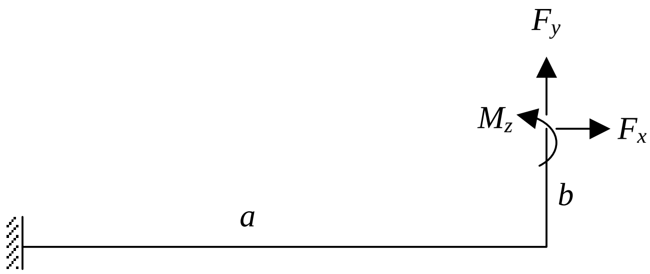
Solution to homework #5

Problem 2



We consider vertical and horizontal forces and the moment at the end because we need to solve for the end being guided in x and y directions.

The bending moment is given by $M = \begin{cases} -bF_x + F_y(a-x) + M_z & \text{for} \quad 0 \le x \le a \\ -F_x(a+b-x) + M_z & \text{for} \quad a \le x \le a+b \end{cases}$

The axial force is given by $P = \begin{cases} F_x & \text{for} \quad 0 \le x \le a \\ F_y & \text{for} \quad a \le x \le a+b \end{cases}$

When the end is guided along the x-axis, the moment and the vertical force need to be determined such that the slope and the vertical displacement of the point are zero.

Likewise, when the end is guided along the y-axis, the horizontal force and the moment need to be determined to make the slope and horizontal displacement zero.

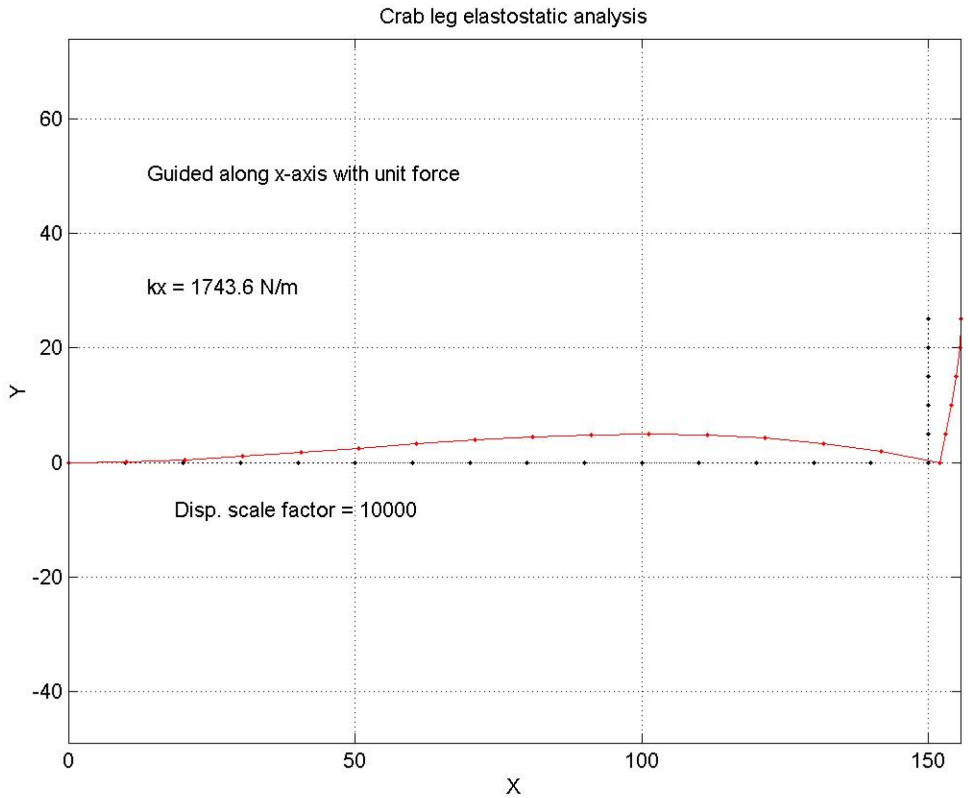
See the Maple script, which used the Castiglianos energy method by accounting for both the bending and axial strain energies, just as the FEA code does.

