Chapter 4

Shape Functions

In the finite element method, continuous models are approximated using information at a finite number of discrete locations. Dividing the structure into discrete elements is called discretization. Interpolation within the elements is achieved through shape functions, which is the topic of this chapter.

4.1 Linear shape functions for bar elements

Let us isolate a bar element from the continuous bar. The deformations at the ends of the elements (called "nodes") are part of the unknowns in the finite element analysis problem. Let us now define shape functions for the bar element in order to linearly interpolate deformation within the element. We will define the shape functions in such a way that they can be used for an element of any size. That is, we will normalize the length of the element by using a new local coordinate system shown below.

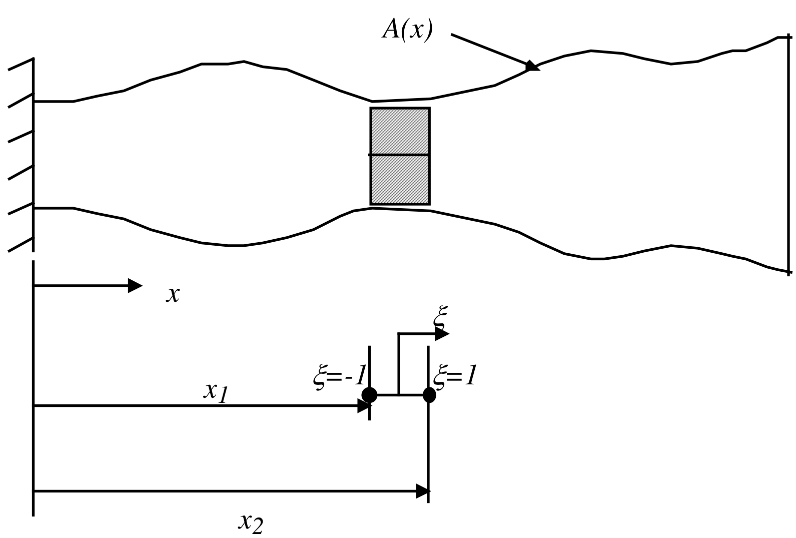

Figure 1 Local coordinate system for a finite bar element

The ξ-coordinate system is defined in such a way that $\xi = -1$ *to* 1 would cover the entire element irrespective of what x_1 and x_2 are for a given element. The following relationship gives that range for ξ as x varies from x_1 to x_2.

$$\xi = \frac{2}{x_2 - x_1}\left(x - x_1\right) - 1 \tag{1}$$

Now we define two shape functions in the ξ-coordinate system shown below.

$$N_1(\xi) = \frac{1-\xi}{2} \quad \text{and} \quad N_2(\xi) = \frac{1+\xi}{2} \tag{2}$$

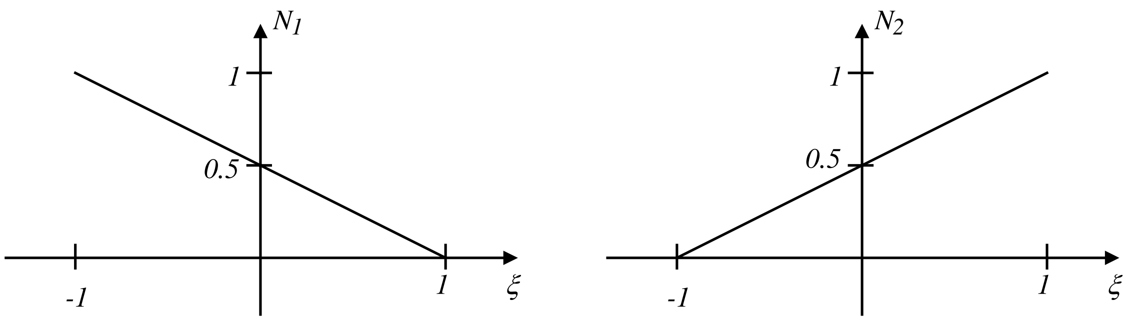

Figure 2 Linear shape functions for a bar element

Using these shape functions, the deformations within the element are interpolated as follows:

$$u = N_1 q_1 + N_2 q_2 \tag{3}$$

where q_1 and q_2 are the deformations at the ends (<u>nodes</u>) of the element. It is easy to see that u varies linearly as shown in Figure 3.

