Chapter 7
Finite Element Formulations for Beams and Frames

Beams and frames can take axial, transverse (i.e., perpendicular to the axis), and moment loads. Unlike truss elements, they undergo bending. In this chapter, we will obtain element stiffness matrix and force vectors for a beam element by following the same procedure as the one used for the axially loaded bars. This will reinforce our understanding of the finite element formulation so that we can discuss the general procedure for any type of element in the coming chapters.

We will first obtain an expression for the strain energy and work potential of a beam. Then, by assuming shape functions of certain form, we will write the strain energy for a beam element in order to obtain the stiffness matrix and force vectors for the element.

7.1 The bending strain energy for a beam

For a beam of length *L*, we write the strain energy density (= half of the product of stress and strain), and then integrate it over the entire volume to get *SE* of the beam.

$$SE = \int_V \frac{1}{2}(stress)(strain)dV = \int_V \frac{1}{2E}(stress)^2\, dV \tag{1a}$$

$$= \int_V \frac{1}{2E}\left(\frac{My}{I}\right)^2 dV \tag{1b}$$

$$= \int_L \left\{ \int_A \left(\frac{M^2}{2EI^2}\right) y^2\, dA \right\} dx \tag{1c}$$

Equation (1b) is obtained by using the fact that the bending stress in a beam is given by

$$stress = \frac{My}{I} \tag{2}$$

Next, we evaluate the area integral in Equation (1c).

$$\int_A \left(\frac{M^2}{2EI^2}\right) y^2\, dA = \left(\frac{M^2}{2EI^2}\right) \int_A y^2\, dA = \left(\frac{M^2}{2EI^2}\right) I = \left(\frac{M^2}{2EI}\right) \tag{3}$$

In Equation (3) above, we noticed that $\int_A y^2 \, dA = I$ is nothing but the area moment of inertia of the beam cross-section. Next we use Bernoulli's beam deflection theorem that states that the bending moment is proportional to the curvature. Since for small strains, the curvature is given by the second derivative of the transverse deflection *v(x)* of the neutral axis, we write:

$$M = EI\frac{d^2v}{dx^2} \tag{4}$$

Substitution of Equation (3) into Equation (1c) and using (4) yields

$$SE = \int_L \frac{M^2}{2EI} dx \tag{5a}$$

$$= \int_L \frac{1}{2EI}\left(EI\frac{d^2v}{dx^2}\right)^2 dx = \int_L \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx \tag{5b}$$

The importance of Equation (5b) lies in the fact that if we know the transverse deflection *v(x),* we can compute the bending strain energy by evaluating the integral—an essential feature for the finite element formulation.

7.2 Shape functions for beam elements

The first step in the finite element formulation is to choose the suitable shape functions. We will consider <u>two-noded beam elements</u>. Each node will have three degrees of freedom, viz. axial and transverse displacements, and the slope. We will first consider only the transverse displacement and the slope, and we include consider the axial displacement later. We do this because, in bar elements we already accounted for the axial displacement. It will be a simple matter to include this effect later, as you will see at the end of this chapter.

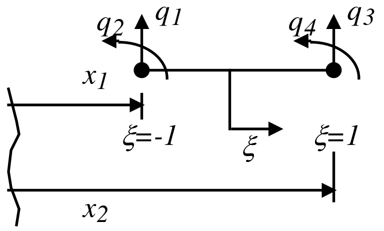


Figure 1 The four degrees of freedom of a beam element in local coordinate system

The degrees of freedom q_1, q_2, q_3, and q_4 are shown in the local ξ-coordinate system in Figure 1. Recall that the purpose of using a local coordinate system is to make the derivation general so that the formulation is applicable to all elements of different sizes and orientations. Recall also the transformation from the local to the global system:

$$\xi = \frac{2}{L_e}\left(x - x_1\right) - 1 \tag{6}$$

We will now define four shape functions N_i $(i = 1,2,3,$ and $4)$ such that $v(x)$ and $\frac{dv(x)}{dx}$ give the transverse displacement and the slope of the beam respectively.

$$v(\xi) = \left\{ N_1 \quad \frac{L_e}{2}N_2 \quad N_3 \quad \frac{L_e}{2}N_4 \right\} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \begin{matrix} \longrightarrow \text{Displacement at node 1} \\ \longrightarrow \text{Slope at node 1} \\ \longrightarrow \text{Displacement at node 2} \\ \longrightarrow \text{Slope at node 2} \end{matrix} \tag{7}$$

where

$$\begin{aligned} N_1 &= \frac{1}{4}(1-\xi)^2(2+\xi) \\ N_2 &= \frac{1}{4}(1-\xi)^2(1+\xi) \\ N_3 &= \frac{1}{4}(1+\xi)^2(2-\xi) \\ N_4 &= \frac{1}{4}(1+\xi)^2(\xi-1) \end{aligned} \tag{8}$$

Note that

$$\frac{dv}{dx} = \frac{dv(\xi)}{d\xi}\frac{d\xi}{dx} \tag{9}$$

where

$$\frac{d\xi}{dx} = \frac{2}{L_e} \tag{10}$$

and

$$\frac{dv}{d\xi} = \left\{ \frac{dN_1}{d\xi} \quad \frac{L_e}{2}\frac{dN_2}{d\xi} \quad \frac{L_e}{2}\frac{dN_3}{d\xi} \quad \frac{dN_4}{d\xi} \right\} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \tag{11a}$$

$$\frac{d^2 v}{d\xi^2} = \begin{Bmatrix} \frac{d^2 N_1}{d\xi^2} & \frac{L_e}{2}\frac{d^2 N_2}{d\xi^2} & \frac{L_e}{2}\frac{d^2 N_3}{d\xi^2} & \frac{d^2 N_4}{d\xi^2} \end{Bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \tag{11b}$$

