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On an optimal property of compliant topologies

A. Saxena and G.K. Ananthasuresh

Abstract An optimal structural property for compliant topologies is presented in this paper for general multicriteria formulations that comprise the conflicting *flexibility* and *stiffness* requirements. The property deduced from the first-order necessary conditions for optimality implies that *the ratio of the mutual potential energy density to the strain energy density is uniform throughout the continuum, but for portions otherwise bounded by gage constraints.* This property is used to develop an optimality criteria method for synthesizing compliant topologies. It is also noted that the multicriteria formulations considered here are nonconvex and can result in nonunique solutions. However, by incorporating a one-variable search along the direction determined by the above optimal property, it is ensured that the converged solution is a minimum. Several synthesis examples are included with linear frame finite elements which are easy for implementation and are capable of appropriately accounting for the bending behaviour in the continuum. Examples with previously reported density based design parameterization using bilinear plane-stress elements are also included to illustrate the synthesis procedure.

Nomenclature

E effective Young's modulus of an element
E^o Young's modulus of the material
$\mathbf{F}$ force vector comprising the actual loads
$\mathbf{F}_d$ force vector comprising the unit dummy load
F_{in} input force
H Convolution operator in the filtering scheme
$\mathbf{H}$ Hessian of the objective function
$\mathbf{K}$ stiffness matrix
MPE mutual potential energy
MPE_i mutual potential energy of element i
SE strain energy
SE_i strain energy of element i
R ratio of MPE and SE
s_i i-th element of search vector, $\mathbf{S}$
$\mathbf{S}$ normalized search direction vector
$\mathbf{U}$ displacement vector due to actual loads
$\mathbf{V}$ displacement vector due to the unit dummy load
$\mathbf{X}, \mathbf{X}_o$ vectors containing the design variables
$\mathbf{k}_i$ stiffness matrix of element i
w weighting factor
x_i design variable for element i
x_ℓ lower limit on the design variables
x_u upper limit on the design variables
Δ_{in} deformation at the input port
Δ_{out} output deformation
$\mathbf{\Omega}$ design domain
α scalar parameter for golden section method
$\boldsymbol{\varepsilon}$ strain field due to the actual load
$\boldsymbol{\sigma}$ stress field due to the actual load
$\boldsymbol{\sigma}_d$ stress field due to the unit dummy load

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A. Saxena and G.K. Ananthasuresh

Department of Mechanical Engineering and Applied Mechanics, University of Pennsylvania, Philadelphia, PA 19104-6315, USA
e-mail: gksuresh@seas.upenn.edu

1 Introduction

A fully compliant mechanism is a flexible continuum of material which acquires force and motion transmission capabilities through elastic deformation. Ease of manufacture, reduced assembly costs, reduced friction, wear, noise, and the ability to accommodate unconventional actuation schemes are some of the numerous advantages of compliant mechanisms (Howell and Midha 1995; Ananthasuresh *et al.* 1994). Two approaches known in the literature for the systematic synthesis of compliant mechanisms are the *kinematics based approach* (Howell and Midha 1996) and the *continuum based approach* (Ananthasuresh *et al.* 1994; Frecker *et al.* 1997; Sigmund 1996; Saxena and Ananthasuresh 1998). In the kinematics based approach, a known compliant topology is represented and synthesized using a rigid-body linkage with joint springs and is called the *pseudo-rigid-body model.*

The *continuum based approach*, on the other hand, focuses on the determination of the *topology*, *shape* and *size* of the mechanisms.

Although the intended functions of compliant mechanisms and stiffest structures are inherently different, structural optimization algorithms can be adapted for the synthesis of compliant topologies. A fundamental difference in the two design problems can be comprehended by considering the continua shown in Figs. 1a and 1b. Although similar in topology, the design intent for the *structure* in Fig. 1a is to support external loads while undergoing minimum deformation, whereas the *unitized mechanism* in Fig. 1b is designed to flex and deliver the required output deformation. In compliant mechanisms, therefore, adequate *flexibility* is deemed essential for their structural reconfiguration to afford the required displacement at the point of interest. Additionally, a compliant mechanism also needs to be stiff enough to be able to sustain external loads. Thus, there are two design objectives to be met simultaneously when designing a compliant mechanism, namely, to seek an optimal continuum (i) *flexible enough to satisfy the kinematic requirements* and (ii) *stiff enough to support external loads.* These design objectives are complementary to each other, for a compliant mechanism with additional flexibility may not uphold large external forces and a stiffer continuum may not comply enough to yield the required deformation. Hence, an optimum balance between the two requirements of flexibility and stiffness is required in the synthesis of compliant mechanisms.