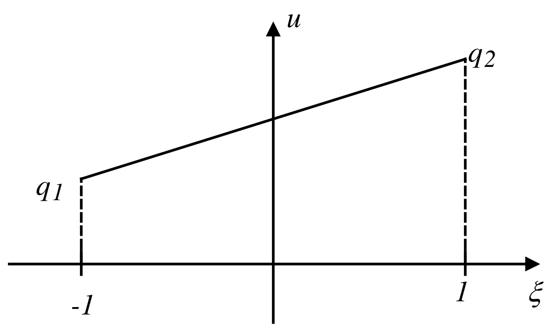

Figure 3 Linear interpolation for u in a bar element

Equation (3) can be written in <u>matrix notation</u> as

$$\mathbf{u} = \{N_1 \quad N_2\}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \mathbf{Nq} \tag{4}$$

The interpolation contained in Equation (4) is the fundamental basis for the <u>piece-wise continuous function-based local approximation</u> in FEM. Once the shape functions are chosen, the rest of the procedure is routine, as we will see again and again in this notes. The shape functions used here are called <u>Lagrangian interpolating functions</u>. Several types of shape functions can be chosen. For instance, we could have chosen quadratic or cubic interpolating functions. The choice of shape functions determines the type of the finite element.

Let us proceed further to write stresses and strains for the element. We need to do this in order to write the *PE* and *WP* of the element. We know that the strain in a bar element is given by

$$\varepsilon = \frac{du}{dx} \tag{5}$$

Since u is a function of N's and N's which are functions of ξ_1, and ξ_2, which are in turn functions of x, we need to use chain-rule differentiation:

$$\varepsilon = \frac{du}{dx} = \frac{du}{d\xi}\,\frac{d\xi}{dx} \tag{6}$$

From equations (4) and (1),

$$\frac{du}{d\xi} = \frac{d}{d\xi}\left(N_1 q_1 + N_2 q_2\right) = \frac{d}{d\xi}\left(\frac{1-\xi}{2}q_1 + \frac{1+\xi}{2}q_2\right) = \frac{q_2 - q_1}{2} \tag{7a}$$

$$\frac{d\xi}{dx} = \frac{2}{x_2 - x_1} \tag{7b}$$

Therefore,

$$\varepsilon = \frac{q_2 - q_1}{x_2 - x_1} \tag{8}$$

Since $(x_2 - x_1) = L_e$, the length of the element, strain in Equation (8) can be re-written in a convenient normalized matrix form as

$$\varepsilon = \frac{1}{L_e}(-q_1 + q_2) = \frac{1}{L_e}\{-1 \quad 1\}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \mathbf{Bq} \tag{9}$$

where

$$\mathbf{B} = \frac{1}{L_e}\{-1 \quad 1\} \tag{10}$$

is called the strain displacement matrix for a bar finite element.

Stress is then given by

$$\sigma = E\varepsilon \tag{11}$$

This can also be written in a general matrix form as

$$\boldsymbol{\sigma} = \mathbf{DBq} \tag{12}$$

where **D** is the <u>stress-strain matrix</u>. In the case of the bar element, it is simply the Young's modulus *E*. In general it will be a matrix. We will come to that later in Chapter 8. What is important to note at this point is that the Equations (4), (9), and (12) are of the general matrix form, and are applicable even for a 3-D solid finite element. Thus, even though the shape functions are discussed only for the bar element here, <u>the procedure is identical for any type of element.</u>

To reinforce our understanding, let us repeat the above exercise for *quadratic interpolating shape functions*. Once again, these are Lagrangian type interpolating functions.