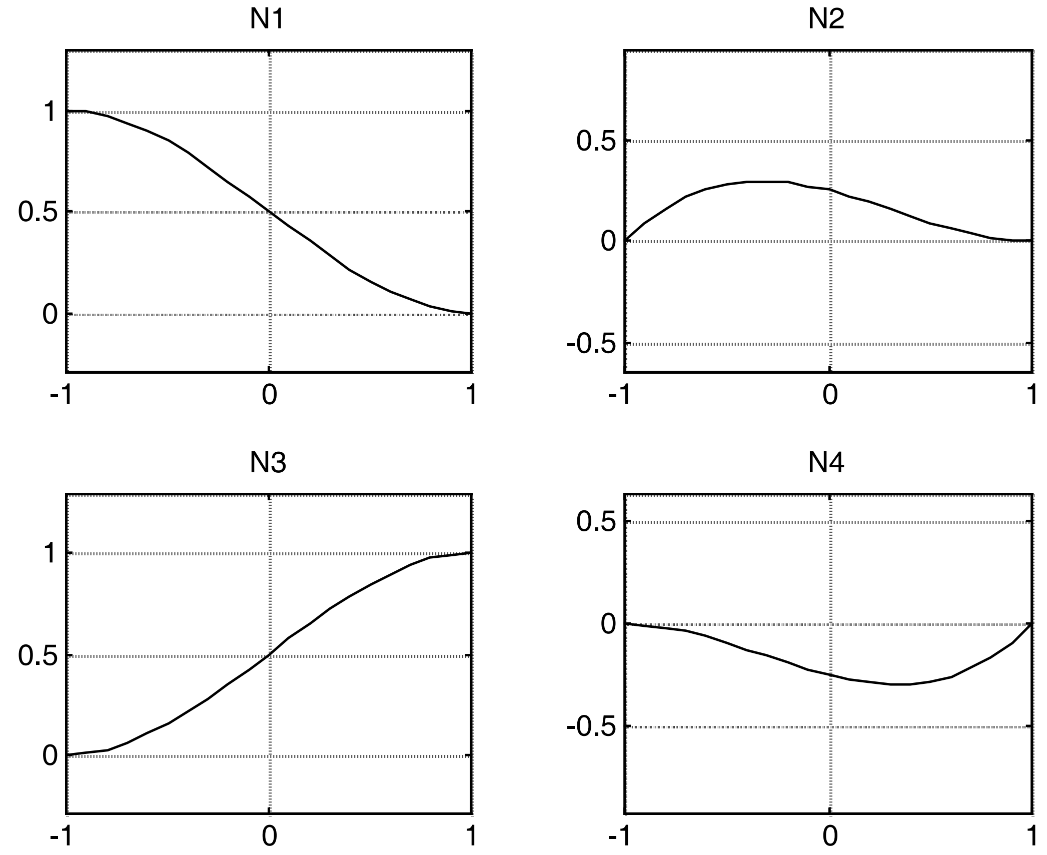


Figure 2 Plots of beam shape functions

We should pause a little here to think about why the shape functions are defined this way. Study Figure 2 carefully where the four shape functions are shown graphically. Note that N_1 has zero slope at the beginning and end, and zero value at the end. Likewise, N_2 has zero values at beginning and the end, and zero slope at the end. Just as N_1 has unit displacement at the beginning, N_2 has unit slope (angle = 45°). N_3 and N_4 exhibit similar features. i.e., *i th* shape function is zero at all other dofs and unity at *i th* dof. Recall from our discussion of shape functions for the bar element. There, the two shape functions were defined such that each shape function is unity only at one at one end and zero at the other. This is to ensure that the interpolating shape functions satisfy the deformations at the ends. Here also, all the four end conditions q_1, q_2, q_3, and q_4 are satisfied, as can be seen in plots in Figure 2. Notice that each shape function is unity at its corresponding degree of freedom while it is zero at the other degrees of freedom. If you understand this property, you can easily sketch the shape functions without looking at their expressions. Please also note that you can also easily derive them then. Let us demonstrate that with the first shape function.

Shape function $N_1(\xi)$ must satisfy four conditions:

$$\begin{aligned} &N_1(\xi=-1)=1 \\ &\frac{dN_1}{d\xi}(\xi=-1)=0 \\ &N_1(\xi=1)=0 \\ &\frac{dN_1}{d\xi}(\xi=1)=0 \end{aligned} \qquad (12)$$

A polynomial curve that satisfies four conditions must be a cubic at the minimum. So, let us take $N_1(\xi)$ to be a cubic in ξ.

$$N_1(\xi)=a_0+a_1\xi+a_2\xi^2+a_3\xi^3 \qquad (13a)$$

Then,

$$\frac{dN_1(\xi)}{d\xi}=a_1+2a_2\xi+3a_3\xi^2 \qquad (13b)$$

Conditions in Equation (12) when substituted into Equation (13a) and (13b) give:

$$\begin{aligned} &a_0-a_1+a_2-a_3=1 \\ &a_1-2a_2+3a_3=0 \\ &a_0+a_1+a_2+a_3=1 \\ &a_1+2a_2+3a_3=0 \end{aligned} \qquad (14)$$

Solving for $a_0, a_1, a_2,$ and a_3 in Equation (14) yields

$$a_0=0.5, a_1=-3/4, a_2=0, \text{ and } a_3=1/4$$

That means

$$N_1=0.5-\frac{3}{4}\xi+\frac{1}{4}\xi^3=\frac{1}{4}\left(2-3\xi+\xi^3\right) \qquad (15)$$

You can easily verify that the first shape function in Equation (8a) is the same as the one in Equation (15) because $\frac{1}{4}(1-\xi)^2(2+\xi)=\frac{1}{4}\left(2-3\xi+\xi^3\right)$.