The deformation at a prescribed output point in a specified direction can be used as a measure for the intended flexibility in a compliant mechanism. The unit dummy load method of computing the required deformation using the principle of virtual work (Barnett 1961) can be used to formulate an energy-like functional to facilitate the use of Euler-Lagrange equations of variational calculus. Shield and Prager (1970) formalized this approach by defining the deformation at a point as the *mutual potential energy* or *MPE.* The strain energy, *SE*, on the other hand serves as the traditional measure for the structural stiffness. Based on the notion that a compliant mechanism is required to meet both flexibility and stiffness requirements, Ananthasuresh *et al.* (1994) presented a multicriteria formulation wherein a weighted linear combination of the two objectives of maximizing *MPE* and minimizing *SE* was used. Frecker *et al.* (1997) and Nishiwaki *et al.* (1998) posed the multicriteria objective as maximizing the ratio of *MPE* and *SE* to improve the convergence behaviour in optimization. Sigmund (1996) proposed to maximize the mechanical advantage of the mechanism with constraints on volume and input displacements while Larsen *et al.* (1996) presented the synthesis of compliant topologies with multiple input and output ports. Saxena and Ananthasuresh (1998) proposed an energy based formulation wherein the available output energy which is proportional to the square of the output displacement was used as a measure for flexibility and was maximized while minimizing the input energy.

2 Motivation

The basis for the multicriteria formulation of the compliant topology problem is the physical intuition that a compliant mechanism should meet both the flexibility and stiffness requirements. Notwithstanding the efficiency, direct implementation of mathematical programming algorithms in solving such multicriteria problems does not provide further physical insight into the nature of the optimal solution. The first-order necessary conditions for optimality reveal an interesting structural property for compliant topologies which can be generalized for a variety of multicriteria formulations. The focus in this paper is to rigorously derive and present the generalized *optimal property* for compliant topologies. This structural property is later employed to develop an efficient algorithm similar to a class of structural optimization schemes called the optimality criteria methods which are known to be very robust in handling large problems in structural optimization (Venkayya 1989; Haftka and Gürdal 1989; Rozvany 1989).

3 Problem formulation

Consider an arbitrary design domain with given loading and boundary conditions shown in Fig. 2; P_1 is the point of application of the input force F_{in}, and Δ_{out} is the expected output deformation at point P_2. Some applications may also require a compliant mechanism to resist an output force when interacting with its surroundings (e.g. a workpiece or an electrostatic force in a Micro Electro Mechanical Systems application), which in general, may not be known *a priori.* For such cases, Ananthasuresh *et al.* (1994) proposed a spring model to approximate this force with a spring of spring constant, k_s at the output port.

The output deformation can be expressed as a functional termed as *mutual potential energy* (*MPE*) (Barnett 1961; Shield and Prager 1970). A pseudo force of unit magnitude is applied at point P_2 in the direction of the desired deformation. For given design specifications, *MPE* is calculated as

$$MPE = \int_{\mathbf{\Omega}} \boldsymbol{\sigma}_d^T \varepsilon \, \mathrm{d}\mathbf{\Omega} \,, \tag{1}$$

where $\boldsymbol{\sigma}_d$ is the stress field in the continuum when only the unit dummy load is applied and $\boldsymbol{\varepsilon}$ is the strain field when only the input load is applied. The strain energy,

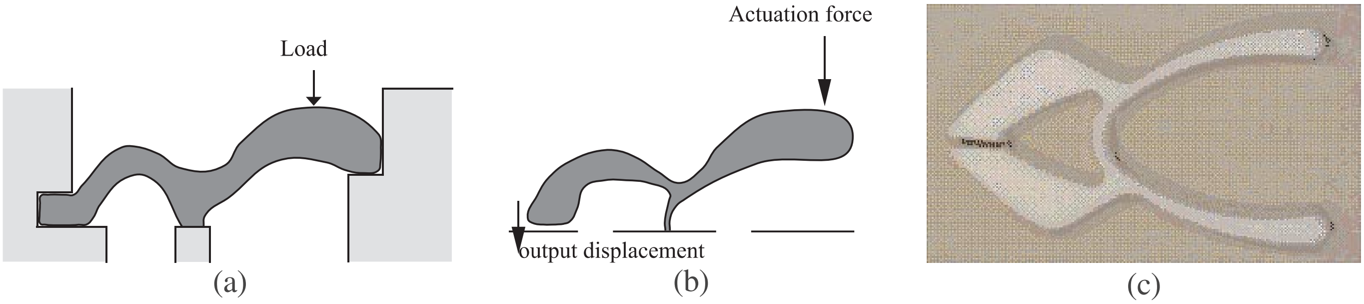


Fig. 1 (a) A structure, (b) symmetric half of the compliant mechanism shown in (c)

