Chapter 5

Finite Element Modeling for Bar Elements Using the MPE Principle

In this Chapter, we will systematically construct the finite element model for the bar element.

5.1 Strain energy for a bar element

Let us recall from Chapter 4 that the strain and stress for a bar element are given by the following expressions when we use linear shape functions.

$$\boldsymbol{\varepsilon} = \mathbf{Bq} \qquad \boldsymbol{\sigma} = \mathbf{DBq}$$
$$\mathbf{B} = \frac{1}{L_e}\{-1 \quad 1\} \qquad \mathbf{q} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \qquad \mathbf{D} = E \tag{1}$$

The strain energy for the bar element is then computed by integrating the strain energy density over the entire volume of the element. Since, the area of cross-section within the element is assumed to be constant, the integration will be carried out over the length of the element.

The strain energy density is half of the product of stress and strain. Although stress and strain are scalar quantities in the case of bar elements, they are vectors* in the general case. So, we will use vector notation here also. The scalar (dot) product of two vectors V_1 and V_2 is given by $V_1^T V_2$. Thus the strain energy density is $\frac{1}{2}\boldsymbol{\sigma}^{\mathbf{T}}\boldsymbol{\varepsilon}$. Therefore,

$$SE = \int_{x_1}^{x_2}\left(\frac{1}{2}\boldsymbol{\sigma}^{\mathbf{T}}\boldsymbol{\varepsilon}\ A_e\right)dx = \frac{A_e}{2}\int_{x_1}^{x_2}(\mathbf{DBq})^T(\mathbf{Bq})dx = \frac{A_e}{2}\int_{x_1}^{x_2}\left(\mathbf{q}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}\mathbf{DBq}\right)dx \tag{2}$$

Note that **D** the stress-strain matrix is symmetric, i.e, $\mathbf{D^T=D}$. Each of the matrix and vector entries in the above equation are expanded out fully for the sake of clarity.

$$\mathbf{q}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}\mathbf{DBq} = \{q_1 \quad q_2\}\ \frac{1}{L_e}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} E_e\ \frac{1}{L_e}\{-1 \quad 1\}\ \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \{q_1 \quad q_2\}\frac{E_e}{L_e^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{3}$$

We can now substitute Equation (3) into Equation (2). In doing so, we take those terms that do not depend on x out of the integral sign. Note that x varies within the element from x_1 and x_2. And, when we do integration on a matrix, we integrate each term in the matrix. Note further that $(x_2-x_1) = L_e$.

* Actually *tensors*, but we will use the vector notation here to keep mathematical manipulations simple.

$$
\begin{aligned}
SE &= \{q_1 \quad q_2\}\left(\frac{A_e E_e}{2L_e^2}\int_{x_1}^{x_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}dx\right)\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}\\
&= \frac{1}{2}\{q_1 \quad q_2\}\left(\frac{A_e E_e}{L_e}\right)\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}\\
&= \frac{1}{2}\mathbf{q}^{\mathbf{T}}\mathbf{K_e}\mathbf{q}
\end{aligned} \tag{4}
$$

We wrote *SE* in matrix form for a good reason, which would become clear later in this chapter. The matrix $\mathbf{K_e}$ re-written below is called the element stiffness matrix. Note that it is symmetric. This is a very important property of the stiffness matrix and is true for any type of element.

$$
\mathbf{K_e} = \frac{A_e E_e}{L_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}\dfrac{A_e E_e}{L_e} & -\dfrac{A_e E_e}{L_e}\\ -\dfrac{A_e E_e}{L_e} & \dfrac{A_e E_e}{L_e}\end{bmatrix} = \begin{bmatrix}k_e & -k_e\\ -k_e & k_e\end{bmatrix} \tag{5}
$$

where $k_e = \dfrac{A_e E_e}{L_e}$. When we expand *SE* in Equation (4), what we get is very familiar to us: it is simply the strain energy of a spring whose spring constant is k_e with its ends having deformations q_1 and q_2.

$$
SE = \frac{1}{2}\mathbf{q}^{\mathbf{T}}\mathbf{K_e}\mathbf{q} = \frac{1}{2}k_e\left(q_2 - q_1\right)^2 \tag{6}
$$

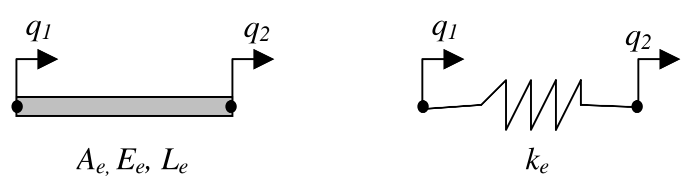

Figure 1 Equivalence of a bar element of linear shape functions and a spring

So, in a way, all we have done in this section is to derive a simple result in a convoluted manner. But, we should not say that. We have followed a very general procedure for constructing *SE* for a finite element. Since, our shape functions were linear interpolators, we ended up with a familiar result in Equation (6). Can you, for instance, write *SE* if we use quadratically interpolating shape functions for the bar element? You probably cannot write it simply by inspection. But, if you follow this general procedure, you can do it systematically.