It is important to notice also that the shape functions N_2 and N_4 are multiplied by $\frac{L_e}{2}$. This is because these two shape functions multiply with slope coordinates which have radians as units. So, this is necessary in order to be consistent with units in Equation (7). Another reason becomes apparent from Equations (10) and (11) if you think about the slope being equal to unity at those degrees of freedom.

There is a name to the four shape functions that we just used. They are called Hermite polynomials. Recall that the shape functions used for bar elements were called Lagrange polynomials. While Lagrange polynomials satisfy only the C_0 continuity (i.e., only the position continuity), Hermite polynomials satisfy C_1 continuity (i.e., the slope is also continuous). In beam elements, since the slope is one degree of freedom, we need slope continuity.

7.3 Element stiffness matrix for beam elements

As shown in Equation (5b), the strain energy of a beam element is given by

$$SE_e=\frac{1}{2}EI\int_{x_1}^{x_2}\left(\frac{d^2v}{dx^2}\right)^2 dx \tag{16}$$

Using the shape functions and the ξ-coordinate system presented above,

$$\begin{aligned}SE_e&=\frac{EI}{2}\int_{-1}^{1}\left(\frac{d^2v}{dx^2}\left(\frac{d\xi}{dx}\right)^2\right)^2\left(\frac{dx}{d\xi}d\xi\right)\\&=\frac{EI}{2}\int_{-1}^{1}\left(\frac{d^2v}{dx^2}\right)^2\left(\frac{d\xi}{dx}\right)^4\frac{dx}{d\xi}d\xi\\&=\frac{EI}{2}\int_{-1}^{1}\left(\frac{d^2v}{dx^2}\right)^2\left(\frac{2}{L_e}\right)^4\frac{L_e}{2}d\xi\\&=\frac{1}{2}\left(\frac{8EI}{L_e^3}\right)\int_{-1}^{1}\left(\frac{d^2v}{dx^2}\right)^2 d\xi\end{aligned} \tag{17}$$

where $\frac{d^2v}{dx^2}$ is given in Equation (11b). To evaluate $\left(\frac{d^2v}{dx^2}\right)^2$, we will use the matrix notation instead of doing that tediously by expanding Equation (11b). Because squaring a matrix entity is equivalent to multiplying its transpose with itself,

$$\left(\frac{d^2v}{dx^2}\right)^2 = \left(\frac{d^2v}{dx^2}\right)^T\left(\frac{d^2v}{dx^2}\right) = \left\{\left\{\frac{d^2N_1}{d\xi^2} \quad \frac{L_e}{2}\frac{d^2N_2}{d\xi^2} \quad \frac{d^2N_3}{d\xi^2} \quad \frac{L_e}{2}\frac{d^2N_4}{d\xi^2}\right\}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}\right\}^T$$
$$\times\left\{\left\{\frac{d^2N_1}{d\xi^2} \quad \frac{L_e}{2}\frac{d^2N_2}{d\xi^2} \quad \frac{d^2N_3}{d\xi^2} \quad \frac{L_e}{2}\frac{d^2N_4}{d\xi^2}\right\}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}\right\} \tag{18}$$

The substitution of Equation (18) into Equation (16), and some simplification yields

$$SE_e = \frac{1}{2}\mathbf{q}^T\left(\int_{-1}^{1}\begin{bmatrix} \frac{9}{4}\xi^2 & \frac{3}{8}\xi(3\xi-1)L_e & -\frac{9}{4}\xi^2 & \frac{3}{8}\xi(3\xi+1)L_e \\ \frac{3}{8}\xi(3\xi-1)L_e & \left(\frac{3\xi-1}{4}\right)^2 L_e^2 & -\frac{3}{8}\xi(3\xi-1)L_e & \frac{9\xi^2-1}{16}L_e^2 \\ -\frac{9}{4}\xi^2 & -\frac{3}{8}\xi(3\xi-1)L_e & \frac{9}{4}\xi^2 & -\frac{3}{8}\xi(3\xi-1)L_e \\ \frac{3}{8}\xi(3\xi+1)L_e & \frac{9\xi^2-1}{16}L_e^2 & -\frac{3}{8}\xi(3\xi-1)L_e & \left(\frac{3\xi+1}{4}\right)^2 L_e^2 \end{bmatrix} d\xi\right)\mathbf{q} \tag{19}$$

Note that the integration on ξ applies to each term in the 4 x 4 matrix above. Note also that

$$\int_{-1}^{1} d\xi = 2 \qquad \int_{-1}^{1} \xi\ d\xi = 0 \qquad \int_{-1}^{1} \xi^2\ d\xi = \frac{2}{3} \tag{20}$$

Upon integration, Equation (19) gives

$$SE_e = \frac{1}{2}\mathbf{q}^T\left(\frac{EI}{L_e^3}\begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix}\right)\mathbf{q} \tag{21}$$

from which we infer that the element stiffness matrix is given by

$$\mathbf{K}_e = \frac{EI}{L_e^3}\begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix} \tag{22}$$

If we also account for axial deformations, and consider a beam with six degrees of freedom as shown in Figure 3, we get the beam element in the local ξ-coordinate system. We denote this with "prime" because it is in the local coordinate system. The element stiffness matrix for this case is given by

$$\mathbf{K}'_e = \begin{bmatrix} \frac{A_e E_e}{L_e} & 0 & 0 & -\frac{A_e E_e}{L_e} & 0 & 0 \\ 0 & 12\frac{EI}{L_e^3} & 6\frac{EI}{L_e^2} & 0 & -12\frac{EI}{L_e^3} & 6\frac{EI}{L_e^2} \\ 0 & 6\frac{EI}{L_e^2} & 4\frac{EI}{L_e} & 0 & -6\frac{EI}{L_e^2} & 2\frac{EI}{L_e} \\ -\frac{A_e E_e}{L_e} & 0 & 0 & \frac{A_e E_e}{L_e} & 0 & 0 \\ 0 & -12\frac{EI}{L_e^3} & -6\frac{EI}{L_e^2} & 0 & 12\frac{EI}{L_e^3} & -6\frac{EI}{L_e^2} \\ 0 & 6\frac{EI}{L_e^2} & 2\frac{EI}{L_e} & 0 & -6\frac{EI}{L_e^2} & 4\frac{EI}{L_e} \end{bmatrix} \qquad (23)$$

