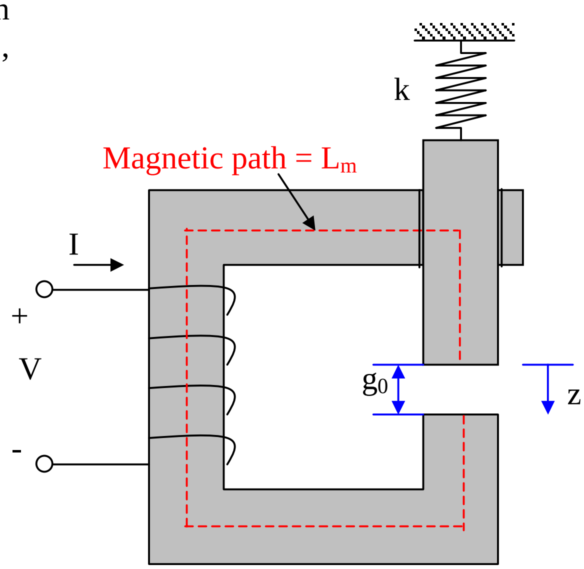
Solution to homework #6

Problem 5

By denoting length of the magnetic path before any movement of the armature by L_{m0}, we can write for any z:



$$F_{MM} = H_\mu (L_{m0} + z) + H_g (g0 - z)$$

Since $H_\mu = \dfrac{\mu_0}{\mu} H_g$,

$$F_{MM} = \frac{\mu_0}{\mu} H_g (L_{m0} + z) + H_g (g0 - z)$$

and from this,

$$\phi = B_g A = \mu_0 H_g A = \frac{\mu_0 A}{\frac{\mu_0}{\mu}(L_{m0} + z) + (g0 - z)} F_{MM}$$

By writing the magnetic co-energy as

$$W^*(F_{MM}, g) = \frac{1}{2} \frac{\mu_0 A}{\frac{\mu_0}{\mu}(L_{m0} + z) + (g0 - z)} F_{MM}^2$$

whose derivative with respect to z gives the negative of the force on the vertically moving armature. As the form of this equation is similar to that of the electrostatic force equation, the behavior of this system is the same as before. Therefore, pull-in occurs here also.

Note also that $F_{MM} = nI$. If we know the electrical circuit's resistance, we can compute the current I to be put into the above equation of co-energy in terms of the given DC voltage, V.

Problem 6

```
> restart;
> Ces := epsilon0*A/(g0-x);
```
Capacitance expression

Voltage across the electromechanical capacitor and the external capacitor

```
Ves2 := V2*(Cs/(Cs+Ces))^2; Vs2 := V2*(Ces/(Ces+Cs))^2;
```

Note that the energy of the external capacitor is also included.

```
Wstar := 0.5*k*x^2 - 0.5*Ces*Ves2 - 0.5*Cs*Vs2;
```

$$Ces := \frac{\varepsilon 0\, A}{g0 - x}$$

$$Ves2 := \frac{V2\ Cs^2}{\left(Cs + \frac{\varepsilon 0\ A}{g0 - x}\right)^2}$$

$$Vs2 := \frac{V2\ \varepsilon 0^2\ A^2}{(g0 - x)^2 \left(Cs + \frac{\varepsilon 0\ A}{g0 - x}\right)^2}$$

$$Wstar := .5\ k\ x^2 - \frac{.5\ \varepsilon 0\ A\ V2\ Cs^2}{(g0 - x)\left(Cs + \frac{\varepsilon 0\ A}{g0 - x}\right)^2} - \frac{.5\ Cs\ V2\ \varepsilon 0^2\ A^2}{(g0 - x)^2\left(Cs + \frac{\varepsilon 0\ A}{g0 - x}\right)^2}$$

```
> Fbalance := simplify(diff(Wstar,x));
CriticalStability := simplify(diff(Fbalance,x));
```

$$Fbalance := .5000000000\ (2.\ k\ Cs^2\ x^3 - 4.\ x^2\ k\ g0\ Cs^2 - 4.\ x^2\ k\ \varepsilon 0\ A\ Cs + 2.\ x\ k\ g0^2\ Cs^2 + 4.\ x\ k\ g0\ \varepsilon 0\ A\ Cs + 2.\ x\ k\ \varepsilon 0^2\ A^2 - 1.\ \varepsilon 0\ A\ V2\ Cs^2) \big/ (-1.\ Cs\ g0 + Cs\ x - 1.\ \varepsilon 0\ A)^2$$

$$CriticalStability := (-3.\ k\ Cs^3\ g0\ x^2 + k\ Cs^3\ x^3 - 3.\ k\ Cs^2\ x^2\ \varepsilon 0\ A + 3.\ k\ Cs^3\ g0^2\ x + 6.\ k\ Cs^2\ g0\ x\ \varepsilon 0\ A + 3.\ k\ Cs\ x\ \varepsilon 0^2\ A^2 - 1.\ k\ Cs^3\ g0^3 - 3.\ k\ Cs^2\ g0^2\ \varepsilon 0\ A - 3.\ k\ g0\ Cs\ \varepsilon 0^2\ A^2 - 1.\ k\ \varepsilon 0^3\ A^3 + \varepsilon 0\ A\ V2\ Cs^3) \big/ (-1.\ Cs\ g0 + Cs\ x - 1.\ \varepsilon 0\ A)^3$$

```
> soln := solve({Fbalance,CriticalStability}, {V2,x});
```