Exercise 5.1

Using the quadratic interpolating functions (Equation 13 in Chapter 4), derive an expression for SE of a bar finite element.

Answer:

$$SE = \frac{1}{2}\{q_1 \quad q_2 \quad q_3\}\frac{A_e E_e}{3L_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

5.2 Work potential for a bar element

The work potential *WP*, as outlined in Chapter 2, is the negative of the work done by the external forces on a structure. But, we need to know what kinds of forces act on a structure. Although writing down *WP* for a bar element is easy, we will follow a general, matrix based approach so that we can do it for other types of elements easily.

Table 1 Different types forces acting on a structure

	Body forces	Surface forces	Point forces
Definition	Defined per unit volume of the structure.	Defined per unit external surface area of the structure.	Defined at a point on the surface – a special case of surface forces.
Symbol/Units	$\mathbf{f}$ N/m^3	$\mathbf{T}$ N/m^2	$\boldsymbol{P}$ N
Explanation	They act at every point inside of the structure	They act at every point on the boundary of the structure	They act a single point. They are also called concentrated forces.
Example	Gravity force	Electrostatic force, fluid pressure force, etc.	Any mechanical push-pull type force.
Equation	$WP = -\int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\; dV$	$WP = -\int_S \mathbf{u}^{\mathrm{T}}\mathbf{T}\; dS$	$WP = -\sum_{i=1}^{N} u_i P_i$ Assuming that there are N point forces.

The integration shown in the above table is to be done over the element. Let us do that separately for each case.

Body forces:

Noting that $\mathbf{u} = \mathbf{Nq}$ (Equation (4) in Chapter 4),

$$\int_{V_e} \mathbf{u}^{\mathbf{T}}\mathbf{f}\ \ dV = \int_{x_1}^{x_2} (\mathbf{Nq})^T \mathbf{f}\ \ A_e\, dx = \int_{x_1}^{x_2} (\mathbf{q}^T \mathbf{N}^T)\ \ \mathbf{f}\ \ A_e\, dx = \mathbf{q}^T \begin{Bmatrix} \int_{x_1}^{x_2} N_1 A_e f\, dx \\ \int_{x_1}^{x_2} N_1 A_e f\, dx \end{Bmatrix} \tag{7}$$

If we assume that the cross-section area A_e and the body force f are constant within the element, we only need to integrate N_1 and N_2. From Chapter 4, we have

$$N_1 = \frac{1-\xi}{2} \quad \text{and} \quad N_2 = \frac{1+\xi}{2}$$

$$\xi = \frac{2}{L_e}\left(x - x_1\right) - 1 \quad \text{and} \quad d\xi = \frac{2}{L_e} dx$$

Thus,

$$\begin{aligned} \int_{x_1}^{x_2} N_1\, dx &= \frac{L_e}{2}\int_{-1}^{1} \frac{1-\xi}{2} d\xi = \frac{L_e}{2} \\ \int_{x_1}^{x_2} N_2\, dx &= \frac{L_e}{2}\int_{-1}^{1} \frac{1-+\xi}{2} d\xi = \frac{L_e}{2} \end{aligned} \tag{8}$$

Substitution of integrals from Equation (8) into Equation (7) yields:

$$\int_{V_e} \mathbf{u}^T \mathbf{f}\, dV = \mathbf{q}^T \mathbf{f}_e = \mathbf{q}^T \begin{Bmatrix} A_e L_e f / 2 \\ A_e L_e f / 2 \end{Bmatrix} \tag{9}$$

where $\mathbf{f_e}$ is called the *element body force vector*.

Equation (9) implies that the total body force acting on the element $A_e L_e f$, is divided equally between the two nodes. Once again, it is a simple result that could have been written by inspection. But, if we use quadratic shape functions, the contribution of the element body force on each of the three nodes is not obvious. Fortunately, the systematic procedure we followed in this section enables is to do that.