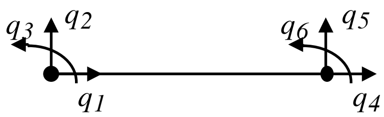
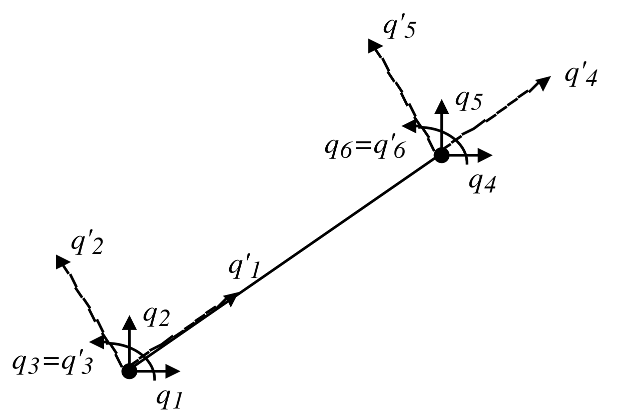


Figure 3 The six degree-of-freedom beam element

Note how we combined the 2 x 2 matrix of the axial stiffness $\begin{bmatrix} \frac{A_e E_e}{L_e} & -\frac{A_e E_e}{L_e} \\ -\frac{A_e E_e}{L_e} & \frac{A_e E_e}{L_e} \end{bmatrix}$, with the 4 x 4 bending stiffness of Equation (23), into a general 6 x 6 matrix in Equation (23). It simply means that $\mathbf{q^T K_e q}$ will give the sum of axial and bending energies.

In general, an element of a curved beam can be in any orientation. So, we need to derive the transformation matrix **L**, as we did in the case of trusses. Refer to Figure 4 and Equation (24) below.



Slope of the element = θ

Figure 4 Arbitrarily oriented beam element

$$\begin{Bmatrix} q'_1 \\ q'_1 \\ q'_1 \\ q'_1 \\ q'_1 \\ q'_1 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_1 \\ q_1 \\ q_1 \\ q_1 \\ q_1 \end{Bmatrix} \tag{24}$$

Now, the stiffness matrix for a six degree-of-freedom beam element is:

$$\mathbf{K}_e = \mathbf{L}^T \mathbf{K}'_e \mathbf{L} \tag{21}$$

This follows from the strain-energy argument that

$$SE_e = \frac{1}{2}\mathbf{q}^T \mathbf{K}'_e \mathbf{q} = \frac{1}{2}\mathbf{q}^T \mathbf{L}^T \mathbf{K}'_e \mathbf{L} \mathbf{q} \tag{22}$$

7.4 Element force vector for beam elements

The general procedure for deriving the element force vectors was described in Section 5.2 in the context of bar elements. We will repeat the same here briefly for beam elements with body force *f* per unit length. As in Equation (7) of Chapter 5, we write the work potential due *f* on the element as

$$WP_e = \int_L f\, v(x)\, dx = \frac{fL_e}{2}\left(\int_{-1}^{1} \left\{ N_1 \quad \frac{L_e}{2} N_2 \quad N_3 \quad \frac{L_e}{2} N_4 \right\} d\xi \right) \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \mathbf{f}_e^T\, \mathbf{q} \tag{23}$$

After performing the integration in matrix form, we get

$$\mathbf{f}_e = \left\{ \frac{fL_e}{2} \quad \frac{fL_e^2}{12} \quad \frac{fL_e}{2} \quad -\frac{fL_e^2}{12} \right\} \tag{24}$$

The force vector due to the distributed surface (or line) force is done in a similar manner. The point forces and moments are directly included in the global system force vector. It is then necessary to create nodes at all the locations that have point forces.

7.5 Solving beam FEM problems

The assembly of the global system, imposing the boundary conditions, and all the post-processing operations are exactly the same as we did for the trusses. Now, we realize the procedure we followed in the previous chapters is general enough to apply to other types of elements. The only thing that is different about different elements are the shape functions and the stress-strain and strain energy relationships. We will discuss the stress-strain relationships in Chapter 8. In the remaining section of this chapter, we will present the Matlab script for beam FEM problems. Study this script carefully and note the similarities (and some differences) between this script and the one for truss elements.

7.6 Matlab script for beam FEM

Four input files are needed for beam.m. These files contain the geometric, physical and materiel property, and force information about the truss to be analyzed.

1) node.dat

This file contains the (X,Y) coordinates of all the node numbers in the following format.

Node#	X-coordinate	Y-Coordinate
1		
2		
3		
etc.		

2) elem.dat

This file contains the element information pertaining to the two nodes that form the element, the cross-section area and Young's modulus of the element.

Element#	Node#1	Node#2	Area	Young's modulus	Inertia
1					
2					
3					
etc.					

3) forces.dat

This file contains the nodal forces to be applied on the truss. You need to specify the node number, the degree-of-freedom (dof) number (1 if force is in the X-direction, and 2 if the force is in the Y-direction), and the value of the force. If you have forces in the X and Y directions at the same node, then you need to put in two lines in the force, one for each direction.

Serial#	Node#	dof#	Force value
1			
2			
etc.			

