Chapter 3

Rayleigh-Ritz Method

As discussed in Chapter 2, one can solve axially loaded bars of arbitrary cross-section and material composition along the length using the lumped mass-spring model. As shown in Figure 12 of Exercise 2.4, one can approach the exact solution very closely by dividing the bar into more elements. One of the disadvantages of the lumped models is that we can only compute the deflection at the locations of the lumped masses (we call these points *nodes*), and we know nothing about what happens within the element. Consequently, if we want to get the smooth shape of the deflection curve, we need to take a very large number of elements. The Raleigh-Ritz method offers an alternative method to overcome these problems. This method also uses the MPE principle.

Referring back to the tapering beam problem, what we were able to do with the lumped model is essentially solving the governing differential equation that represents the deflection of axially loaded bars. Our method of solution was of course numerical. It is worthwhile to study the differential equation that we just solved numerically in Chapter 2.

Thus, the objectives of this Chapter are: (i) Derive the differential equation of an axially loaded bar using the force-balance method (ii) Derive the same equation using the MPE principle (iii) Discuss the Rayleigh-Ritz method.

3.1 Derivation of the governing differential equation of an axially loaded bar using the force-balance method

Let $A(x)$, the cross-section area of the bar at x, be given. There is a body-force (gravity-like force), $f(x)$, per unit volume of the bar. $\sigma(x)$, the axial stress and $u(x)$, the axial deflection, are two unknown functions. We would like to derive a differential equation that describes the axially loaded bar so that we can solve for $\sigma(x)$ and $u(x)$.

Consider a differential element of length dx at some x. The stress and area at the left end of the differential element are $\sigma(x)$ and $A(x)$. At $(x+dx)$, the right end, the same quantities can be approximated as $\left(\sigma(x)+\frac{d\sigma(x)}{dx}dx\right)$ and $\left(A(x)+\frac{dA(x)}{dx}dx\right)$. The free-body-diagram of the infinitesimally small differential element shows that the internal forces (stresses multiplied by areas of cross-section) balance the body-force acting to the right. The body force acting on the differential element is given by $f(x)A(x)dx$. Let us now expand and simplify the internal force acting to the right.

$$\left(\sigma(x)+\frac{d\sigma(x)}{dx}dx\right)\left(A(x)+\frac{dA(x)}{dx}dx\right)$$

$$= \sigma(x)A(x)+\sigma(x)\frac{dA(x)}{dx}dx+A(x)\frac{d\sigma(x)}{dx}dx+\underbrace{\cancel{\left(\frac{d\sigma(x)}{dx}\right)\left(\frac{dA(x)}{dx}\right)dx^2}}_{0} \tag{1}$$

The last term in the above expression is a small second-order term and hence it can be ignored as shown stricken by an arrow in Equation (1). The first term balances the internal force acting on the left end of the differential element. So, the second and third terms and the body-force term should sum to zero for equilibrium

$$\sigma(x)\frac{dA(x)}{dx}dx+A(x)\frac{d\sigma(x)}{dx}dx+f(x)A(x)dx=0 \tag{2a}$$

You can easily check that after canceling dx although in the above equation, the two terms on the left hand side can be collapsed as one term as shown below.

$$\frac{d\left(\sigma(x)A(x)\right)}{dx}+f(x)A(x)dx=0 \tag{2}$$

This leads to the following differential equation in *σ(x)*.

$$\frac{d}{dx}\left(\sigma(x)\,A(x)\right)+f(x)A(x)=0 \tag{3}$$

Next, we would like to express *u(x)* in terms of *σ(x)* so that we can get the governing differential equation in *u(x)*. From the definition of axial strain (change in length divide by the original length), we get the following expression for strain, $\varepsilon(x)=\frac{du(x)}{dx}$, where *du(x)* is the deflection of the differential element of length *dx*. We also know the relationship between stress and strain: $\sigma(x)=E\,\varepsilon(x)$ where *E* is the Young's modulus. By substituting these relationships into Equation (3), we get the governing differential equation:

$$\frac{d}{dx}\left(E\,A(x)\frac{du(x)}{dx} \right) + f(x)A(x) = 0 \tag{4}$$

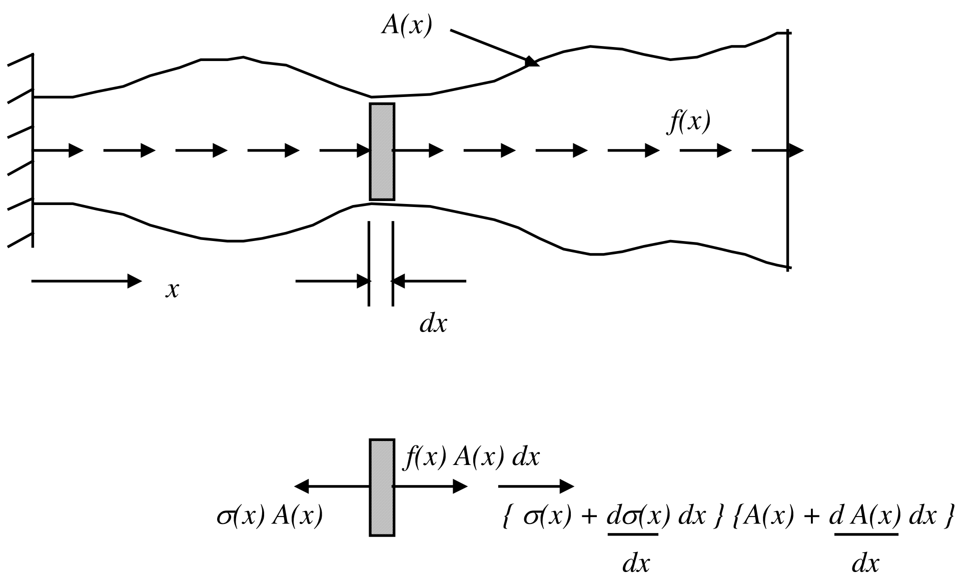

Figure 1 Force balance of a differential element in an axially loaded bar

We had observed in Chapter 2 that the equilibrium equations could be written using the force balance method as well as the MPE principle. For the continuous model of an axially loaded bar, we just derived the equilibrium differential equation using the force-balance method. We will obtain the same equation using the MPE principle now.

3.2 Derivation of the governing equation using the *MPE* principle

In this method, first we need to write down the *PE* of the system. Since this is a continuous model, both *SE* and *WP* are integrals over the length of the bar. Note that

$$SE = \int_{dV} (strain\,energy\,density)\,dV = \int_{dV} \frac{1}{2}(stress)(strain)\,dV$$

$$= \int_0^L \frac{1}{2}\left(E\frac{du(x)}{dx}\right)\left(\frac{du(x)}{dx}\right) A(x)dx \tag{5}$$

$$WP = -\int_0^L f(x)A(x)u(x)\,dx \tag{6}$$

By denoting $\frac{du(x)}{dx}$ by u', from Equations (5) and (6), the *PE* can be written as the sum of *SE* and *WP*.

$$PE = SE + WP = \int_0^L \frac{1}{2}A(x)Eu'^2 dx - \int_0^L f(x)A(x)u(x)dx \tag{7}$$

As before, we have to minimize *PE* with respect to the deformation variables. Here, the deflection variable, *u(x)* is a continuous function, and the PE is an integral. In fact, *PE* in Equation (7) is called a functional — in this case an integral whose integrand is a function (in this case a differential relation) of some function *u(x).*

Next we will show that if *PE* is minimized with respect to all kinematically admissible displacement *u(x)*, then that *u(x)* satisfies the differential equation (4). To show this, consider the kinematically admissible displacement $\tilde{u}(x) = u(x) + \alpha\,\delta u(x)$ where the variation from the exact solution *u(x)* is given by the function $\delta u(x)$ times the parameter α. Since $\tilde{u}(x)$ must satisfy the same kinematical boundary conditions as *u(x)*, it follows that $\delta u(x=0) = 0$. With $\tilde{u}(x)$ substituted in the place of *u(x)* in the *PE* expression in Equation (7), for a given $\delta u(x)$, we can regard the potential energy to be a function of the parameter α, i.e., $PE(\alpha)$. Then, minimizing $PE(\alpha)$ with respect to α and setting $\alpha = 0$ gives the desired governing differential equation:

$$PE(\alpha) = \int_0^L \frac{1}{2}EA(x)(u' + \alpha\,\delta u')^2 dx - \int_0^L f(x)A(x)\big(u + \alpha\,u\big)dx$$

$$\frac{d(PE)}{d\alpha} = \int_0^L EA(x)(u' + \alpha\,\delta u')\;\delta u'\,dx - \int_0^L f(x)A(x)\big(\delta u\big)dx = 0$$

