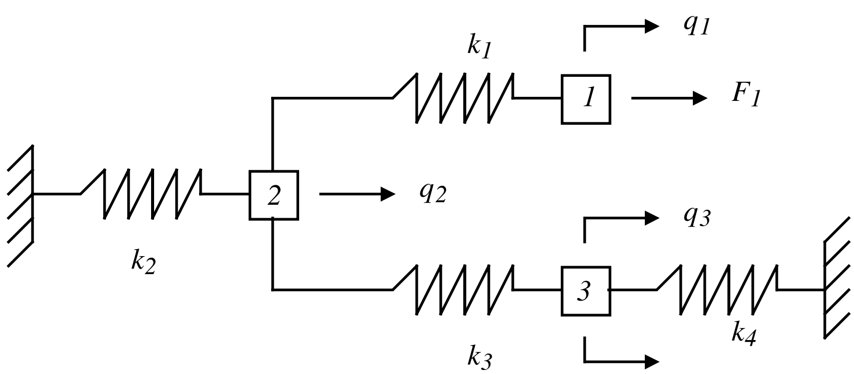


Figure 5 A spring-mass system with three degrees of freedom

We will use the *MPE* principle to solve for the equilibrium values of q_1, q_2, and q_3 when forces F_1 and F_3 are applied (Note that one can also apply F_2, but in this problem we assume that there is no force on mass 2). In order to write the *SE* for the springs, we need to write the deflection (elongation or contraction) of the springs in terms of the degrees of freedom q_1, q_2, and q_3.

$$\begin{aligned} u_1 &= q_1 - q_2 \\ u_2 &= q_2 \\ u_3 &= q_3 - q_2 \\ u_4 &= -q_3 \end{aligned} \tag{6}$$

The *PE* for this system can now be written as

$$PE = \left(\frac{1}{2}k_1 u_1^2 + \frac{1}{2}k_2 u_2^2 + \frac{1}{2}k_3 u_3^2 + \frac{1}{2}k_4 u_4^2 \right) + \left(-F_1 q_1 - F_3 q_3 \right)$$

$$PE = \left(\frac{1}{2} k_1 (q_1 - q_2)^2 + \frac{1}{2} k_2 q_2^2 + \frac{1}{2} k_3 (q_3 - q_2)^2 + \frac{1}{2} k_4 q_3^2 \right) + \left(-F_1 q_1 - F_3 q_3 \right) \tag{7}$$

For equilibrium, PE should be an extremum with respect to all three q's. That is,

$$\frac{\partial (PE)}{\partial q_i} = 0 \qquad \text{for} \qquad i = 1, 2, \text{ and } 3. \tag{8a}$$

i.e.,
$$\frac{\partial (PE)}{\partial q_1} = k_1 (q_1 - q_2) - F_1 = 0 \tag{8b}$$

$$\frac{\partial (PE)}{\partial q_2} = -k_1 (q_1 - q_2) + k_2 q_2 - k_3 (q_3 - q_2) = 0 \tag{8c}$$

$$\frac{\partial (PE)}{\partial q_3} = k_3 (q_3 - q_2) + k_4 q_3 - F_3 = 0 \tag{8d}$$

Noting the relationship between q's and u's from Equation (6), we can readily see that the equilibrium equations obtained in Equations (8) can be directly obtained from force-balance on the three masses as shown in Figure 6.

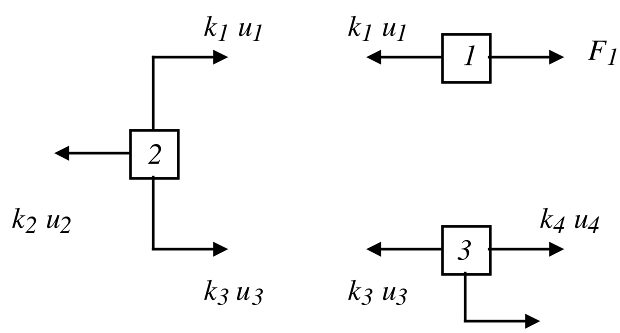


Figure 6 Force-balance free-body-diagrams for the system in Figure 5

It is important to note that Equations (8) were obtained routinely from the *MPE* principle where as force-balance method requires careful thinking about the various forces (including the internal spring reaction forces and their directions. Thus, for large and complex systems, the *MPE* method is clearly advantageous, especially for implementation on the computer.