4) disp.dat

This file contains the information about the displacement boundary conditions, i.e. which nodes and dof's are fixed to be zero. dof# is to be treated in the same way as you do in the case of force specification.

Serial#	Node#	dof#
1		
2		
etc.		

Matlab Scripts for beam FEM

```
---------start of fembeam.m----------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Finite Element MATLAB script for
% deflection and stress analysis of beams
% Written by: G.K. Ananthasuresh
% for MEAM 310 class in Spring 1997.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 clg                       % Clear graphics window
 clear all                 % Clear all variables
 clc                       % Clear command window
 hold off                  % No hold on the graphics window
%
% This script needs the following scripts to run
%    matcut.m   --> trims a matrix
%    veccut.m   --> trims a vector
%    femtruss.m --> FEA for trusses
% It also needs the following input files
%    node.dat   -->  nodal data
%    elem.dat   -->  element data
%    forces.dat -->  force data
%    disp.dat   -->  displacement boundary condition data
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% READ INPUT from files
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% Read in nodal and element connectivity data from
% files: nodes.dat and elem.dat
load node.dat
load elem.dat
%
% Read in force data from the file forces.dat
load forces.dat
% Read in displacement boundary condition data from the file disp.dat
load dispbc.dat
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% PRE-PROCESSING
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Identify the number of nodes, X and Y Coordinates of the nodes
NNODE = size(node,1);
nx = node(:,2);
ny = node(:,3);
%
% Identify the number of elements and form an element connectivity
array,
% the cross-section and Young's modulus arrays.
NELEM = size(elem,1);
ncon = elem(:,[2 3]);
A = elem(:,4);
E = elem(:,5);
Inertia=elem(:,6);
```

```
%
% Arrange force information into a force vector, F
F = zeros(3*NNODE,1);            % Initialization
Nforce = size(forces,1);
for i = 1:Nforce,
   dof = (forces(i,2)-1)*3 + forces(i,3);
   F(dof) = forces(i,4);
end
%
% Displacement boundary conditions
Nfix = size(dispbc,1);
j = 0;
for i = 1:Nfix,
    j = j + 1; dispID(j) = (dispbc(i,2)-1)*3+dispbc(i,3);
end
dispID = sort(dispID);
%
% Compute the lengths of the elements
for ie=1:NELEM,
  eye = ncon(ie,1);
  jay = ncon(ie,2);
  L(ie) = sqrt ( (nx(jay) - nx(eye))^2 + (ny(jay) - ny(eye))^2 );
end
%
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% SOLUTION
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Call fembeam.m to solve for the following.
%     Deflections: U
%     Reaction forces at the constrained nodes: R
%     Internal forces in each truss member: Fint
%     Global stiffness matrix: Kglobal
%     Strain energy: SE
fembeam(A,L,E,nx,ny,ncon,NELEM,NNODE,F,dispID,Inertia);
%
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% POST-PROCESSING
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plotting
clg

for ip=1:NELEM,
  pt1 = ncon(ip,1); pt2 = ncon(ip,2);
  dx1 = U(3*(pt1-1)+1);
  dy1 = U(3*(pt1-1)+2);
  dx2 = U(3*(pt2-1)+1);
  dy2 = U(3*(pt2-1)+2);
%
  plot([nx(pt1) nx(pt2)], [ny(pt1) ny(pt2)],'--c');
  hold on
  plot([nx(pt1) nx(pt2)], [ny(pt1) ny(pt2)],'.w');
  plot([nx(pt1)+dx1 nx(pt2)+dx2], [ny(pt1)+dy1 ny(pt2)+dy2],'-y');
  plot([nx(pt1)+dx1 nx(pt2)+dx2], [ny(pt1)+dy1 ny(pt2)+dy2],'.w');
  hold on


end
xlabel('X');
ylabel('Y');
axis('equal');
----------end of beam.m-----------

----------start of mcut.m-----------
function D = mcut(C,i)
% To remove ith row and ith column from C (size: NxN) and
% return D (size: N-1xN-1)
[m,n] = size(C);
d1 = C(1:i-1,1:i-1);
d2 = C(1:i-1,i+1:n);
d3 = C(i+1:m,1:i-1);
d4 = C(i+1:m,i+1:n);
D = [d1 d2; d3 d4];
----------end of mcut.m-----------

----------start of veccut.m-----------
function D = veccut(C,i)
% To remove member i from C and return D with size 1 less.
[m,n] = size(C);
if m == 1
   d1 = C(1:i-1);
   d2 = C(i+1:n);
   D = [d1 d2];
end
if n == 1
   d1 = C(1:i-1);
   d2 = C(i+1:m);
   D = [d1; d2];
end
----------end of veccut.m-----------

----------start of fembeam.m-----------

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This m-file forms the finite element model of a beam
% system and solves it.
% G.K. Ananthasuresh for MEAM 310 class in Spring 1997.
% Inputs:
%     Ae    -->  A vector containing the cross-section areas of all
%                the elements
%     Le    -->  A vector containing the lengths of all the elements
%     Ee    -->  A vector containing the Young's moduli of all the
%                elements
%     nx    -->  A vector containing the X-coordinates of all
%                the nodes
%     ny    -->  A vector containing the Y-coordinates of all
%                the nodes
%     ncon  -->  A matrix containing the nodal-connectivity of all
%                the elements
%     F     -->  A vector containing the applied external forces
%     dispID--> A vector containing the displacement boundary
%                 condition information
%     Inertia-->A vector containing the Inertia of all the elements
%
% Outputs:
%     u     -->  The displacements of all the dof
%     Rdisp -->  The reaction forces at the dof which are specified
%                to be fixed
%     King  -->  The global stiffness matrix before the application
%                of the boundary conditions.
%     SE    --> The strain energy in the entire truss
%
function  [u,Rdisp,P,Ksing,SE]  =  fembeam(Ae,  Le,  Ee,  nx,  ny,  ncon,
NELEM, NNODE, F, dispID,Ine)

K = zeros(3*NNODE,3*NNODE);                %     Initialize     global