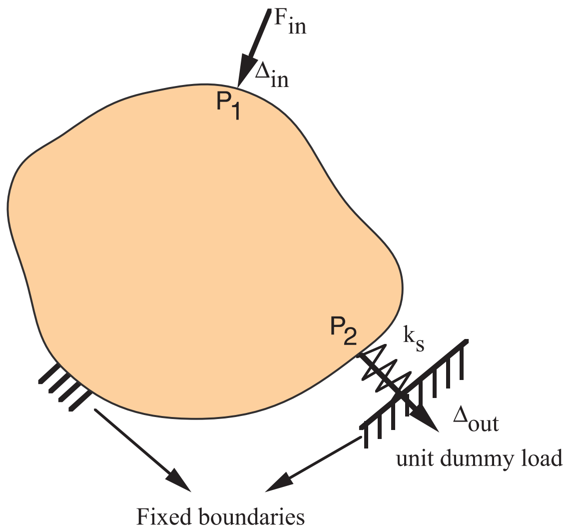


Fig. 2 Design domain and problem specifications

SE stored in the continuum can be expressed as

$$SE = \frac{1}{2} \int_{\boldsymbol{\Omega}} \boldsymbol{\sigma}^T \varepsilon \, \mathrm{d}\boldsymbol{\Omega} \,, \tag{2}$$

where $\boldsymbol{\sigma}$ is the stress field due to the input load only. To compute MPE and SE numerically, the continuum can be approximated using the finite element method. Then for given boundary conditions, the discretized displacement field, $\mathbf{U}$ due to the input force, F_{in} can be solved using

$$\mathbf{KU} = \mathbf{F} \,, \tag{3}$$

where $\mathbf{K}$ is a matrix representing the combined stiffness of the structure and the output spring, k_s, and $\mathbf{F}$ is the force vector comprising the input force. With the same boundary conditions, the displacement field, $\mathbf{V}$ due to the unit dummy load can be obtained from

$$\mathbf{KV} = \mathbf{F}_d \,, \tag{4}$$

where $\mathbf{F}_d$ is the force vector comprising the dummy load. The mutual potential energy or the output displacement, Δ_{out} can then be computed as

$$MPE = \Delta_{\text{out}} = \mathbf{V}^T \mathbf{KU} \,, \tag{5}$$

while the strain energy stored in the continuum can be written as

$$SE = \frac{1}{2} \mathbf{U}^T \mathbf{KU} \,, \tag{6}$$

Ananthasuresh *et al.* (1994) proposed to simultaneously accomplish the two requirements of maximizing MPE and minimizing SE with the multicriteria objective defined as their weighted linear combination. That is,

$$\text{minimize:} \ -w\,MPE + (1-w)SE \,, \ 0 \leq w \leq 1 \,. \quad \text{(P1)}$$

where w is a user specified control parameter that assigns relative weights to the two objectives. Quite often in this formulation, the orders of magnitudes of the two objectives may not be comparable and one of the objectives tends to dominate the other. This effect can be compensated by choosing an appropriate weighting factor which may vary from one problem to another. In general, choosing a value of the normalized weight, w in the global sense is very difficult. To overcome this limitation, Frecker *et al.* (1997) proposed an alternative multicriteria objective of maximizing the ratio of the output deformation and the mean compliance. In other words,

$$\text{minimize:} \quad -\frac{MPE}{SE} \,. \quad \text{(P2)}$$

Saxena and Ananthasuresh (1998) proposed an energy based method to accomplish the three objectives of maximizing the flexibility, stiffness and the mechanical advantage of the mechanism simultaneously. The underlining notion here is that a compliant mechanism should deliver maximum available energy at the output port while storing minimum energy within its deformed continuum. Mathematically, the energy based objective can be posed as

$$\text{minimize:} \quad -\operatorname{sign}(MPE)\frac{\frac{1}{2}k_s MPE^2}{SE} \,. \quad \text{(P3)}$$

where $\frac{1}{2}k_s MPE^2$ is the energy stored in the output spring. The term, *sign*(*MPE*) is introduced to restore the direction of the output deformation which is lost in the squared term in the numerator of (P3). In a more general setup, the flexibility requirement can be defined by a monotonically increasing function of *MPE*, i.e. *f*(*MPE*), where $\frac{\partial f(MPE)}{\partial MPE} > 0$. Similarly, an increasing function of *SE*, *g*(*SE*), where $\frac{\partial g(SE)}{\partial SE} > 0$ can be used as a measure for the structural stiffness. The multicriteria formulations comprising these measures can then be studied under two groups. The first group involves the *linear combination* type formulations wherein the objective is to maximize the linear combination of the two functions, namely,

$$\text{minimize:} - f(MPE) + g(SE)\,. \quad \text{(P4)}$$

A special case of this group is the formulation (P1) where *f*(*MPE*) and *g*(*SE*) are linear functions of *MPE* and *SE*, respectively. The second group comprises the *ratio type* formulations where it is intended to maximize the ratio of the two measures, i.e.

$$\text{minimize:} \quad -\frac{f(MPE)}{g(SE)}\,. \quad \text{(P5)}$$

Special cases of the ratio type formulations are those in (P2) and (P3) where *f*(*MPE*) is, respectively, linear and quadratic in *MPE* and *g*(*SE*) is linear in *SE*. The multicriteria formulation (P5) is used in the following section to derive the optimality property for compliant topologies which is subsequently generalized. This property is later employed for the topological synthesis of compliant mechanisms. Such a method of synthesis is very similar to optimality criteria methods which seek an optimal solution by directly solving the equations resulting from the optimality conditions.

