Practice problems: #1

Problem 1

Consider a beam of length L, width w, and thickness t, that is clamped on both longitudinal ends. Imagine a plane below the beam separated by a gap g_0. Assume that a voltage V is applied between the beam and the plane. This creates an electrostatic force between the two. If the plane is held fixed, the beam will elastically deform under this force.
For now, assume that the electrostatic force acting per unit length of the beam is given by $(\varepsilon_0 wV^2)/(2g^2)$ where ε_0 is the permittivity of free space.

- Check the dimensionality of the above formula for the electrostatic force by using the following fundamental dimensions: M (mass), L (length), T (time), and C (Coulomb for charge).

Fix the proportions of the beam and gap as: w = L/20, t = g = L/50. Let δ be the maximum deflection of the beam under the force. Assume silicon to be the material with an Young's modulus of 150 GPa.

- For V = 5 V, plot δ/L for different sizes (i.e., vary *L* over a wide range from 10 m to 100nm. Use log-scale.
- For δ/L = 0.02, plot the required voltage vs. the length. Again, vary the length over a wide range as before.

Do your results indicate why electrostatics is the preferred mode of actuation in MEMS but not in the macro world?

● In order to check the dimensions of the following formula, let us first get the exponents of the fundamental dimensions for the quantities involved in it.

Electrostatic force per unit length of a parallel-plate capacitor = $f = \dfrac{\varepsilon_o wV^2}{2g^2}$

ε_0 : Consider the formula for the electrostatic force between two charges.

$$F = \left[MLT^{-2}\right] = \frac{q_1 q_2}{4\pi\varepsilon_0 d^2} = \frac{[C][C]}{[?]\left[L^2\right]} \Rightarrow \left[M^{-1}L^{-3}T^2C^2\right] \text{ are the dimensions for } \varepsilon_0 .$$

V : Consider the electric power formula involving voltage and current.

Power = $\left[ML^2T^{-3}\right] = VI = [?]\left[T^{-1}C\right] \Rightarrow \left[ML^2T^{-2}C^{-1}\right]$ are the dimensions for V .

Now, for the force per unit length formula, we can write the dimensions as follows.

$$f = \frac{\varepsilon_o wV^2}{2g^2} = \frac{\left[M^{-1}L^{-3}T^2C^2\right][L]\left[M^2L^4T^{-4}C^{-2}\right]}{\left[L^2\right]} = \left[MT^{-2}\right] = \frac{\left[MLT^{-2}\right]}{L}$$ are the dimensions for force per unit length.

● For a fixed-fixed beam with uniform distributed force of q per unit length, the maximum deflection occurs at the center of the beam. It is given by

$$\delta = \frac{fL^4}{384EI}$$

For a rectangular cross-section beam, $I = \frac{wt^3}{12}$.

Taking the force from the given electrostatic force formula above, we get

$$\delta = \frac{12L^4}{384Ewt^3} \cdot \frac{\varepsilon_0 w V^2}{2g^2} = \frac{\varepsilon_0 V^2 L^4}{64Et^3 g^2}$$

With given proportions, $t = g = \frac{L}{50}$, we get $\delta = \frac{50^5 \varepsilon_0 V^2}{64EL}$.

By substituting

ε_0 = 8.8542E-12 m^{-3} kg^{-1} s^4 A^2

E =150E9 Pa

we get,

$$\delta = \frac{50^5 \varepsilon_0 V^2}{64EL} = 2.8822E-16\frac{V^2}{L}.$$

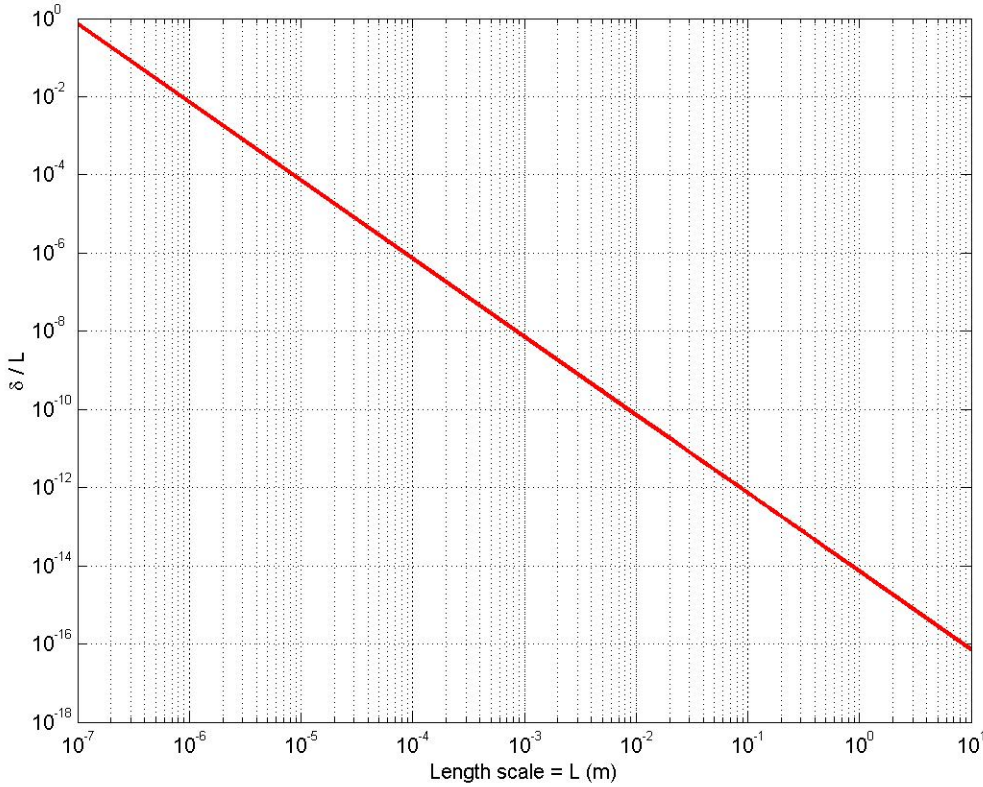
For the first plot, we have $\frac{\delta}{L} = \frac{7.2056E-15}{L^2}$.