Surface forces:

Following the same procedure as above, we get

$$\int_{S_e} \mathbf{u}^T \mathbf{T}\, dS = \mathbf{q}^T \mathbf{T}_e \tag{10}$$

where $\mathbf{T}_e = \begin{Bmatrix} L_e T/2 \\ L_e T/2 \end{Bmatrix}$ is called the element surface force vector. In Equation (10), T is treated as force per unit length. This is because, we have assumed that the area of cross-section along the element is a constant, and therefore there is no need to include the perimeter into the definition of T. This is a subtle point and it will become clearer when you look at 3-D problems. It suffices to note at this point that for 1-D and 2-D problems T is really a distributed "line force" defined per unit length. This makes sense, because the boundary for 1-D and 2-D objects is really a curve and not a surface.

Point forces:

The quantity u_i in $\sum_{i=1}^{N} u_i P_i$ is the displacement at the point where the point force P_i is acting. Since we have the interpolating function for the deformation u, we can compute u_i even if the point is within the element. However, it is recommended that nodes be placed at the points where point forces are acting. As you will see later, there is some convenience in doing this. Thus, assuming that point forces are allowed to act only at the nodes,

$$\sum_{i=1}^{N} u_i P_i = \sum_{i=1}^{N_{dof_{element}}} q_i P_i = \mathbf{q}^T \mathbf{P} \tag{11}$$

where

$N_{dof_{element}}$ is the number of degrees of freedom (dof) of the element. For a bar element dof is 2, viz. q_1 and q_2.

The concept of dof for a finite element is very important. The value of $N_{dof_{element}}$ determines the size of the element matrices and vectors. The element stiffness matrix is of size ($N_{dof_{element}}$ x $N_{dof_{element}}$), while the force and deformation vectors are of the size ($N_{dof_{element}}$ x 1).

Now, we are in a position to write WP for the bar element using Equations (9), (10, and (11).

$$WP = -\mathbf{q}^T \mathbf{f}_e - \mathbf{q}^T \mathbf{T}_e - \mathbf{q}^T \mathbf{P} = -\mathbf{q}^T \mathbf{F}_e \tag{12}$$

where

$$\mathbf{F}_e = \mathbf{f}_e + \mathbf{T}_e + \mathbf{P} \tag{13}$$

5.3 Potential energy of the bar element

The potential energy of the bar element, from Equations (4) and (13), can be written as

$$PE_e = \frac{1}{2}\mathbf{q}^T\mathbf{K}_e\mathbf{q} - \mathbf{q}^T\mathbf{F}_e \tag{14}$$

5.4 Total potential energy of the entire bar

We can compute the *PE* for each element as per Equation (14). By adding the individual PE_e's, we can compute the total potential energy of the entire bar.

$$PE = \sum_{i=1}^{N_e} \frac{1}{2}\mathbf{q}_i^T\mathbf{K}_{e_i}\mathbf{q}_i - \mathbf{q}_i^T\mathbf{F}_{e\,i} \tag{15}$$

where

N_e is the total number of elements in the discretized bar.

If we have a large number of elements, Equation (15) can become very long. Note that PE will be a function of ($N_{dof} = N_n \text{ x } N_{dof\,\text{node}}$) q's, if N_n is the number of nodes in the discretized structure. According to the *MPE* principle, we have to take the derivative of *PE* in Equation (15) with respect to all of these q's. It looks like a daunting task, but there is really a nice structure to the finite element formulation. The importance of the matrix notation becomes useful at this juncture.

Equation (15), after summing, takes the following form:

$$PE = \frac{1}{2}\mathbf{Q}^T\mathbf{K}\mathbf{Q} - \mathbf{Q}^T\mathbf{F} \tag{16}$$

where

$\mathbf{Q}$ is the global deformation vector of size (N_{dof} x 1)

$\mathbf{K}$ is the global stiffness matrix (N_{dof} x N_{dof})

and $\mathbf{F}$ is the global force vector of size (N_{dof} x 1)

The <u>assembly process</u> that leads to the <u>global system matrix</u> and <u>vectors</u> warrants careful understanding. We will discuss this in the next section.

We can now apply the *MPE* principle to Equation (16). If we take the derivative of *PE* with respect to $\mathbf{Q}$, we get

$$\frac{\partial PE}{\partial \mathbf{Q}} = \mathbf{KQ} - \mathbf{F} = 0 \quad \Rightarrow \quad \mathbf{KQ} = \mathbf{F} \tag{17}$$

It is important to understand the differentiation done to the matrix form of Equation (16). You may want to verify Equation (17) by doing the differentiation on the expanded (i.e., multiply out matrices and vectors and reduce the right hand side entity in Equation (16) to a scalar expression) version of *PE*, then taking the derivative with respect to each of the *q*'s, and then converting back to the matrix form. It is recommended that you do it for a small N_{dof}, say 2. But, once you are comfortable taking derivatives with the matrices themselves, it is not tedious at all. This is one advantage of the matrix notation.

$$\mathbf{KQ} = \mathbf{F} \tag{18}$$

is the matrix form of equilibrium equations. It consists of N_{dof} linear equations. The unknown $\mathbf{Q}$ can be computed by solving Equation (18).