By substituting $\alpha = 0$, we get

$$\left.\frac{d(PE)}{d\alpha}\right|_{\alpha=0} = \int_0^L EA(x)(u')\;\delta u'\,dx - \int_0^L f(x)A(x)\big(\delta u\big)dx = 0$$

Integrating the expression in the last equation by parts and using the boundary conditions on $\delta u(x)$, we arrive at (note: we substitute $u' = \frac{du(x)}{dx}$ to get back to our original notation)

$$\int_0^L \left(\frac{d}{dx}\left(EA(x)(\frac{du(x)}{dx}) \right) + f(x)A(x) \right)\delta u dx = 0 \tag{8}$$

Since this last integral must vanish for all kinematically admissible δu when the potential energy of the deformed beam is minimized, it follows that the integrand itself must vanish, i.e.:

$$\frac{d}{dx}\left(EA(x)(\frac{du(x)}{dx}) \right) + f(x)A(x) = 0 \tag{9}$$

which is the same as Equation (4).

We have demonstrated above that the MPE principle can be applied to continuous elastic systems as well. In fact, in doing so, we have utilized a fundamental mathematical approach in the *calculus of variations*. We could also have derived Equation (9) by applying what is known as Euler-Lagrange equation of calculus of variations. The Euler-Lagrange equation helps us minimize a functional (the *PE* expression in Equation (7) in our case) with respect to a function (in our case *u(x)*). It is given by

$$\frac{d}{dx}\left(\frac{\partial(PE)}{\partial u'} \right) - \frac{\partial(PE)}{\partial u} = 0 \tag{10}$$

You should verify that Equation (10) also leads to Equation (9).

Once again, the *MPE* principle gave us the solution with less work and more systematically as compared to the force-balance method. It is systematic in the following sense. If you were to derive the governing equilibrium differential equation for a beam, all you need is its *PE*, as opposed to the force-balance method where you need to know much more about the internal forces. Much of the theoretical basis for the finite element method is rooted in the method we used above. In particular, Equation (10) is a fundamental equation in calculus of variations – an important mathematical tool in FEM formulations. Refer to any book on calculus of variations for more details. References to two books are given in the bibliography at the end.

3.3 Rayleigh-Ritz method

In Chapter 2, we solved a problem numerically the differential equation of which we derived in this chapter. We noted that the lumped-model method gives us deflections at only some discrete points (nodes), and we know nothing in between the nodes. Rayleigh-Ritz method is an alternative numerical method to solve the same equation in a simple way to know what happens in between as well.

There is one more thing to bear in mind. The lumped-model method gave us a nice set of linear equations, which we can easily solve. Also, we reduced a continuous system to a discretized system so that we can easily implement it on the computer. We don't want to lose these advantages in the Rayleigh-Ritz method. Thus, the Rayleigh-Ritz method is another way to discretize the continuous model.

Let us refer to Equation (7). We need to minimize *PE* to find $u(x)$. If $u(x)$ were to be a scalar variable, we could have minimized *PE* very easily as we did several times in Chapter 2. So, we have to employ a trick to get $u(x)$ to become scalar variables somehow. We can do that as follows.

Note from Figure 12 of Chapter 2 that as we increased the number of elements, the deflection curve converged to a continuous shape. And that shape looks like a parabola. So, the unknown function $u(x)$ can be assumed to be a quadratic equation of the form shown below.

$$u(x) = a_0 + a_1 x + a_2 x^2 \tag{10}$$

But, what we don't know are three scalars viz. a_0, a_1, and a_2. That is perfectly agreeable to us, because we can substitute for $u(x)$ from Equation (10) into the expression for *PE* given in Equation (7). Then, we get *PE* in terms of scalar quantities as we wanted. Now invoke the *MPE* principle.

$$\textit{Extremize} \quad PE(a_0, a_1, a_2) \quad \textit{with respect to} \quad a_0, a_1, \& a_2 \tag{11}$$

The conditions for solving the above are:

$$\frac{\partial(PE)}{\partial a_i} = 0 \qquad i = 0, 1, 2 \tag{12}$$

Equations (12) result in three linear equations in a_0, a_1, and a_2, which can easily be solved. In fact, you would note at once that $a_0 = 0$ as $u(x=0) = 0$. That is our assumed function for $u(x)$ should satisfy the boundary condition. Or in other words, it should be a kinematically admissible deformation. If you didn't appreciate kinematic admissibility in Chapter 2, here is the second chance!

Exercise 3.1

For the same tapered bar problem considered in Chapter 1, use the Rayleigh-Ritz method. That is, write Equations (7), and (12) to solve for a_0, a_1, and a_2.