$$soln := \{x = .3333333333\ \frac{Cs\ g0 + \varepsilon 0\ A}{Cs},\ V2 = .2962962963\ \frac{k\ (Cs^3\ g0^3 + 3.\ Cs^2\ g0^2\ \varepsilon 0\ A + 3.\ Cs\ g0\ \varepsilon 0^2\ A^2 + \varepsilon 0^3\ A^3)}{\varepsilon 0\ A\ Cs^3}\}$$

```
> assign(soln);
> eqCs := g0 - x;
```

$$eqCs := g0 - \frac{.3333333333\ (Cs\ g0 + \varepsilon 0\ A)}{Cs}$$

```
> solve(eqCs=0, Cs);
```

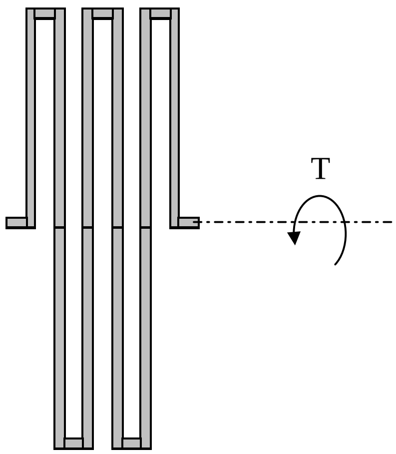
$$.4999999999\ \frac{\varepsilon 0\ A}{g0}$$

```
>
```

Notice that the external capacitance in the limiting case is half of the electromechanical capacitor at zero displacements. Since the two capacitors are in series, the total capacitance will be one third the zero-displacement capacitance. This means that the initial gas is three times the original value. So, we get g_0 as the stable gap before pull-in occurs. So, we could have written this result intuitively without having to go through this algebraic exercise.

Problem 7

The torsional stiffness of the serpentine structure is calculated on the basis of the bending of the vertical beams when a torque is applied about the axis as shown below. Since the horizontal beams in the figure are very small, their twists are neglected.



There are two vertical beams of length p and four of length $2p$. The angular rotation (twist of the serpentine spring) for a vertical beam due to torque T is given by

$$\theta = \frac{Ml}{EI} = \frac{Tp}{EI} \; or \; \frac{T(2p)}{EI}$$

where $I = \frac{bt^3}{12}$.

The total angular displacement for torque T is the summation of the rotations of all the six (four long and two short) beams. Then, we can calculate the angular stiffness κ constant as follows.

$$\theta_{total} = \frac{2Tp}{EI} + \frac{4T(2p)}{EI} = \frac{10Tp}{EI} \Rightarrow \kappa = \frac{EI}{10p}.$$

Since there are two serpentine springs, one on either side, it will be 2κ for the spring constant for the rotation about x and y axes.

The rotational inertia J of the disk is given by

$$J = J_x = J_y = \frac{1}{12} M\left(3\frac{d^2}{4} + t^2\right) = \frac{1}{12}\frac{\pi d^2 t}{4}\left(3\frac{d^2}{4} + t^2\right)$$

In this modeling, we neglect the vertical forces and vertical deflections of the disk and consider only the rotations about x and y axes. The equations of motions are given by

$$J\ddot{\phi}_x + \kappa\phi_x = T_x$$
$$J\ddot{\phi}_y + \kappa\phi_y = T_y$$

The coupling between the two axes of rotation arises due to the torque created by the electrostatic force on the disk. For this, we simply calculate the net torque by numerically integrating the electrostatic force computed using the parallel-plate approximation. For this we need to know the z -height of the point in the disk for given ϕ_x and ϕ_y .

Consider the rotation matrix approach to find the z -height of a point (x , y) of the disk.

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_x & \sin\phi_x \\ 0 & -\sin\phi_x & \cos\phi_x \end{bmatrix} \begin{bmatrix} \cos\phi_y & 0 & -\sin\phi_y \\ 0 & 1 & 0 \\ \sin\phi_y & 0 & \cos\phi_y \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

This gives the coordinates of all point in the disk. Note that $x = r\cos\alpha, y = r\sin\alpha, z = g_0$ for the disk in the zero position. When the disk rotates, these coordinates will change according to the rotation matrices above. The integration (summation in the discretized sense) is done over the area of the disk under which the electrode is activated. Use $\Delta\alpha = 5\pi/360$ and $\Delta r = a/20$ for the purpose of discretization. Then, the area of the parallel plate for each discretized point is $(\Delta r)(r\Delta\alpha)$. Then, the toques are given as

$$T_x = \sum \frac{\varepsilon_0 (\Delta r)(r\Delta\alpha)V^2}{2(z')^2} y'$$

$$T_y = \sum \frac{\varepsilon_0 (\Delta r)(r\Delta\alpha)V^2}{2(z')^2} x'$$