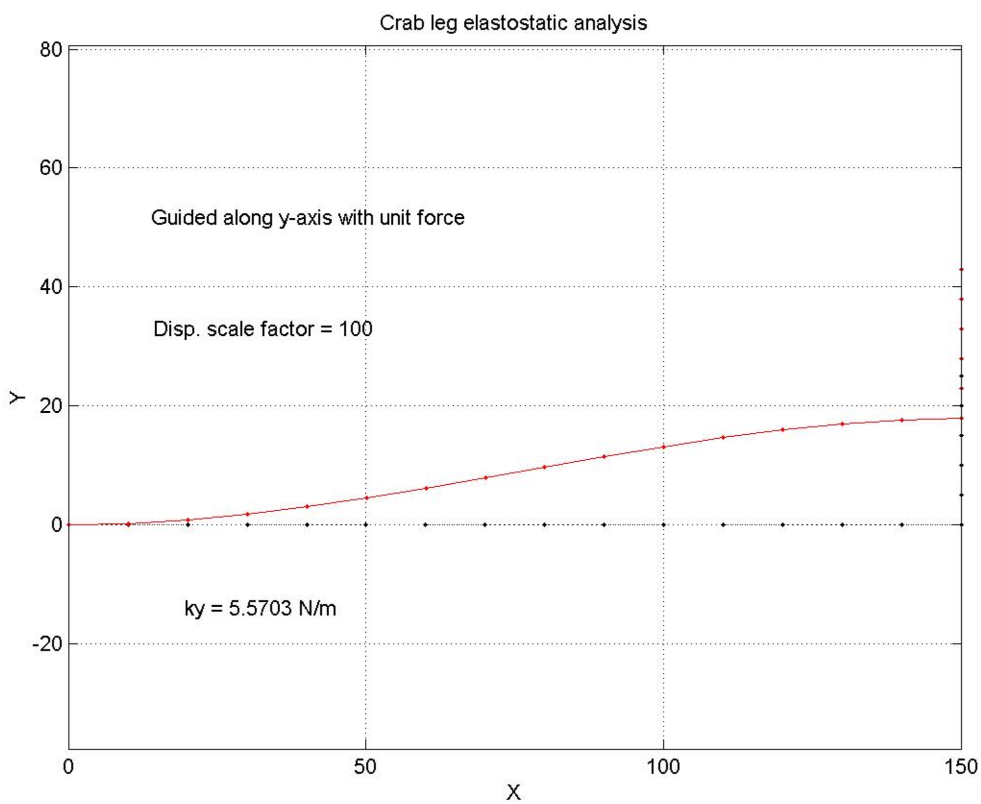
Beam FEM solution

Input files:

node.dat	elem.dat
1 0.0000 0.0000 2 10.0000 0.0000 3 20.0000 0.0000 4 30.0000 0.0000 5 40.0000 0.0000 6 50.0000 0.0000 7 60.0000 0.0000 8 70.0000 0.0000 9 80.0000 0.0000 10 90.0000 0.0000 11 100.0000 0.0000 12 110.0000 0.0000 13 120.0000 0.0000 14 130.0000 0.0000 15 140.0000 0.0000 16 150.0000 0.0000 17 150.0000 5.0000 18 150.0000 10.0000 19 150.0000 15.0000 20 150.0000 20.0000 21 150.0000 25.0000	1 1 2 1 5 155E3 2 2 3 1 5 155E3 3 3 4 1 5 155E3 4 4 5 1 5 155E3 5 5 6 1 5 155E3 6 6 7 1 5 155E3 7 7 8 1 5 155E3 8 8 9 1 5 155E3 9 9 10 1 5 155E3 10 10 11 1 5 155E3 11 11 12 1 5 155E3 12 12 13 1 5 155E3 13 13 14 1 5 155E3 14 14 15 1 5 155E3 15 15 16 1 5 155E3 16 16 17 1 10 155E3 17 17 18 1 10 155E3 18 18 19 1 10 155E3 19 19 20 1 10 155E3 20 20 21 1 10 155E3
dispbc.dat for guiding the along along x	forces.dat for guiding the tip along x
1 1 1 0.00 2 1 2 0.00 3 1 3 0.00 4 21 2 0.00 5 21 3 0.00	1 21 1 1
dispbc.dat for guiding along y	forces.dat for guiding the tip along y
1 1 1 0.00 2 1 2 0.00 3 1 3 0.00 4 21 1 0.00 5 21 3 0.00	1 21 2 1