4 Optimality property for compliant topologies

From the stationarity conditions, the function gradients must vanish at an optimum. Differentiating the objective in (P5) with respect to the design variable, x_i and equating it to zero yields,

$$\frac{\partial(MPE)/\partial x_i}{\partial(SE)/\partial x_i} = \frac{f(MPE)}{g(SE)}\frac{g'(SE)}{f'(MPE)}\,, \tag{7}$$

where $f'(MPE)$ represents $\frac{\partial f(MPE)}{\partial MPE}$ and $g'(SE)$ is $\frac{\partial g(SE)}{\partial SE}$. Using (5), $\partial(MPE)/\partial x_i$ can be expressed as

$$\frac{\partial(MPE)}{\partial x_i} = \frac{\partial \mathbf{V}^T}{\partial x_i}\mathbf{KU} + \mathbf{V}^T\frac{\partial \mathbf{KU}}{\partial x_i}\,. \tag{8}$$

Since, $\mathbf{F}$ which is also equal to $\mathbf{KU}$ in (3), is independent of the design variables, (8) can be simplified as

$$\frac{\partial(MPE)}{\partial x_i} = \frac{\partial \mathbf{V}^T}{\partial x_i}\mathbf{KU}\,. \tag{9}$$

Further, since $\mathbf{F}_d$ in (4) is independent of the design variables, differentiation of (4) with respect to x_i yields

$$\frac{\partial \mathbf{V}^T}{\partial x_i}\mathbf{K} = -\mathbf{V}^T\frac{\partial \mathbf{K}}{\partial x_i}\,. \tag{10}$$

On combining (9) and (10), we obtain

$$\frac{\partial(MPE)}{\partial x_i} = -\mathbf{V}^T\frac{\partial \mathbf{K}}{\partial x_i}\mathbf{U}\,. \tag{11}$$

Computation of $\partial\mathbf{K}/\partial x_i$ can be simplified by choosing the design variables, x_i in a manner that the stiffness matrix, $\mathbf{K}$ is linear in x_i. Since the *i*-th design variable occurs only in its corresponding element stiffness matrix, $\mathbf{k}_i$, this helps in simplifying the right-hand side of (11) to

$$\frac{\partial(MPE)}{\partial x_i} = -\mathbf{v}_i^T\frac{\partial \mathbf{k}_i}{\partial x_i}\mathbf{u}_i\,, \tag{12}$$

where $\mathbf{u}_i$ and $\mathbf{v}_i$ are the displacement vectors for the *i*-th element due to the load vectors, $\mathbf{F}$ and $\mathbf{F}_d$, respectively, and $\mathbf{k}_i$ is the stiffness matrix of that element. As $\mathbf{k}_i$ is linear in x_i, (12) can be rewritten as

$$\frac{\partial(MPE)}{\partial x_i} = -\mathbf{v}_i^T\frac{\mathbf{k}_i}{x_i}\mathbf{u}_i\,. \tag{13}$$

Following similar steps to compute $\partial(SE)/\partial(x_i)$, we obtain

$$\frac{\partial(SE)}{\partial x_i} = -\frac{1}{2}\mathbf{u}_i^T\frac{\mathbf{k}_i}{x_i}\mathbf{u}_i\,. \tag{14}$$

Equations (7), (13) and (14) can finally be combined to obtain

$$\frac{\mathbf{v}_i^T\mathbf{k}_i\mathbf{u}_i}{\frac{1}{2}\mathbf{u}_i^T\mathbf{k}_i\mathbf{u}_i} = \frac{f(MPE)}{g(SE)}\frac{g'(SE)}{f'(MPE)}\,. \tag{15}$$

Here, $\mathbf{v}_i^T\mathbf{k}_i\mathbf{u}_i$ represents the contribution of the *i*-th element to the desired output deformation, *MPE*. This can be termed as the element mutual potential energy, MPE_i of the *i*-th element. Similarly, $\frac{1}{2}\mathbf{u}_i^T\mathbf{k}_i\mathbf{u}_i$ is the contribution of the *i*-th element to the strain energy, SE of the continuum. This can be termed as the element strain energy, SE_i of that element. Equation (15) thus becomes

$$\frac{MPE_i}{SE_i} = \frac{f(MPE)}{g(SE)}\frac{g'(SE)}{f'(MPE)} = R\,, \tag{16}$$

where R is a global constant. Following identical steps for the *linear combination* type formulations in (P4), the stationarity criteria yield a similar result where R is equal to $\frac{g'(SE)}{f'(MPE)}$. On dividing both the numerator and denominator of (15) by the element volume, v_i and taking the limit as v_i tends to zero, the ratio, MPE_i/SE_i can be interpreted as the ratio of mutual potential energy density to the strain energy density at a point in the continuum. Equation (16) therefore reveals an interesting structural property for compliant topologies which can be stated as