The linear Equations (8) can be written in the form of matrix system as follows:

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ F_3 \end{Bmatrix} \tag{9a}$$

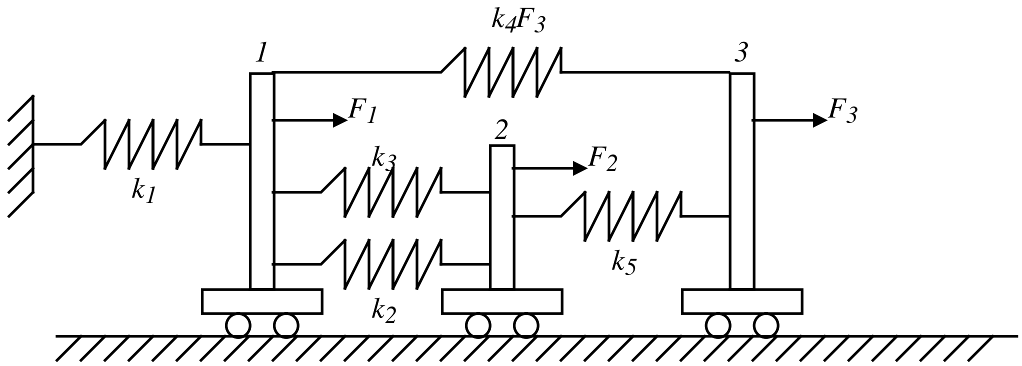
or
$$\mathbf{Kq} = \mathbf{F} \tag{9b}$$

(bold letters indicate that they are either vectors or matrices.)

The matrix $\mathbf{K}$ is referred to as the <u>stiffness matrix</u> of a structural system. Any linear elastic structural system can be represented as Equation (9b). We will see later that the finite element method enables us to construct the matrix $\mathbf{K}$, and vectors $\mathbf{q}$ and $\mathbf{F}$ systematically for any complex structure.

Exercise 2.1

Use MPE principle and the force-balance method to obtain the equilibrium equations shown in the matrix representation in Figure 7.



$$\begin{bmatrix} k_1+k_2+k_3+k_4 & -k_2-k_3 & -k_4 \\ -k_2-k_3 & k_2+k_3+k_5 & -k_5 \\ -k_4 & -k_5 & k_4+k_5 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Figure 7 A spring-mass system and its equilibrium matrix equation

2.3 Modeling axially loaded bars using the spring-mass models

The spring-mass model is useful in arriving at the equilibrium equations for an axially loaded bar as shown in Figure 8. For a bar of uniform cross-section A, homogeneous material with Young's modulus E, and total length l, the spring constant k is given by

$$k = \frac{AE}{l} \tag{10}$$

In order to see how we wrote Equation (10), consider the following equations.

$$stress = \frac{F}{A}; \quad strain = \frac{\delta}{l}; \quad E = \frac{stress}{strain} = \frac{Fl}{A\delta}; \quad \Rightarrow F = \left(\frac{AE}{l}\right)\delta = (k)\delta \tag{11}$$

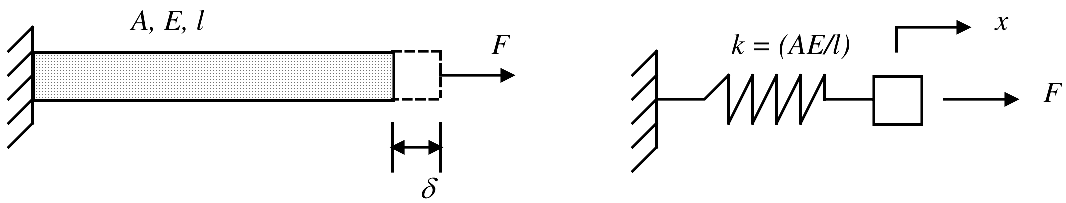


Figure 8 Axially loaded bar as a spring-mass system

Now, we can also analyze a stepped bar (a bar with two different cross-section areas) under two concentrated forces F_1 and F_2 as shown in Figure 9.