Analytical solution is obtained using Maple. See Maple script next. The expressions of Maple were pasted into Matlab script. See Matlab script. When you run the Matlab script, you will see that the results as those obtained with FEM code are obtained—and this is what we expected as both the FEA code and energy method are based on the same type of modeling of the beam (Euler-Bernoulli beam theory).


Crab leg elastostatic analysis
Guided along x-axis with unit force
kx = 1743.6 N/m
Disp. scale factor = 10000
Y
X
60
40
20
0
-20
-40
0
50
100
150


Crab leg elastostatic analysis
Guided along y-axis with unit force
Disp. scale factor = 100
ky = 5.5703 N/m
Y
X
80
60
40
20
0
-20
0
50
100
150

Maple script

```
> #HW 5, problem 2; Crab leg suspension problem
restart;
M1 := (-b*Fx + Fy*(a-x) + Mz)^2 /2/E/I1;
M2 := (-Fx*(a+b-x)+Mz)^2 /2/E/I2;
P1 := Fx^2/2/A1/E;
P2 := Fy^2/2/A2/E;
```

$$M1 := \frac{1}{2}\frac{(-b\,Fx + Fy\,(a-x) + Mz)^2}{E\,I1}$$

$$M2 := \frac{1}{2}\frac{(-Fx\,(a+b-x) + Mz)^2}{E\,I2}$$

$$P1 := \frac{1}{2}\frac{Fx^2}{A1\,E}$$

$$P2 := \frac{1}{2}\frac{Fy^2}{A2\,E}$$

```
> SE := int(M1,x=0..a) + int(M2, x=a..a+b) + int(P1, x=0..a)
+ int(P2,x=a..a+b);
```

$$SE := \frac{1}{6}\frac{Fy^2\,a^3}{E\,I1} - \frac{1}{2}\frac{(-b\,Fx + Fy\,a + Mz)\,Fy\,a^2}{E\,I1} + \frac{\frac{1}{2}(-b\,Fx + Fy\,a + Mz)^2\,a}{E\,I1} + \frac{\frac{1}{6}Fx^2\,((a+b)^3 - a^3)}{E\,I2} + \frac{\frac{1}{2}(-Fx\,(a+b) + Mz)\,Fx\,((a+b)^2 - a^2)}{E\,I2} + \frac{\frac{1}{2}(-Fx\,(a+b) + Mz)^2\,b}{E\,I2} + \frac{\frac{1}{2}Fx^2\,a}{A1\,E} + \frac{\frac{1}{2}Fy^2\,b}{A2\,E}$$

```
> SEFx := diff(SE,Fx); SEFy := diff(SE,Fy); SEMz :=
diff(SE,Mz);
```

$$SEFx := \frac{1}{2}\frac{b\,Fy\,a^2}{E\,I1} - \frac{(-b\,Fx + Fy\,a + Mz)\,a\,b}{E\,I1} + \frac{\frac{1}{3}Fx\,((a+b)^3 - a^3)}{E\,I2} + \frac{\frac{1}{2}(-a-b)\,Fx\,((a+b)^2 - a^2)}{E\,I2} + \frac{\frac{1}{2}(-Fx\,(a+b) + Mz)\,((a+b)^2 - a^2)}{E\,I2} + \frac{(-Fx\,(a+b) + Mz)\,b\,(-a-b)}{E\,I2} + \frac{Fx\,a}{A1\,E}$$

$$SEFy := -\frac{1}{6}\frac{a^3\,Fy}{E\,I1} + \frac{\frac{1}{2}(-b\,Fx + Fy\,a + Mz)\,a^2}{E\,I1} + \frac{Fy\,b}{A2\,E}$$