Property: 1
For an optimal compliant topology that satisfies both flexibility and stiffness requirements, the ratio of the mutual potential energy density and the strain energy density is uniform throughout the continuum provided the continuum stiffness is linear in design variables.

The optimality property holds true for arbitrary increasing functions, $f(MPE)$ and $g(SE)$ chosen as respective measures of flexibility and stiffness. Noting that the formulations in (P4) and (P5) are general, the property is independent of the multicriteria formulations and therefore is truly a structural attribute of optimal compliant topologies. The structural property is also independent of the finite element type used in continuum discretization as long as the structural stiffness matrix, $\mathbf{K}$ is linear in the design variables. It should be noted, however, that some design variables may assume values (negative values for instance) that are not meaningful to satisfy the optimality property as they represent the physical attributes of the elements such as the cross-sectional areas of truss elements or widths of frame elements. Suitable lower and upper bounds on the variables are therefore essential to realize a practical solution. Such constraints are called *gage constraints*. The lower bound on the variables is typically a very small positive number to avoid singularity problems in the structural stiffness matrix. An upper bound can result due to limited availability of the material for topology synthesis. This requirement is different from the resource (weight/volume) constraint and merely implies the nonavailability of an infinite amount of material. Thus, a minimum solution is guaranteed in the multicriteria formulations and satisfies the following property.

Property: 2
For realistic optimal compliant topologies with linearly varying continuum stiffness, the ratio of the mutual potential energy density to the strain energy density is uniform throughout the continuum, but for portions otherwise bounded by gage constraints.

4.1 Numerical verification

Synthesis examples solved using a mathematical programming scheme are presented to verify the stationarity criterion in (16). The input-output specifications for an inverter mechanism, an example used by Sigmund (1996), are indicated in Fig. 3a. Linear frame elements are used to model the domain. Choosing the out-of-plane widths of the elements as design variables renders the stiffness matrix, $\mathbf{K}$ linear in design variables. The discretized domain is shown in Fig. 3b with element numbers. Respective widths are initialized to 0.25 cm for which the lower and upper limits are 10^{-4} cm and 0.5 cm, respectively. Young's modulus, E is taken as 2×10^6 N/cm^2 and the thickness is 0.1 cm. An input force, F_{in} of 10 N is used for mechanism actuation.

Three cases of the multicriteria formulation, (P5) are chosen to illustrate that the structural property is indeed satisfied by optimal compliant topologies. First $-\frac{MPE}{SE}$, a special case of (P5), is minimized using the Sequential Quadratic Programming (SQP) method (Matlab 1997). Figures 3c and 3d show the resultant topology and the deformed profile of an optimal structure. It can be observed by inspection that the primary deformation at the output port is along the direction specified. Table 1 compares the energy ratios of the individual elements with that of the continuum. It is observed that not all design variables satisfy the stationarity criteria but only those which remain within the specified bounds. This is an expected result from Property (2). Both the gage constraints for such variables are inactive and the corresponding Lagrange multipliers are zero. For the design variables reaching their upper or lower bounds, the energy ratio criterion is not satisfied since the corresponding gage constraints are active for which the Lagrange multipliers are positive as expected from the first-order necessary conditions. The structural property is also verified for the second and third functions $-sign(MPE)\frac{MPE^2}{SE}$ and $-\frac{MPE}{e^{SE}}$ as special cases of (P5) for which similar trends are observed. Figures 3e and 3f show the optimal topologies and the deformed profiles for the second multicriteria function while Figs. 3g and 3h show the same for the third case. Tables 2 and 3 exhibit the energy ratios for the elements for the functions $-sign(MPE)\frac{MPE^2}{SE}$ and $-\frac{MPE}{e^{SE}}$, respectively.