stiffness matrix
k = zeros(6,6);                            %      Initialize      local
stiffness matrix

for ie=1:NELEM,
  eye = ncon(ie,1);
  jay = ncon(ie,2);

% Form the transformation matrix, Lambda.
  L = Le(ie);
  A = Ae(ie);
  E = Ee(ie);
  In= Ine(ie);
  lox = (nx(jay)-nx(eye))/L; mox = (ny(jay)-ny(eye))/L;
  loy = -mox; moy = lox;
  Lambda = [ lox mox  0   0   0   0   ; ...
            -mox lox  0   0   0   0   ; ...
             0   0   1   0   0   0   ; ...
             0   0   0  lox  mox 0   ; ...
             0   0   0 -mox  lox 0   ; ...
             0   0   0   0   0   1  ];
% Form local element matrix
  k(1,1)=E*A/L;            k(1,4)=-k(1,1);
  k(2,2)=12*E*In/(L^3);          k(2,3)=6*E*In/(L^2);
  k(2,5)=-k(2,2);          k(2,6)=k(2,3);
  k(3,2)=k(2,3);      k(3,3)=4*E*In/L;
  k(3,5)=-k(2,3);          k(3,6)=2*E*In/L;
  k(4,1)=-k(1,1);          k(4,4)=k(1,1);
  k(5,2)=-k(2,2);          k(5,3)=-k(2,3);
  k(5,5)=k(2,2);      k(5,6)=-k(2,3);
  k(6,2)=k(2,3);      k(6,3)=k(3,6);
  k(6,5)=-k(2,3);          k(6,6)=k(3,3);

  klocal = Lambda' * k * Lambda;


% Form ID matrix to assemble klocal into the global stiffness matrix,
K.
  id1 = 3*(eye-1) + 1;
  id2 = id1 + 1;
  id3 = id2 + 1;
  id4 = 3*(jay-1) + 1;
  id5 = id4 + 1;
  id6 = id5 + 1;

  K(id1,id1) = K(id1,id1) + klocal(1,1);
  K(id1,id2) = K(id1,id2) + klocal(1,2);
  K(id1,id3) = K(id1,id3) + klocal(1,3);
  K(id1,id4) = K(id1,id4) + klocal(1,4);
  K(id1,id5) = K(id1,id5) + klocal(1,5);
  K(id1,id6) = K(id1,id6) + klocal(1,6);

  K(id2,id1) = K(id2,id1) + klocal(2,1);
  K(id2,id2) = K(id2,id2) + klocal(2,2);
  K(id2,id3) = K(id2,id3) + klocal(2,3);
  K(id2,id4) = K(id2,id4) + klocal(2,4);
  K(id2,id5) = K(id2,id5) + klocal(2,5);
  K(id2,id6) = K(id2,id6) + klocal(2,6);

  K(id3,id1) = K(id3,id1) + klocal(3,1);
  K(id3,id2) = K(id3,id2) + klocal(3,2);
  K(id3,id3) = K(id3,id3) + klocal(3,3);
  K(id3,id4) = K(id3,id4) + klocal(3,4);
  K(id3,id5) = K(id3,id5) + klocal(3,5);
  K(id3,id6) = K(id3,id6) + klocal(3,6);

  K(id4,id1) = K(id4,id1) + klocal(4,1);
  K(id4,id2) = K(id4,id2) + klocal(4,2);
  K(id4,id3) = K(id4,id3) + klocal(4,3);
  K(id4,id4) = K(id4,id4) + klocal(4,4);
  K(id4,id5) = K(id4,id5) + klocal(4,5);
  K(id4,id6) = K(id4,id6) + klocal(4,6);

  K(id5,id1) = K(id5,id1) + klocal(5,1);
  K(id5,id2) = K(id5,id2) + klocal(5,2);
  K(id5,id3) = K(id5,id3) + klocal(5,3);
  K(id5,id4) = K(id5,id4) + klocal(5,4);
  K(id5,id5) = K(id5,id5) + klocal(5,5);
  K(id5,id6) = K(id5,id6) + klocal(5,6);

  K(id6,id1) = K(id6,id1) + klocal(6,1);
  K(id6,id2) = K(id6,id2) + klocal(6,2);
  K(id6,id3) = K(id6,id3) + klocal(6,3);
  K(id6,id4) = K(id6,id4) + klocal(6,4);
  K(id6,id5) = K(id6,id5) + klocal(6,5);
  K(id6,id6) = K(id6,id6) + klocal(6,6);
end

% Store unaltered K as Ksing before applying the boundary conditions.
  Ksing = K;

%det(K)
%inv(K);
%pause

% Imposing displacement boundary conditions
% ----------------------------------------
% dispID array contains the dof which are assigned zero values.
[sm,sn] = size(dispID);
Ndbc = sn;
for nd=1:Ndbc,
 K = matcut(K,dispID(nd)-nd+1);
 F = veccut(F,dispID(nd)-nd+1);
end

% To solve for unknown displacements.
U = inv(K)*F;
SE = .5*U'*K*U;

% Results
% --------------
% "u" for all nodes (including those where values were specified)
u = zeros(3*NNODE,1);

for iu=1:Ndbc,
  u(dispID(iu)) = 12345.12345;
end

iuc = 0;
for iu=1:3*NNODE,
    if u(iu) == 12345.12345
     iuc = iuc+1;
  else
     u(iu) = U(iu-iuc);
  end
end

for iu=1:Ndbc,
  u(dispID(iu)) = 0;
end

dx = zeros(1,NNODE);
dy = zeros(1,NNODE);
```

```
for iu=1:NNODE,
  dx(iu) = u(3*(iu-1)+1);
  dy(iu) = u(3*(iu-1)+2);
end

%-----------------------------------------------
% Computation of reactions at constrained nodes
%-----------------------------------------------
R = Ksing*u;
Rdisp = zeros(1,Ndbc);
for iu=1:Ndbc,
  Rdisp(iu) = R(dispID(iu));
end

%--------------------------------------------
% Computation of internal reaction forces
% and storing in P(NNODE,4)
%--------------------------------------------
%M = zeros(NNODE,1);

for ie=1:NELEM,

  eye = ncon(ie,1);
  jay = ncon(ie,2);


% Form the transformation matrix, Lambda.
  L = Le(ie); A = Ae(ie);
  lox = (nx(jay)-nx(eye))/L; mox = (ny(jay)-ny(eye))/L;
  loy = -mox; moy = lox;
  Lambda = [ lox mox  0   0   0   0   ; ...
            -mox lox  0   0   0   0   ; ...
             0   0   1   0   0   0   ; ...
             0   0   0  lox  mox 0   ; ...
             0   0   0 -mox  lox 0   ; ...
             0   0   0   0   0   1  ];
% Form local element matrix
  k(1,1)=E*A/L;             k(1,4)=-k(1,1);
  k(2,2)=12*E*In/(L^3);          k(2,3)=6*E*In/(L^2);
  k(2,5)=-k(2,2);           k(2,6)=k(2,3);
  k(3,2)=k(2,2);      k(3,3)=4*E*In/L;
  k(3,5)=-k(2,3);           k(3,6)=2*E*In/L;
  k(4,1)=-k(1,1);           k(4,4)=k(1,1);
  k(5,2)=-k(2,2);           k(5,3)=-k(2,3);
  k(5,5)=k(2,2);      k(5,6)=-k(2,3);
  k(6,2)=k(2,3);      k(6,3)=k(3,6);
  k(6,5)=-k(2,3);           k(6,6)=k(3,3);

  klocal = Lambda' * k * Lambda;
```

```
% Form ID matrix to identify respective displacements.
  id1 = 3*(eye-1) + 1;
  id2 = id1 + 1;
  id3 = id2 + 1;
  id4 = 3*(jay-1) +1;
  id5 = id4 + 1;
  id6 = id5 + 1;

  ulocal = [u(id1) u(id2) u(id3) u(id4) u(id5) u(id6)];

  Rint = klocal*ulocal';

  P(ie,1) = Rint(1);
  P(ie,2) = Rint(2);
  P(ie,3) = Rint(3);
  P(ie,4) = Rint(4);
  P(ie,5) = Rint(5);
  P(ie,6) = Rint(6);

  nxd1 = nx(eye) + u(id1);
  nyd1 = ny(eye) + u(id2);
  nxd2 = nx(jay) + u(id4);
  nyd2 = ny(jay) + u(id5);

  Ld = sqrt( (nxd2-nxd1)^2 + (nyd2-nyd1)^2 );
  if Ld > L P(ie,5) = 1; else P(ie,5) = -1; end
  if Ld==L P(ie,5) = 0; end
end

----------end of fembeam.m-----------
```