We have now obtained what we need, but we haven't discussed yet how we assembled the global matrices and vectors. We will discuss that next.

5.5 Assembly of global matrices and vectors

Consider three bar elements consisting of four nodes viz. q_{i-1}, q_i, q_{i+1}, and q_{i+2} as shown in Figure 2a. Each of the three elements will have an element stiffness matrix, and element deformation and force vectors (Equations (5) and (13)). Their *placement* in the *global system* is shown in Figure 2b. The assembly takes place in the following manner. The contribution of the stiffness matrix of an element made of nodes *i* and *j* goes to a sub-matrix of the global matrix that is formed by rows *i* and *j*, and columns *i* and *j*. Each entry in the global stiffness matrix will receive contribution from two elements. Obviously there are going to be many zero entries in the matrix. In fact, it is a tri-diagonal matrix. That is, only the diagonal and the ones immediately next to them are the only non-zero entries. The assembly of deformation and force vectors takes place in a similar manner.

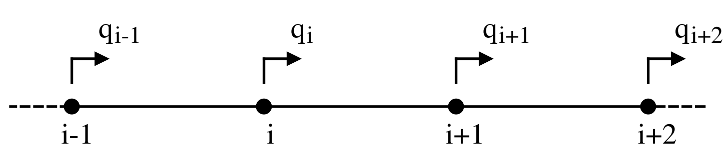

Figure 2a Sample of three bar elements

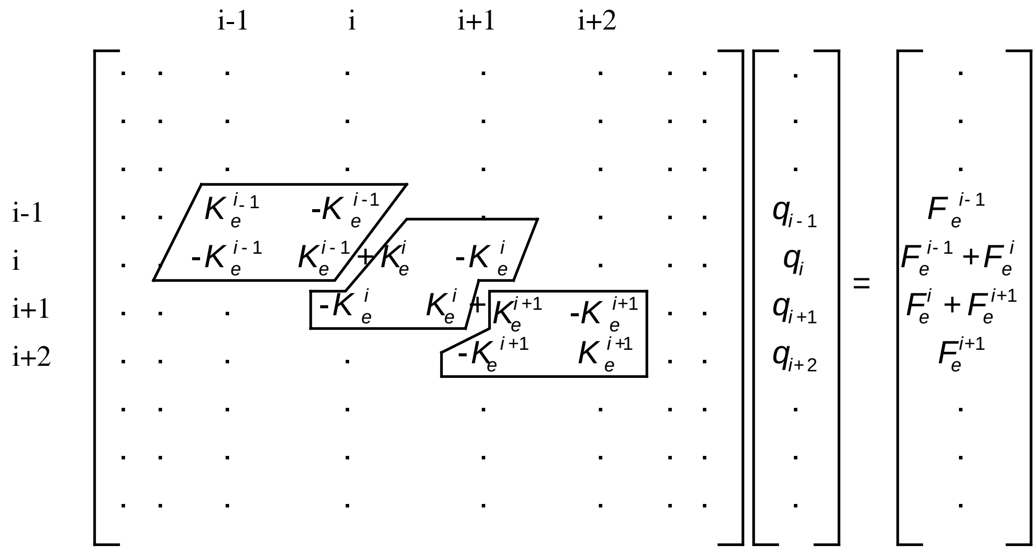

Figure 2b Assembly of global system for the three elements in Figure 2a

Assembly of bar elements is very simple, as you just saw. Other types of elements will need some more work and organization. We will consider this aspect when we deal with truss and beam elements later.

5.6 Imposing the displacement boundary conditions

After assembly of the global system stiffness matrix, and displacement and force vectors we get the equation shown below:

$$\mathbf{KQ} = \mathbf{F} \tag{19}$$

The matrix **K** in the above equation will be singular (i.e., the determinant is zero) as it is not constrained from moving like a rigid body (i.e., no deformation). Every structure that we analyze will be constrained in one way or the other. Constraining a structure implies that some q's within **Q** will be set to a known value. When the set value is zero, the corresponding degree of freedom is fixed. Non-zero value implies that there is a known displacement at that degree of freedom. This situation arises when you pull a beam and displace it by some known amount. We will discuss both the zero and the non-zero cases next.