Table 1 The energy ratios of the elements satisfying the optimality property for $-\frac{MPE}{SE}$ (Fig. 3c)

Element number	x_i	$\frac{MPE_i}{SE_i}$	$\mathbf{R} = \frac{MPE}{SE}$
1	0.01441	0.5842	0.5842
4	0.01981	0.5842	0.5842
24	0.16329	0.5842	0.5842
26	0.40026	0.5842	0.5842

5 Optimality criteria approach

Optimality criteria methods find the optimal solution by directly solving the equations resulting from the optimal conditions. These conditions can be intuitive stipulations; for instance the *Fully Stressed Design* technique in traditional structural optimization wherein it is intended to remove the material from the members which are not fully stressed unless prevented by the minimum size con-

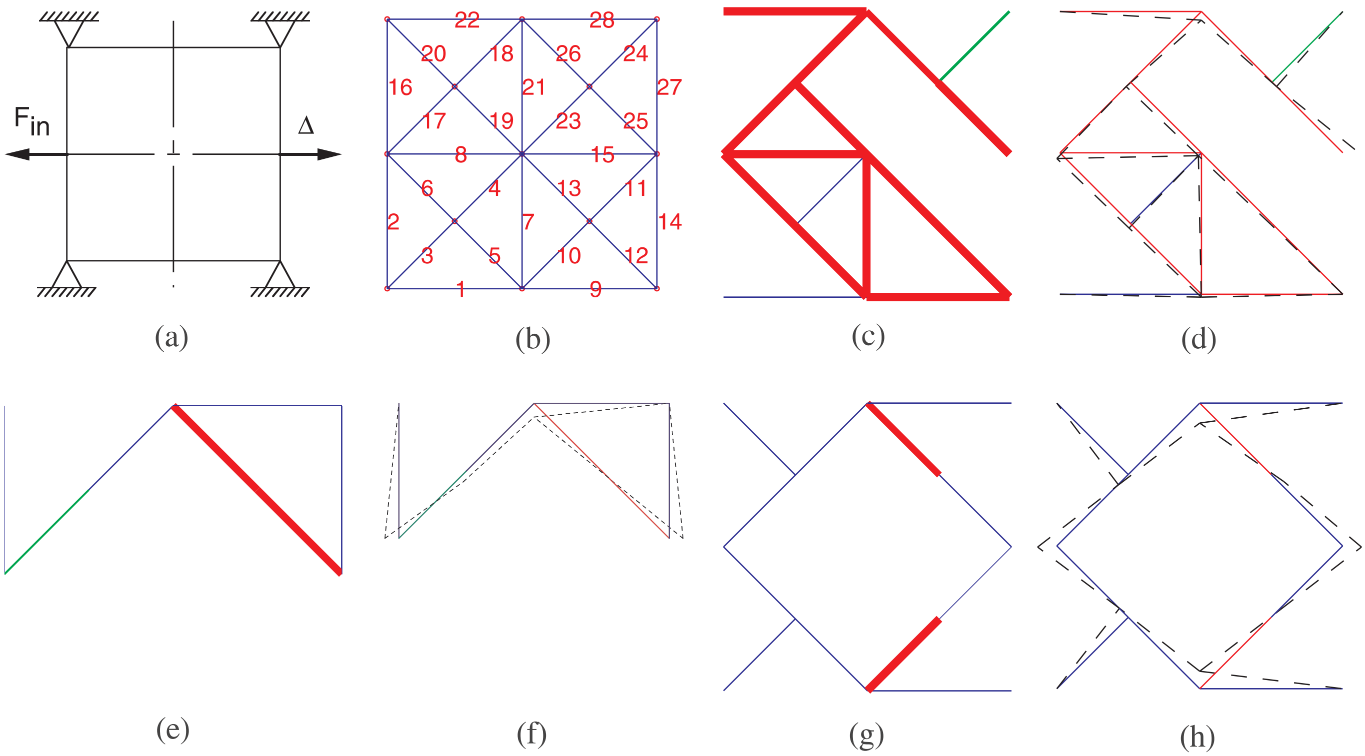


Fig. 3 A synthesis example of the displacement inverter using sequential quadratic programming. (a) Design domain for compliant inverter, (b) a frame elements mesh, (c)–(d) optimal topology of displacement inverter for $-\frac{MPE}{SE}$ and the deformed profile, (e)–(f) optimal topology and deformed profile for $-sign(MPE)\frac{MPE^2}{SE}$, (g)–(h) optimal topology and deformed configuration for $-\frac{MPE}{e^{SE}}$

Table 2 The energy ratios of the elements satisfying the structural property for $-sign(MPE)\frac{MPE^2}{SE}$ (Fig. 3e)

Element number	x_i	$\frac{MPE_i}{SE_i}$	$\mathbf{R} = \frac{MPE}{2SE}$
9	0.00080	0.08896	0.08896
16	0.01299	0.08896	0.08896
17	0.1285	0.08896	0.08896
18	0.00763	0.08896	0.08896
24	0.00081	0.08896	0.08896
27	0.00903	0.08896	0.08896
28	0.02052	0.08896	0.08896

Table 3 The energy ratios of the elements satisfying the optimality criteria for $-\frac{MPE}{e^{SE}}$ (Fig. 3g)

Element number	x_i	$\frac{MPE_i}{SE_i}$	$\mathbf{R} = MPE$
1	0.03645	0.16786	0.16785
6	0.03109	0.16786	0.16785
10	0.00644	0.16785	0.16785
11	0.01033	0.16785	0.16785
12	0.00057	0.16786	0.16785
17	0.03099	0.16785	0.16785
22	0.03645	0.16786	0.16785
24	0.00057	0.16785	0.16785
25	0.01033	0.16785	0.16785
26	0.00644	0.16785	0.16785

straint. Optimality conditions can also be derived using the stationarity conditions. An example is that of obtaining a stiff structure with minimum weight using the criteria that the *strain energy density* is uniform throughout the structure. This condition can be derived for an objective of minimizing the strain energy with the volume constraint.