- Work it out by hand so that you can understand more.
- Try it out with Maple also so that you can solve more interesting and larger problems.
- Check the Rayleigh-Ritz solution with the lumped-model solution with a large number of elements.

Exercise 3.2

Consider the overhanging simply supported beam shown below in Figure 2. In order to use the Rayleigh-Ritz method, we would like to approximate the deflected profile, $v(x)$ as $\left\{a\cos\left(\frac{2\pi x}{L}\right)\right\}$ where L is the length of the beam. Use the minimum potential energy principle to compute the unknown constant, a.

(a) Draw the assumed deflected profile. Is it a kinematically admissible function?

(b) Write down the expression for the strain energy of the beam.

(c) What is the work potential due to each force (use $y_{x=0}$, $y_{x=40}$, and $y_{x=80}$)?

(d) Compute the expression for the total potential energy in terms of a.

(e) Compute the value of a.

Note: $\int_0^L \cos^2\left(\frac{2\pi x}{L}\right)dx = \frac{L}{2}$

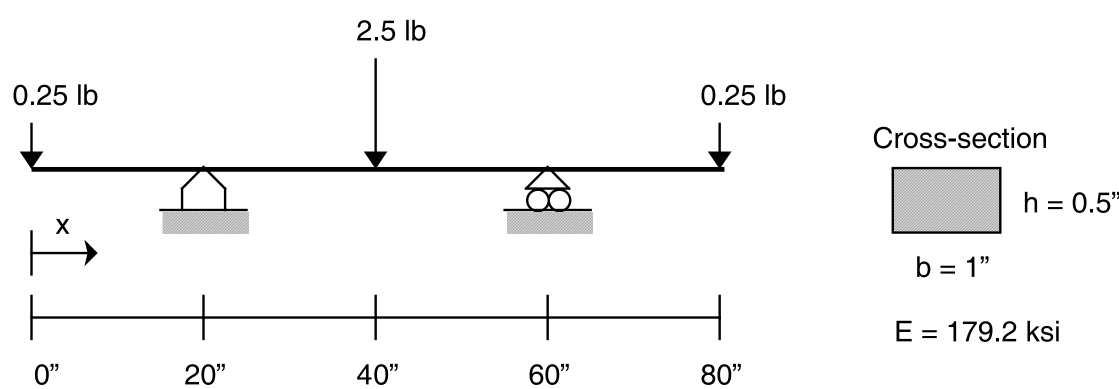

Figure 2 Overhanging simply-supported beam

If a single assumed function is not adequate to represent the deformation, one can use more than one function for different parts of the structure. Each of these functions will have unknown coefficients which can be determined by minimizing *PE*. If more than one function is used, one needs to ensure continuity of the functions at points where they connect with each other. The following exercise uses this technique.

Exercise 3.3

Repeat the tapered bar problem if the area of cross-section varies as follows. Area at the top is the same as before (i.e., A_0). The cross-section area remains constant up to the middle of the bar ($x=0.5$), and then increases parabolically to become three times A_0 at the bottom.

$$A_1(x) = A_0 \qquad \text{for} \qquad 0 \le x \le 0.5$$

$$A_2(x) = A_0(3-8+8x^2) \qquad \text{for} \qquad 0.5 \le x \le 1$$

Use two different polynomials for the ranges ($0 \le x \le 0.5$) and ($0.5 \le x \le 1$) to approximate $u(x)$ with two piece-wise continuous polynomials. Note that you should ensure continuity at $x = 0.5$ so that $u(x)$ and its derivative are continuous.

Exercise 3.4

Comfy Beds, Inc. is considering a new design for the box-spring system. It consists of top and bottom grids of thin strips of metal connected by linear helical springs. A portion of this new box-spring system is shown in the figure. Use Rayleigh-Ritz method to determine the maximum deflections of the top and bottom beams. (see Figure 3).

Use $\begin{array}{c} y_1 = a_1 x_1 (x_1 - l_1) \\ y_2 = -a_2 x_2^2 \end{array}$ as the basis functions where y_1 and y_2 are the deformations of the top and bottom beams respectively. x_1 and x_2 are zero at the left end of each beam.

(a) Do the above basis functions satisfy the kinematic admissibility conditions? Explain how.

(b) The strain energy for a beam is given by $\int_0^L \frac{EI}{2}\left(\frac{d^2 y}{dx^2}\right)^2 dx$. Write the total strain energy stored in the two beams and the spring in terms of a_1 and a_2.