$SEMz :=$

$$-\frac{1}{2}\frac{Fy\,a^2}{E\,I1} + \frac{(-b\,Fx + Fy\,a + Mz)\,a}{E\,I1} + \frac{\frac{1}{2}Fx\,((a+b)^2 - a^2)}{E\,I2} + \frac{(-Fx\,(a+b) + Mz)\,b}{E\,I2}$$

```
> Fx:=1: Fy:=0: Mz:=0: Fxdx := SEFx: Fxdy := SEFy: Fxthz :=
SEMz: Fx:='Fx': Fy :='Fy': Mz :='Mz':
> Fx:=0: Fy:=1: Mz:=0: Fydx := SEFx: Fydy := SEFy: Fythz :=
SEMz: Fx:='Fx': Fy :='Fy': Mz :='Mz':
> Fx:=0: Fy:=0: Mz:=1: Mzdx := SEFx: Mzdy := SEFy: Mzthz :=
SEMz: Fx:='Fx': Fy :='Fy': Mz :='Mz':
> #Guiding along y
eq1 := Rx*Fxdx + Fy*Fydx + RMz*Mzdx:
eq2 := Rx*Fxthz + Fy*Fythz + RMz*Mzthz:
soln1 := solve({eq1=0, eq2=0},{Rx, RMz}); assign(soln1);
Fyy := simplify(Rx*Fxdy + Fy*Fydy + RMz*Mzdy);
```

$$soln1 := \{RMz = \frac{(b^3\,A1 - 6\,a\,I2)\,Fy\,a^2\,I2}{4\,b^3\,a\,I2\,A1 + I1\,b^4\,A1 + 12\,I1\,b\,a\,I2 + 12\,a^2\,I2^2},$$

$$Rx = 3\,\frac{b^2\,Fy\,a^2\,I2\,A1}{4\,b^3\,a\,I2\,A1 + I1\,b^4\,A1 + 12\,I1\,b\,a\,I2 + 12\,a^2\,I2^2}\}$$

$$Fyy := \frac{1}{3}Fy\,(b^3\,a^4\,I2\,A1\,A2 + a^3\,A2\,I1\,b^4\,A1 + 12\,a^4\,A2\,I1\,b\,I2 + 3\,a^5\,A2\,I2^2 + 12\,I1\,b^4\,a\,I2\,A1 + 3\,I1^2\,b^5\,A1 + 36\,I1^2\,b^2\,a\,I2 + 36\,I1\,b\,a^2\,I2^2) \Big/ ((4\,b^3\,a\,I2\,A1 + I1\,b^4\,A1 + 12\,I1\,b\,a\,I2 + 12\,a^2\,I2^2)\,E\,I1\,A2)$$

```
> #Guiding along x
eq1 := Fx*Fxdy + Ry*Fydy + RMzx*Mzdy:
eq2 := Fx*Fxthz + Ry*Fythz + RMzx*Mzthz:
soln1 := solve({eq1=0, eq2=0},{Ry, RMzx}); assign(soln1);
Fxx := simplify(Fx*Fxdx + Ry*Fydx + RMzx*Mzdx);
```

$$soln1 := \{RMzx = \frac{Fx\,b\,(2\,I1\,b\,a^3\,A2 + 6\,I1^2\,b^2 + a^4\,I2\,A2 + 12\,I1\,b\,a\,I2)}{a^4\,I2\,A2 + 4\,I1\,b\,a^3\,A2 + 12\,I1^2\,b^2 + 12\,I1\,b\,a\,I2},$$