First, note that each row in the matrix Equation (19) is a linear equation in q's. If a certain q is known to be zero, say i th q, then we don't need to include i th row in our solution. The reasoning is as follows. Originally we had N_{dof} q's. So, we need N_{dof} equations to solve for q's/ Now we have one less q (which is known to be zero). So, we can throw away the corresponding equation, i.e., the corresponding row. Note also that the i th column also becomes useless now, because the matrix entries in the i th column multiply with i th q. But, that is zero. So, we discard i th column as well. Therefore, whenever a degree of freedom is specified to be zero, we simply discard the corresponding row and the column.

Let us take a simple example to illustrate the above point. Consider a small finite element model with only four degrees of freedom. Its matrix equation before imposing the boundary conditions is shown below.

$$\mathbf{KQ} = \mathbf{F} \;\Rightarrow\; \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} \tag{20}$$

Note that we used the fact that the <u>stiffness matrix is symmetric</u> in writing the entries of **K**. The Equation (20) is a set of four linear equations:

$$\begin{aligned} k_{11}q_1 + k_{12}q_2 + k_{13}q_3 + k_{14}q_4 &= f_1 \\ k_{12}q_1 + k_{22}q_2 + k_{23}q_3 + k_{24}q_4 &= f_2 \\ k_{13}q_1 + k_{23}q_2 + k_{33}q_3 + k_{34}q_4 &= f_3 \\ k_{14}q_1 + k_{24}q_2 + k_{34}q_3 + k_{44}q_4 &= f_4 \end{aligned} \tag{21}$$

Let us say that q_2 is specified to be zero. Then Equations (21) become

$$\begin{aligned} k_{11}q_1 + k_{13}q_3 + k_{14}q_4 &= f_1 \\ k_{12}q_1 + k_{23}q_3 + k_{24}q_4 &= f_2 \\ k_{13}q_1 + k_{33}q_3 + k_{34}q_4 &= f_3 \\ k_{14}q_1 + k_{34}q_3 + k_{44}q_4 &= f_4 \end{aligned} \tag{22}$$

We only need to solve for q_1, q_3, and q_4. So, we need only the first, third, and fourth equations in Equations (22). Those three can be represented in matrix form as

$$\begin{bmatrix} k_{11} & k_{13} & k_{14} \\ k_{13} & k_{33} & k_{34} \\ k_{14} & k_{34} & k_{44} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_3 \\ f_4 \end{Bmatrix} \tag{23}$$

which is obtained by discarding the second row and column of matrix in Equation (20).

When the specified value is non-zero, we still discard the corresponding row, but we don't simply discard the corresponding column. This is because the matrix entries in the column multiply with a non-zero valued specified q which is still a part of each row, This quantity, a known number now, should be taken to the right hand side and added to the force on that row. In other words, a component of the reaction force due to the non-zero valued displacement boundary condition goes to every row (of course, if a matrix entry in the *i* th column is zero, there won't be any contribution to the right hand side). Let us observe this with the same example of Equation (20).

If q_2 is specified to be a non-zero value v_2, Equations (21) become

$$\begin{aligned} k_{11}q_1 + k_{12}v_2 + k_{13}q_3 + k_{14}q_4 &= f_1 \\ k_{12}q_1 + k_{22}v_2 + k_{23}q_3 + k_{24}q_4 &= f_2 \\ k_{13}q_1 + k_{23}v_2 + k_{33}q_3 + k_{34}q_4 &= f_3 \\ k_{14}q_1 + k_{24}v_2 + k_{34}q_3 + k_{44}q_4 &= f_4 \end{aligned} \tag{24}$$

Now, we don't need the second equation in Equations (24) as we already know what q_2 is. So, we take only the first, third, and fourth equations to solve for q_1, q_3, and q_4. But, now we see that there is a known numerical quantity $k_{i2}v_2$ (i=1, 3, and 4), which should be taken to the right hand side as shown below in the matrix form:

$$\begin{bmatrix} k_{11} & k_{13} & k_{14} \\ k_{13} & k_{33} & k_{34} \\ k_{14} & k_{34} & k_{44} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} f_1 - k_{12}v_2 \\ f_3 - k_{23}v_2 \\ f_4 - k_{24}v_2 \end{Bmatrix} \tag{25}$$

Thus, when a non-zero value is specified, we still reduce the matrix size by 1, but need to subtract something from the force vector. This is essentially a component of the reaction force affecting other degrees of freedom due to fixing q_2 to a known value.

Another important point is worth noting here: whenever we fix a degree of freedom, we get a reaction force. That can be easily calculated. In the above example, when we fixed q_2 to v_2, we go back to the second equation in Equations (24). The f_2 that results after solving for q_1, q_3, and q_4 is the reaction force at the second degree of freedom.