4.2 Quadratic shape functions for bar elements

Consider

$$N_1 = \frac{\xi(\xi-1)}{2} \qquad N_2 = (1+\xi)(1-\xi) \qquad N_3 = \frac{\xi(\xi+1)}{2} \tag{13}$$

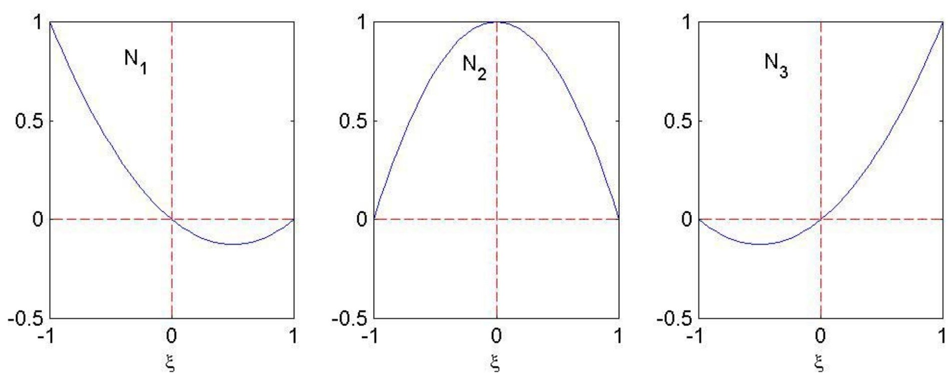

Figure 4 Quadratic shape functions

We need a third mid-side node now in addition to the two end nodes. Thus, our element is composed of three nodes with deformations q_1, q_2, and q_3. As can be seen, The first shape function is 1 at the left side node and zero at the other two nodes. The second shape function is zero at either end, but is 1 at the mid point.. The shape function 3 also has the same property in that it is one at the right node and zero at the left and mid-side nodes. This is in fact the property using which we can easily construct Lagrangian interpolation functions of any order (say, cubic, quartic, etc.).

By virtue of the property of shapefunctions, when u is constructed as shown below (Equation (14)), q_1, q_2, and q_3 will be precisely satisfied at the three nodes. The interpolated deformation within the element is given by

$$u = \{N_1 \quad N_2 \quad N_3\}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \mathbf{Nq} \tag{14}$$

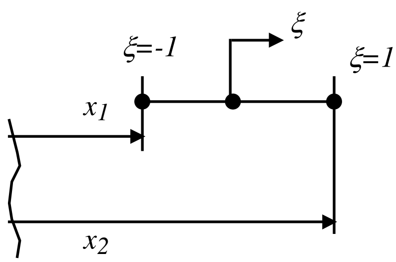

Figure 5 The three nodes of a quadratic bar element

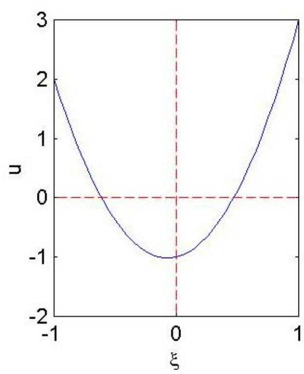

Figure 6 Quadratically interpolated u **using the quadratic shape functions for**

$q_1=2$, $q_2=-1$, $q_3=3$

As done before, the strain is computed using the chain rule (Equation (6)), and the <u>strain displacement matrix</u> is obtained as

$$\varepsilon = \frac{1}{L_e}\{2\xi - 1 \quad -4\xi \quad 2\xi + 1\}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \mathbf{Bq} \tag{15}$$

The matrix **D** is still only the scalar E as we are still dealing with the bar element.

We will consider other shape functions for beam and plane stress elements later on. Our immediate concern is to use the shape functions and formulate the finite element model for the bar elements. That is the focus of Chapter 5.