$$Ry = 3\,\frac{a^2\,A2\,b^2\,Fx\,I1}{a^4\,I2\,A2 + 4\,I1\,b\,a^3\,A2 + 12\,I1^2\,b^2 + 12\,I1\,b\,a\,I2}\}$$

$$Fxx := \frac{1}{3}Fx\,(b^3\,a^4\,I2\,A1\,A2 + a^3\,A2\,I1\,b^4\,A1 + 12\,a^4\,A2\,I1\,b\,I2 + 3\,a^5\,A2\,I2^2 + 12\,I1\,b^4\,a\,I2\,A1 + 3\,I1^2\,b^5\,A1 + 36\,I1^2\,b^2\,a\,I2 + 36\,I1\,b\,a^2\,I2^2) \Big/ (E\,I2\,A1\,(a^4\,I2\,A2 + 4\,I1\,b\,a^3\,A2 + 12\,I1^2\,b^2 + 12\,I1\,b\,a\,I2))$$

```
> kx := simplify(1/Fxx);
>
```

$$kx := 3\,E\,I2\,A1\,(a^4\,I2\,A2 + 4\,I1\,b\,a^3\,A2 + 12\,I1^2\,b^2 + 12\,I1\,b\,a\,I2) \Big/ (Fx\,(b^3\,a^4\,I2\,A1\,A2 + a^3\,A2\,I1\,b^4\,A1 + 12\,a^4\,A2\,I1\,b\,I2 + 3\,a^5\,A2\,I2^2 + 12\,I1\,b^4\,a\,I2\,A1 + 3\,I1^2\,b^5\,A1 + 36\,I1^2\,b^2\,a\,I2 + 36\,I1\,b\,a^2\,I2^2))$$

```
> ky := simplify(1/Fyy);
```

$$ky := 3\,((4\,b^3\,a\,I2\,A1 + I1\,b^4\,A1 + 12\,I1\,b\,a\,I2 + 12\,a^2\,I2^2)\,E\,I1\,A2) \Big/ (Fy\,(b^3\,a^4\,I2\,A1\,A2 + a^3\,A2\,I1\,b^4\,A1 + 12\,a^4\,A2\,I1\,b\,I2 + 3\,a^5\,A2\,I2^2 + 12\,I1\,b^4\,a\,I2\,A1 + 3\,I1^2\,b^5\,A1 + 36\,I1^2\,b^2\,a\,I2 + 36\,I1\,b\,a^2\,I2^2))$$

```
> kxbyky := simplify(kx/ky);
```

$$kxbyky := \frac{I2\,A1\,(a^4\,I2\,A2 + 4\,I1\,b\,a^3\,A2 + 12\,I1^2\,b^2 + 12\,I1\,b\,a\,I2)\,Fy}{Fx\,(4\,b^3\,a\,I2\,A1 + I1\,b^4\,A1 + 12\,I1\,b\,a\,I2 + 12\,a^2\,I2^2)\,I1\,A2}$$