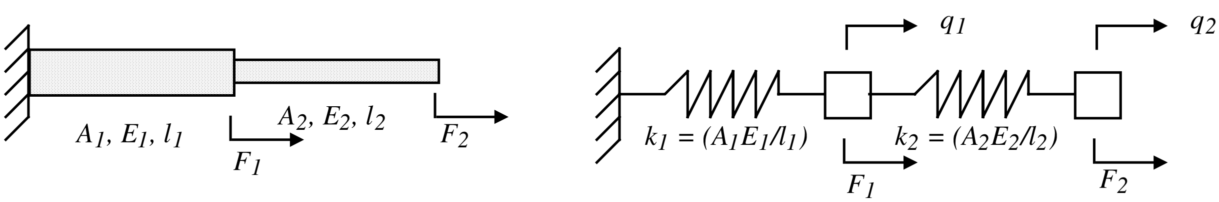


Figure 9 Axially loaded stepped bar and its lumped spring-mass model

Exercise 2.2a

Solve for q_1 and q_2 for the system shown in Figure 9 using the MPE principle.

Answer:
$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}^{-1} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{12}$$

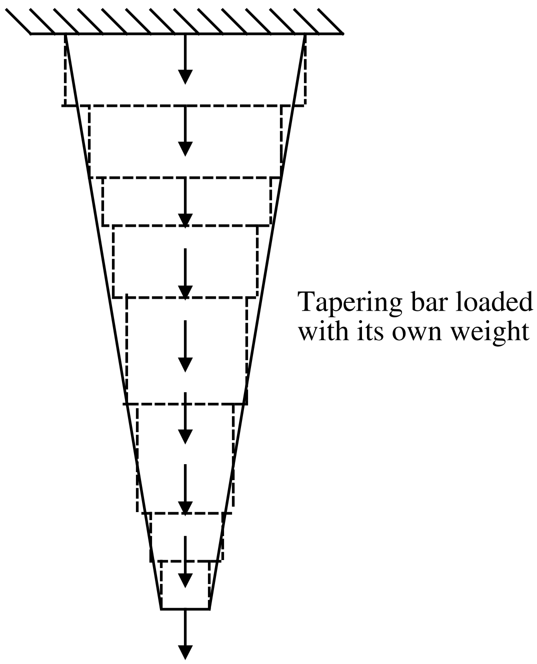
Exercise 2.2b

Repeat Exercise 2.2a when there are three segments. That is, determine the displacements q_1, q_2, and q_3.

Answer:
$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}^{-1} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \tag{13}$$

Do you see any pattern emerging after working out Exercises 2.2a and b? If you work through more number of segments, we will see the <u>tridiagonal</u> pattern in the stiffness matrix. Let us now proceed to use this for a more realistic problem.

Consider the linearly tapering bar loaded with its own weight. This can be easily modeled as a spring mass system. This type of <u>lumped modeling</u> gives only an approximate solution, and as you can imagine, the accuracy improves with increased number of segments. As we increase the number of segments, the number of degrees of freedom increases (i.e., more q's) and the size of the stiffness matrix increases. But, the procedures for doing two bar segments in Figure 9 or many bar segments in Figure 10 are exactly the same, except that it is repetitive and tedious as the number of segments increases. However, it is ideal for implementation on a computer. Notice that k for each segment is of the identical form.



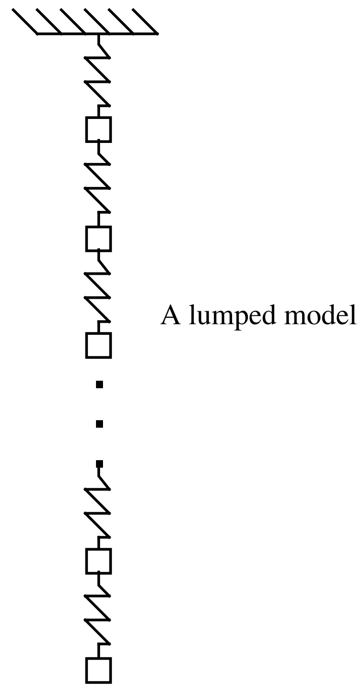


Figure 10 A tapering bar loaded with its own weight, and its lumped spring-mass model

This example illustrates two important concepts.