For the second plot, we have

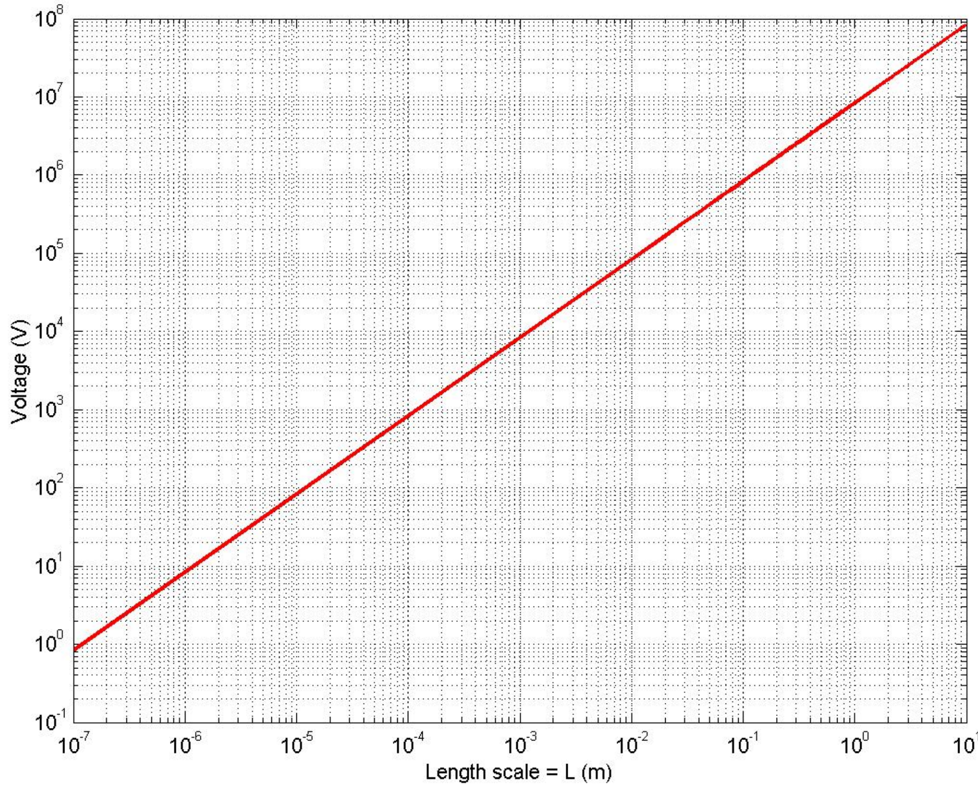
$$V = \sqrt{\frac{64EL\delta}{50^5 \varepsilon_0}} = \sqrt{\frac{64EL^2}{50^5 \varepsilon_0}\left(\frac{\delta}{L}\right)} = 8.3301E6\,L$$

The plots and the Matlab script used for it are given below.

```
% Matlab script for ME 237
% Jan.-Apr., 2005/Ananthasuresh
L = 100E-9:0.01:10;
deltabyL = 7.2056E-15./L.^2;
V = 8.3301E6*L;
figure(1)
clf
loglog(L,deltabyL,'-r','LineWidth',2);
grid
hold on
xlabel('Length scale = L (m)');
ylabel('\delta / L');
figure(2)
clf
loglog(L,V,'-r','LineWidth',2);
xlabel('Length scale = L (m)');
ylabel('Voltage (V)');
grid
```



It is clear from this plot and the equation above that the ratio of the maximum beam deflection to the length of the beam, at a given voltage, decreases rapidly with increasing length scale. Hence, the electrostatic force is favorable at small length scales.