>

Matlab script

```
% HW #5, Problem 2
a = 150;
b = 25;
w = 1; % Thickness of the layer
t1 = 5;
t2 = 10;
A1 = w*t1; A2 = w*t2;
I1 = w*t1^3/12; I2 = w*t2^3/12;
E = 155E3;
% Displacements under unit Fx
Fx = 1;
Fxdx = 1/3*Fx*(3*b^2*a*I2*A1+b^3*I1*A1+3*a*I1*I2)/(E*I1*I2*A1)
Fxdy = -1/2*b*Fx*a^2/(E*I1)
Fxthz = -1/2*b*Fx*(2*a*I2+I1*b)/(E*I1*I2)
%
% Displacements under unit Fy
Fy = 1;
Fydx = -1/2*b*Fy*a^2/(E*I1)
Fydy = 1/3*Fy*(a^3*A2+3*b*I1)/(E*I1*A2)
Fythz = 1/2*Fy*a^2/(E*I1)
%
% Displacements under unit Mz
Mz = 1;
Mzdx = -1/2*Mz*b*(2*a*I2+b*I1)/(E*I1*I2)
Mzdy = 1/2*Mz*a^2/(E*I1)
Mzthz = Mz*(a*I2+b*I1)/(E*I1*I2)
% When guided along y, Rx and RMz reactions are to be found such that
% Rx*Fxdx + Fy*Fydx + RMz*Mzdx = 0
% Rx*Fxthz + Fy*Fythz + RMz*Mzthz = 0
% Ay = [Fxdx Mzdx; Fxthz Mzthz];
```

```
% by = -[Fy*Fydx; Fy*Fythz];
% s1 = inv(Ay)*by;
% Rx = s1(1)
% Mz = s1(2)
Rx = 3*b^2*Fy*a^2*I2*A1/(4*a*I2*b^3*A1+12*a^2*I2^2+I1*b^4*A1+12*I1*b*a*I2);
RMz = (b^3*A1-
6*a*I2)*Fy*a^2*I2/(4*a*I2*b^3*A1+12*a^2*I2^2+I1*b^4*A1+12*I1*b*a*I2);
% Displacements of the tip under this condition
% Fy = 1;
% ydx = Rx*Fxdx + Fy*Fydx + RMz*Mzdx
% ydy = Rx*Fxdy + Fy*Fydy + RMz*Mzdy
% ythz = Rx*Fxthz + Fy*Fythz + RMz*Mzthz
%
Fy = 1;
ydyAnal =
1/3*Fy*(b^3*a^4*I2*A1*A2+3*a^5*A2*I2^2+a^3*A2*I1*b^4*A1+12*a^4*A2*I1*b*I
2+ ...

12*I1*b^4*a*I2*A1+36*I1*b*a^2*I2^2+3*I1^2*b^5*A1+36*I1^2*b^2*a*I2)/((4*a*I2
*b^3*A1+ ...
   12*a^2*I2^2+I1*b^4*A1+12*I1*b*a*I2)*E*I1*A2)
%
Fx = 1;
xdxAnal =
1/3*Fx*(b^3*a^4*I2*A1*A2+3*a^5*A2*I2^2+a^3*A2*I1*b^4*A1+12*a^4*A2*I1*b*I
2+ ...

12*I1*b^4*a*I2*A1+36*I1*b*a^2*I2^2+3*I1^2*b^5*A1+36*I1^2*b^2*a*I2)/(E*I2*A
1*(a^4*I2*A2+ ...
   12*I1*b*a*I2+4*I1*b*a^3*A2+12*I1^2*b^2))
% Stiffness of the crab leg
kx =
3*E*I2*A1*(a^4*I2*A2+4*I1*b*a^3*A2+12*I1^2*b^2+12*I1*b*a*I2)/(Fx*(b^3*a^4*
I2*A1*A2+ ...

a^3*A2*I1*b^4*A1+12*a^4*A2*I1*b*I2+3*a^5*A2*I2^2+12*I1*b^4*a*I2*A1+3*I1^
2*b^5*A1+ ...
   36*I1^2*b^2*a*I2+36*I1*b*a^2*I2^2))
ky =
3*(4*b^3*a*I2*A1+I1*b^4*A1+12*I1*b*a*I2+12*a^2*I2^2)*E*I1*A2/(Fy*(b^3*a^4*
I2*A1*A2+ ...

a^3*A2*I1*b^4*A1+12*a^4*A2*I1*b*I2+3*a^5*A2*I2^2+12*I1*b^4*a*I2*A1+3*I1^
2*b^5*A1+ ...
   36*I1^2*b^2*a*I2+36*I1*b*a^2*I2^2))
% Since there are four legs, kx and ky need to be multiplied by 4
```