• *Continuous systems can be approximated as lumped segments. This is called* *<u>discretization</u>—an important concept in FEM. The segments are called <u>"finite elements"</u>".*

• *All elements have the identical form. So, a general method can be developed to handle large and complex structures. That is, by discretizing the structure into identical elements, the whole structure can be analyzed in a repetitive manner systematically.*

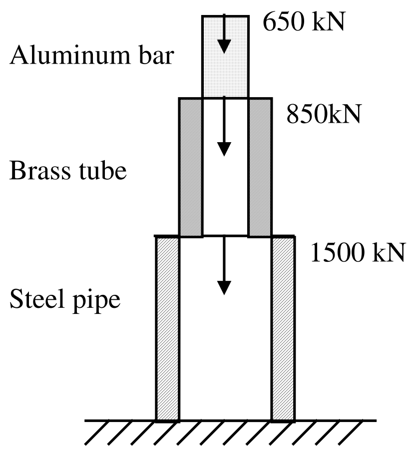
What we have done in this Chapter is not FEM yet. It suffices to note at this point that FEM provides a systematic way of discretizing a complex structure to get an approximate solution. In addition to the intuitive notion presented in this chapter, there is a firm theoretical basis for FEM. We will examine that in the later Chapters. Before we embark upon FEM formulation, it is worthwhile to discuss another important concept method called the Rayleigh-Ritz method. That is the topic of the next Chapter.

2.4 Implementation of mass-spring systems in Matlab

The finite element method is a numerical method. It is important to understand the practical implementation of it in addition to gaining a theoretical understanding of it. This notes emphasizes this aspect and includes finite element programs written in Matlab. You can do this in Maple, Mathematica, MathCad or anything else you are comfortable with. In order to be prepared to handle the finite element programs later, let us get started here with a simple Matlab script to solve the problem shown in Figure 11.

Exercise 2.3

Write down the matrix equation system for the system shown in Figure 11 and study its implementation in the attached Matlab script. Run the script to get experience with Matlab.



The steel pipe ID and OD are 125 mm and 200 mm.
The brass tube ID and OD are 100 mm and 150 mm.
The aluminum bar cross-section diameter is 100 mm.

E-steel = 210 GPa
E-brass = 100 GPa
E-aluminum = 73 GPa

Find the displacements at the points where forces are applied.

Figure 11 A composite axially loaded system

Matlab script 1 for Exercise 2.3

```
clear all
clc
clg
hold off
axis normal

% Aluminum bar
E1 = 73E9;                              % Pa
A1 = (pi/4)*(100E-3)^2;                      % m^2
L1 = 1.0;                               % m
```

```
% Brass tube
E2 = 100E9;                                      % Pa
A2 = (pi/4)*( (150E-3)^2 - (100E-3)^2 ) ;        % m^2
L2 = 1.25;                                  % m

% Steel pipe
E3 = 210E9;                                      % Pa
A3 = (pi/4)*( (200E-3)^2 - (125E-3)^2 );         % m^2
L3 = 0.75;                                  % m

% Forces
F = [-650 -850 -1500]*1e3;                  % N

% Compute the spring constants
k1 = A1*E1/L1;
k2 = A2*E2/L2;
k3 = A3*E3/L3;

% Construct the stiffness matrix of the system
K(1,1) = k1;
K(1,2) = -k1;
K(1,3) = 0;

K(2,1) = -k1;
K(2,2) = k1+k2;
K(2,3) = -k2;

K(3,1) = 0;
K(3,2) = -k2;
K(3,3) = k2+k3;

% Solve for displacements
u = inv(K)*F'
```

Exercise 2.4

Solve the linearly tapering bar problem by using a Matlab script. The advantage of writing in Matlab (or other similar software) is that we can vary the number of elements (i.e., <u>the "fineness" of discretization</u>) and observe what happens. Assume the following data.

The bar is made of aluminum (E = 73 GPa, mass density = 2380 Kg/m^3), and has a circular cross-section with beginning diameter of 100 mm and tip diameter of 20 mm. The length of the bar is 1 m.