The equation used to plot the above graph makes it clear that the required voltage to get the same fraction of deflection increases linearly with the length scale. So, it is impractical to get substantive deflection with electrostatic force at larger length scales. On the other hand, at the MEMS scale and below, it is quite attractive.

Problem 2

Consider a square-prismatic bar of the following proportions. Length of the side of the square = L; Height of the prism = 15L. Assume that it is made of aluminium. Use standard material properties of aluminum taken from textbooks and any reference books. Please write down the source from which you took the material properties. In order to see the scaling effects, repeat the following calculations for L = 1E0 m, 1E-1 m, 1E-2 m, 1E-3 m, 1E-4 m, 1E-5 m, 1E-6 m, and 1E-7 m. Prepare a table for all eight cases for each of the three following calculations.

- Compute the first natural frequency (in Hertz) of free undamped vibration if both the square faces of the aluminum prism are fixed. It is now a fixed-fixed beam. This simple mechanical structure has numerous applications as you will see later in the course.
- Compute the maximum temperature (in K) if a DC voltage is applied between the two square faces of the aluminum prism when the square faces are both at the room temperature of 300 K. The prism is now a resistive heater with its own range of applications. Neglect convective and radiative heat transfer for now. Keep the total power consumed to be constant (1 W) for all the eight cases. In your table, include resistance, current, voltage, and maximum temperature for each case.
- Now imagine a channel, which has the same dimensions as the aluminum prism. Compute the Reynolds number if water is flowing through the channel from one square-face end to the other at a speed 10 times the side of the square per second. Micro-fluidic effects are quite interesting; Reynolds number is just one indicator.

Electrical, mechanical, and thermal properties of aluminium:

Density: $\rho = 2.7 \times 10^3 \ {}^{kg}\!/_{m^3}$

Young's modulus: E = 69 GPa

Thermal conductivity: k= 222 ${}^{W}\!/_{mK}$

Electrical resistivity: $\rho' = 2.83 \times 10^{-8}\,\Omega m$

Properties of water:

Density: $\rho_{water} = 1.0 \times 10^3 \ {}^{kg}\!/_{m^3}$

Viscosity: $\eta = 1.0 \times 10^{-3} \ {}^{kg}\!/_{ms}$

The material properties are taken from:

G.T. Murray,Handbook of Materials Selection for Engineering Applications, Marcel Dekker,Inc,New York,1997.pp.86-87

Octave Levenspiel,Enginerring Flow and Heat Exchange(Revised Edition), Plenum Press,New York,1998.pp.361

(a) The first natural frequency of a fixed-fixed beam is: f = $\dfrac{22.4}{(15L)^2 \times 2\pi}\sqrt{\dfrac{EI}{\rho A}} \propto \dfrac{1}{L}$

So, the natural first frequencies for each case are:

1 (m)	1E0	1E-1	1E-2	1E-3	1E-4	1E-5	1E-6	1E-7
The first natural frequency (Hz)	23.123	231.223	2.3123 $\times 10^3$	2.3123 $\times 10^4$	2.3123 $\times 10^5$	2.3123 $\times 10^6$	2.3123 $\times 10^7$	2.312 $\times 10^8$

(b) The electrical resistance of the aluminum prism is: R= $\rho\dfrac{15L}{A} = \rho\dfrac{15L}{L^2} \propto \dfrac{1}{L}$

Current is: I = $\sqrt{W/R} \propto \sqrt{L}$

Voltage V is: V = $W/I \propto \sqrt{\dfrac{1}{L}}$

Maximum temperature above the room temperature = $\Delta T = \dfrac{W \times 15L}{kA} \propto \dfrac{1}{L}$

1 (m)	1E0	1E-1	1E-2	1E-3	1E-4	1E-5	1E-6	1E-7
Resistance (Ω)	4.25 E-7	4.25 E-6	4.25 E-5	4.25 E-4	4.25 E-3	4.25 E-2	4.25 E-1	4.24
Current (A)	1.53 E3	485.35	153.48	48.53	15.34	4.85	1.53	0.48
Voltage (V)	6.51 E-4	0.0021	0.0065	0.0206	0.0652	0.206	0.6515	2.0603
Maximum Temperature (K)	0067	0.67	6.75	67.56	675.67	6.756 E3	6.756 E4	6.756 E5

(c) The Reynolds number is:

$$\text{Re} = \frac{\rho_{water} UL}{\eta} = \frac{\rho_{water} 10L^2}{\eta} \propto L^2$$

1 (m)	1E0	1E-1	1E-2	1E-3	1E-4	1E-5	1E-6	1E-7
Reynolds number	1.0E7	1.0E5	1000	10	0.1	0.001	1.0E-5	1.0E-7