Exercise 7.3

For the curved beam example, input files are provided below to solve it using the Matlab FEM script. Verify that the vertical deflection at the point of load application is $\frac{13}{192}\left(\frac{FL^3}{EI}\right)$.

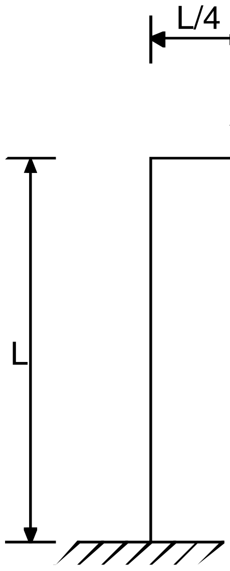


Figure 5 A curved beam example

Data files:

node.dat

```
1    0    0
2    0    1
3    0    2
4    0    3
5    0    4
6    0    5
7    0    6
8    0    7
9    0    8
10   0    9
11   0    10
```

```
12        0         11
13        0         12
14        0         13
15        0         14
16        0         15
17        0         16
18        0         17
19        0         18
20        0         19
21        0         20
22  0.25 20
23  0.5 20
24  0.75 20
25  1.00 20
26  1.25 20
27  0.5 20
28  1.75 20
29  2.00 20
30  2.25 20
31  2.5 20
32  2.75 20
33  3.00 20
34  3.25 20
35  3.5 20
36  3.75 20
37  4.00 20
38  4.25 20
39  4.5 20
40  4.75 20
41  5.00 20
```

---

elem.dat

```
1        1        2        .05      1e7      1.95e-4
2        2        3        .05      1e7      1.95e-4
```

3	3	4	.05	1e7	1.95e-4
4	4	5	.05	1e7	1.95e-4
5	5	6	.05	1e7	1.95e-4
6	6	7	.05	1e7	1.95e-4
7	7	8	.05	1e7	1.95e-4
8	8	9	.05	1e7	1.95e-4
9	9	10	.05	1e7	1.95e-4
10	10	11	.05	1e7	1.95e-4
11	11	12	.05	1e7	1.95e-4
12	12	13	.05	1e7	1.95e-4
13	13	14	.05	1e7	1.95e-4
14	14	15	.05	1e7	1.95e-4
15	15	16	.05	1e7	1.95e-4
16	16	17	.05	1e7	1.95e-4
17	17	18	.05	1e7	1.95e-4
18	18	19	.05	1e7	1.95e-4
19	19	20	.05	1e7	1.95e-4
20	20	21	.05	1e7	1.95e-4
21	21	22	.05	1e7	1.95e-4
22	22	23	.05	1e7	1.95e-4
23	23	24	.05	1e7	1.95e-4
24	24	25	.05	1e7	1.95e-4
25	25	26	.05	1e7	1.95e-4
26	26	27	.05	1e7	1.95e-4
27	27	28	.05	1e7	1.95e-4
28	28	29	.05	1e7	1.95e-4
29	29	30	.05	1e7	1.95e-4
30	30	31	.05	1e7	1.95e-4
31	31	32	.05	1e7	1.95e-4
32	32	33	.05	1e7	1.95e-4
33	33	34	.05	1e7	1.95e-4
34	34	35	.05	1e7	1.95e-4
35	35	36	.05	1e7	1.95e-4
36	36	37	.05	1e7	1.95e-4

```
37      37      38      .05     1e7     1.95e-4
38      38      39      .05     1e7     1.95e-4
39      39      40      .05     1e7     1.95e-4
40      40      41      .05     1e7     1.95e-4
```