(c) What is the work potential due to the applied force, F of 5 lb? (again in terms of a_1 and a_2).

(d) Use the principle of the minimum potential energy to find the equilibrium values of a_1 and a_2.

Both beams have rectangular cross-section of thickness 0.1 in and a width of 1 in. The Young's modulus is 30E6 psi, and the spring constant, k is 10 lb/in. The applied force F is 5 lb. l_1 and l_2 are respectively 40 in and 30 in.

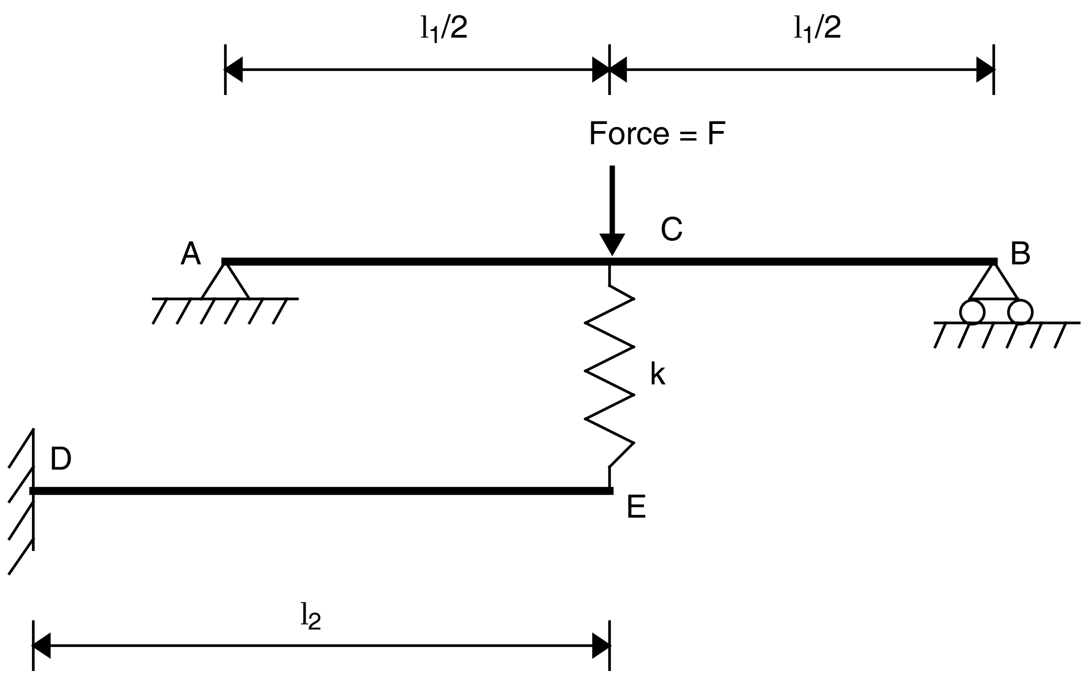

Figure 3 The schematic of the springs used by Comfy Beds, Inc.

The Rayleigh-Ritz method is a powerful method to use if we know *a priori,* the nature of the function for the deformation. However, we may not be able to guess such a function or several piece-wise functions for any given problem. The FEM enables us to come up with such functions systematically. Those functions are called shape functions. They serve the following purpose.

- Approximate the continuous deformation using piece-wise functions defined over elements.
- Shape functions depend on some scalar quantities and those scalar quantities are nothing but the value of the deformation at the nodes.
- *Interpolation*, i.e., knowing what happens within the element is readily available through shape functions.

The following Table summarizes the basic concepts we laid out in Chapters 2 and 3. In the next chapter, we will study the shape functions and apply this concept to the axially loaded bars once again. This is the real beginning of our FEM discussion.

Table 1 Comparison of three approaches to deformation analysis

	Lumped-model	Rayleigh-Ritz	FEM
Discretization	Divide into segments ("element"). The value of the deformation at the discrete points ("nodes") are the unknown scalar quantities to be determined using the MPE principle.	Discretization concept is different. You do convert a continuous problem into a discrete problem. But, the discrete (scalar) unknowns are coefficients of the assumed polynomials (basis functions).	In principle, it is the same as the lumped model, i.e., the discretization is physical.
Interpolation	Not possible.	You need to know the nature of the function so that you can approximate the deformation curve with one or more trial (guess) functions globally. The procedure is not systematic.	The procedure is systematic. Shape functions are used for interpolation locally for small elements.