Since their inception (Prager and Taylor 1968; Venkayya *et al.* 1968), optimality criteria methods have been well-developed in literature and found to be very successful in optimizing structures for minimum weight and/or mean compliance. A critical gap, however, has been the absence of appropriate existence and convergence theorems for the solution sought in discrete systems. This is because both *convexity* of the design space and *uniqueness* issues are difficult to address analytically for the corresponding continuum models (Strang and Kohn 1982). For the multicriteria formulations, the design space is not always convex, i.e. there may exist more than one stationary points, and the objective can yield more than one minima for the same design specifications if searched from different initial guesses. In other words, the optimal solution may not be necessarily unique. Two examples are presented below to illustrate these issues related to the multicriteria formulations.

Consider a two-variable numerical example with beam elements shown in Fig. 4a. A downward vertical force, F of $1N$ is applied at node 3 and it is desired to maximize the deflection at node 2 along the downward vertical direction. The design variables, x_1 and x_2 are the respective widths of the elements for which the stiffness matrix, $\mathbf{K}$ is linear in design variables. The thickness and lengths of the elements are 0.1 cm and 1 cm, respectively, and the Young's modulus is 2×10^6 N/cm^2. Using (5) and (6), analytical expressions for the displacement at node 2 and the strain energy can be computed as

$$MPE = 0.001 \frac{9x_1 + 2x_2}{x_1 x_2} ,$$

$$SE = 0.001 \frac{7x_1 + x_2}{x_1 x_2} . \quad (17)$$

The lower and upper bounds of 10^{-4} cm and 0.1 cm are imposed on the design variables. The problem of maximizing the displacement at a desired point and minimizing the strain energy can be considered as equivalent to minimizing the mean compliance in the presence of a displacement constraint. The displacement constraint in this example is the equality constraint with a value of 0.5. The contours of the strain energy are shown with dotted curves in Fig. 4b and the displacement constraint (MPE = 0.5) is shown with the solid curve. It is evident from the figure that there is no unconstrained extremum for the objective stated above. Further, using the first-order necessary conditions, it can be shown that there exist two stationary points, A and B. At A, the strain energy is maximized to a value of 0.388 for x_1 as 0.100 and x_2 as 0.018, while at B, the strain energy is minimized to a value of 0.275 for values of x_1 and x_2 as 0.005 and 0.100, respectively.

The example for the displacement inverter solved before to verify the structural property is used here to demonstrate nonuniqueness in optimal solutions. Using the formulation in (P2), a special case of (P5), for the same design specifications in Figs. 3a, Fig. 5a shows one of the resultant optimal topologies for which the output displacement at optimum is 0.09 cm and the strain energy is 0.046 N-cm. The objective function is minimized to a value of −1.95. With the same initial specifications and a different initial guess for the design variables (this guess is generated using random numbers in the range of 10^{-4} and 0.5), a second optimal solution is obtained shown in Fig. 5c. For this case, at optimum, the output displacement is 0.05 cm and the strain energy is 0.009 N-cm. The value of the objective for this topology is −5.37. Both the deformed configurations (dashed lines) in Figs. 5b and d of the two topologies, respectively, satisfy the force-deflection specifications in Fig. 3a.

5.1 Resizing algorithm

The resizing approach in the optimality criteria methods differ from those in the mathematical programming schemes; the former uses sweeping techniques while the latter uses point to point search in the design space (Venkayya 1989). Based on the optimality criterion in (16), a resizing scheme can be developed for the synthesis of compliant mechanisms. Multiplying both sides of this equation by x_i, the resulting expression can be written in an iterative form as

$$x_i^{k+1} = \frac{1}{R} \frac{MPE_i}{SE_i} \quad x_i^k , \quad (18)$$

where x_i^k represents the k-th iteration value of the i-th design variable and x_i^{k+1} is the updated (k+1)-th iteration value for the same. Due to the nonconvexity and nonuniqueness behaviour of the multicriteria formulation seen in previous examples, using this updating scheme alone may not necessarily lead the design variables towards a minimum. To guarantee convergence, a tradeoff can be made by combining the *efficiency* of optimality criteria methods and *reliability* of mathematical programming schemes. This can be accomplished in two steps. The first step is to formulate a search vector as the difference between the updated and the old design variables and subsequently normalizing it. The second step is then to perform a robust one dimensional search along the normalized search vector. An efficient line-search is needed to minimize the number of function evaluations since each evaluation requires a finite element analysis. For a normalized search vector, $\mathbf{S}$ and the initial set of design variables, $\mathbf{X}_o$, any other set, $\mathbf{X}$ can be written as