Matlab script 2 for Exercise 2.4

```
clear all
clc
%clg
%hold off
axis normal

% Tapering aluminum bar under its own weight
E = 73E9;                       % Pa
A0 = (pi/4)*(100E-3)^2;              % m^2
At = (pi/4)*(20E-3)^2;               % m^2
L = 1.0;                        % m
rho = 2380*9.81;                % N/m^3

echo on
N = 2;                          % Number of elements
% Change the number of elements and see the how the accuracy
% of the solution improves. You need to run the script many
% times by changing the number of element N, above.
% Note that the hold on graphics is on.
echo off

% Compute element length, area, k and force
Le = L/N;
for i = 1:N,
      Atop = A0 -(A0-At)/N*(i-1);
      Abot = A0 - (A0-At)/N*i;
      A(i) = (Atop+Abot)/2;
      x(i) = L/N*i;
      k(i) = A(i)*E/Le;
      F(i) = A(i)*Le*rho;
end

% Assembly of the stiffness matrix using k's.
K = zeros(N,N);

K(1,1) = k(1) + k(2);
K(1,2) = -k(2);

for i = 2:N-1,
    K(i,i-1) = -k(i);
      K(i,i) = k(i) + k(i+1);
      K(i,i+1) = -k(i+1);
end

K(N,N-1) = -k(N);
K(N,N) = k(N);

% Solve for displacements {q}. It is a column vector.
q = inv(K)*F';

plot([0 x],[0; q],'-w',x,q,'c.');
hold on
```

```
title('Effect of Discretization');
xlabel('X -- the length of the bar (m)');
ylabel('Axial deformation (m)');
```

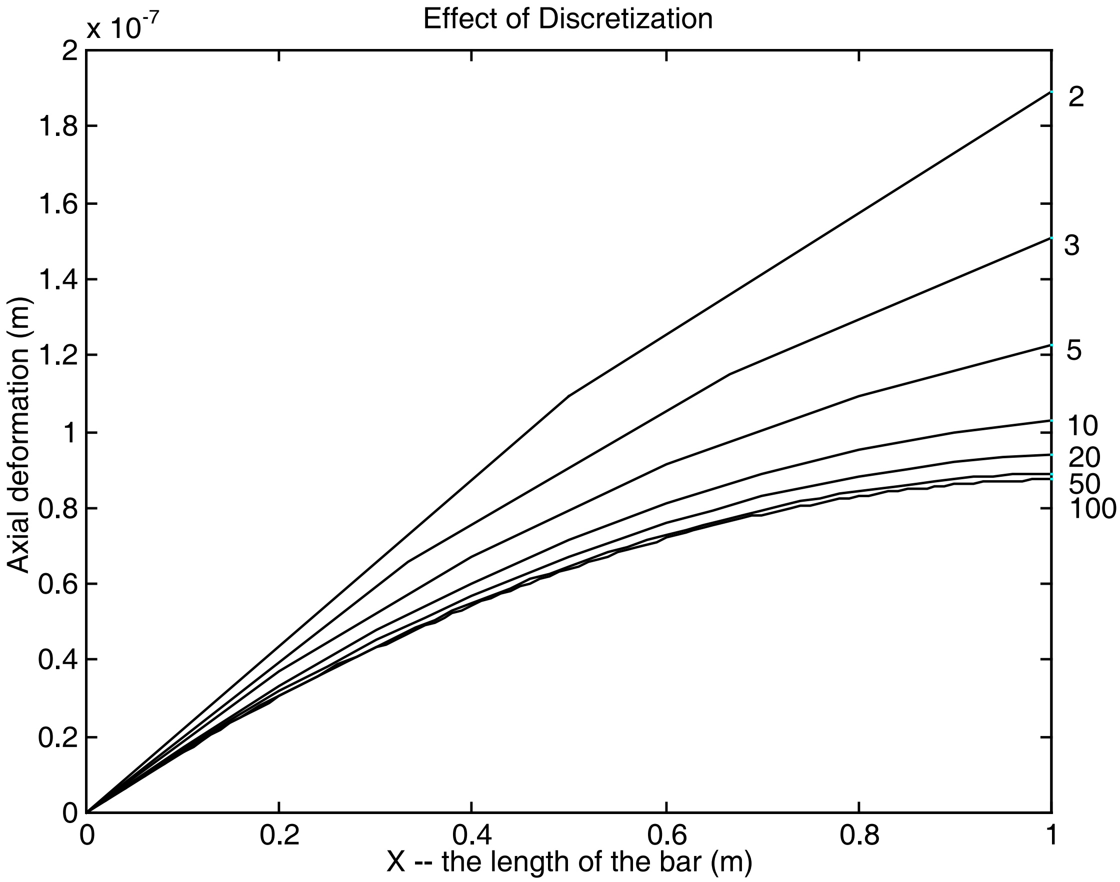


Figure 12 Effect of discretization in the lumped-model

It can be seen in the figure that as the number of elements increases, the solution begins to converge to the exact solution. More about this in the next chapter.