---

forces.dat

```
1       41      2       -2
```

---

disp.dat

```
1       1       1
2       1       2
3       1       3
```

Exercise 7.4

The design of a garment hanger is not a big deal. But, it is a good curved-beam problem and plastic garment hangers do fail occasionally. Using the dimensions shown in Figure 6 and assuming that point A has a pin-support, Consider the two load cases marked (a) and (b) in the figure. Assume that the hanger is made out of tubular plastic with 3 mm ID and 7 mm OD. Assume appropriate values for all other data you need. perform the deflection analysis of the hanger using the Matlab FEM script.

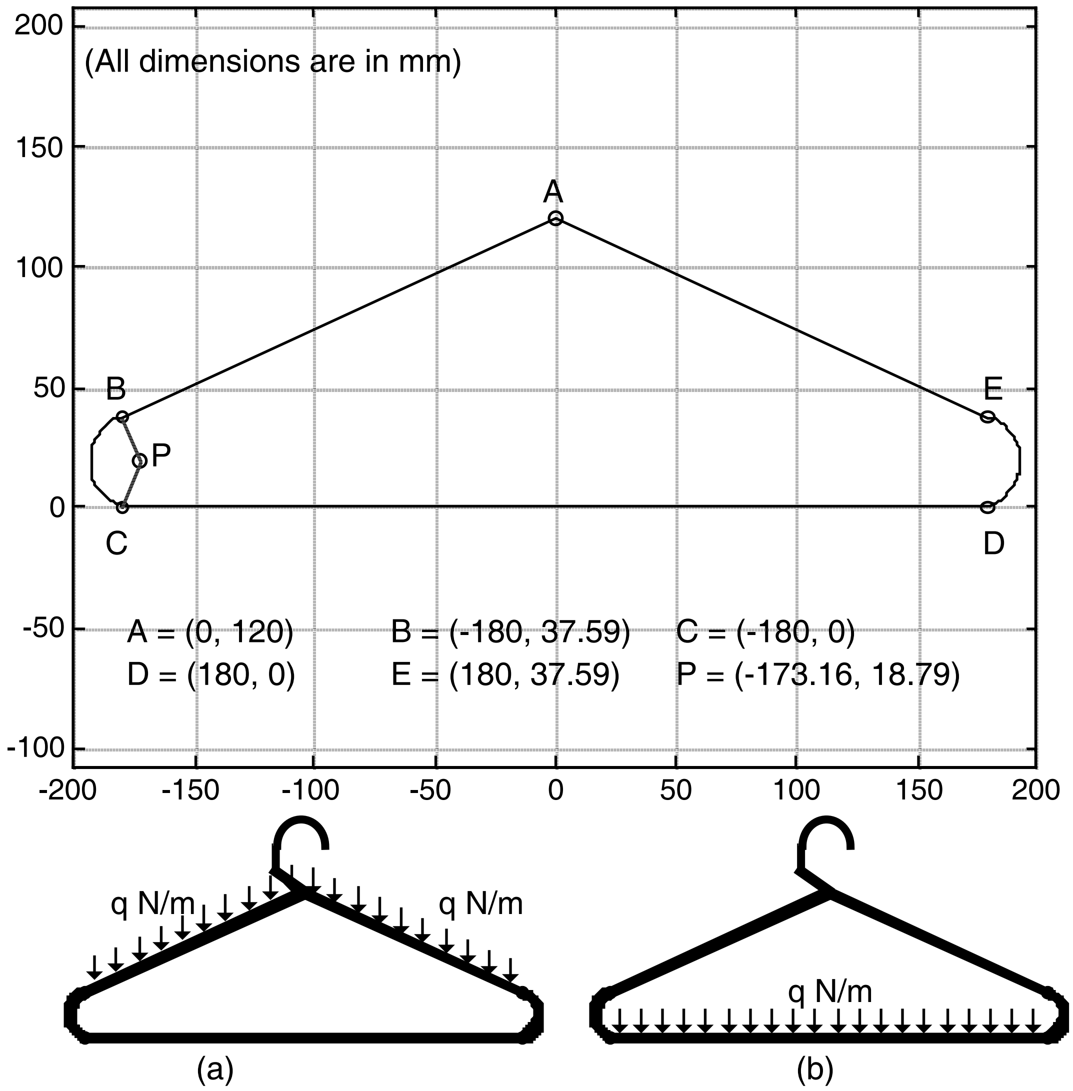


Figure 6 Approximate dimensions for a garment hanger

Exercise 7.5

Modify the Matlab FEM script for beams to compute the stresses. And then use the modified script to estimate stresses in the hanger described in Example 7.2

Exercise 7.6

Derive the shape functions for a higher order beam element that has a mid-side node at $\xi = 0$ in addition to the nodes at $\xi = -1$ and $\xi = 1$. Using those shape functions, construct the element stiffness matrix in the local coordinate system of the beam element.