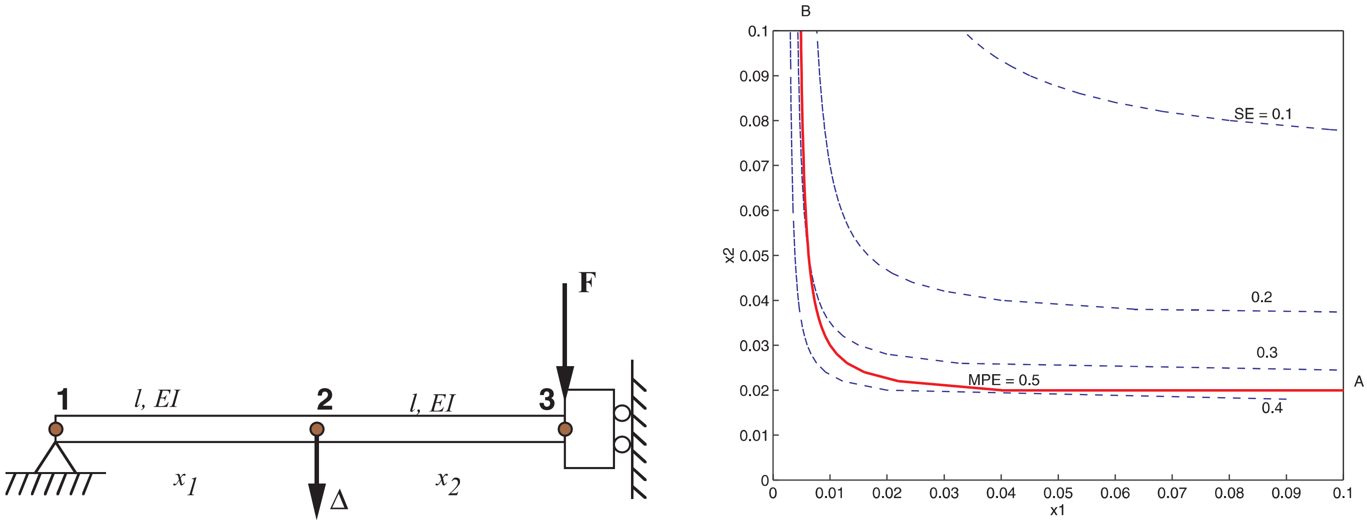
$$\mathbf{X} = \mathbf{X}_o + \alpha \mathbf{S} , \quad (19)$$

where

$$s_i = \frac{\left(x_i^{k+1} - x_i^k\right)}{\max\left(x_i^{k+1} - x_i^k\right)} ,$$

and α is a scalar variable to be determined. Equation (19) allows the design variables and thus the objective function to be expressed in terms of a single variable, α. Minimization of the function then requires determining an appropriate value of α, i.e. α^* within a specified interval.

Among the many one-variable optimization methods available, Brent's algorithm (Press *et al.* 1991) is chosen here for its robustness in implementation and the elimination of the need for derivative calculations. The algorithm performs combined parabolic and Golden Section search (Rao 1984) to determine a minimum within a prespecified interval. When implementing this search with the resizing scheme in (18), the design variables are ensured to remain within their specified bounds in each step. An issue of concern here is that the size of the interval in the scalar variable, α is not known *a priori*. Selecting a large interval size at the outset is not appropriate as the function may not be unimodal, i.e. it may have more than one minima in the interval. Al-



(a) (b)

Fig. 4 (a) The beam example, (b) design space for the beam example

ternatively, selective line searches are employed with checks on the monotonicity of the function. That is, a line search is initially commenced by choosing a small interval size. If the function is monotonic, α^* assumes a value of either the lower or upper limit of the interval. In such a case, the subsequent interval is expanded and minimization is continued. Otherwise, the line search is stopped. Variables obtained after the search are updated again using (18) and the process is iterated until convergence is achieved. A detailed schematic of the algorithm is provided in Fig. 6. The proposed method combines both the efficiency of the optimality criteria method and reliability of mathematical search schemes which requires only the function evaluations. Using a robust line search algorithm with selective interval search further enhances the performance of the method resulting in the reduction of the number of function evaluations.

5.2 Global control on structural compliance and stiffness

The multicriteria formulations previously mentioned allow only an indirect control over the structural flexibility and stiffness which is accomplished when specifying the increasing functions *f*(*MPE*) and *g*(*SE*) as measures of flexibility and stiffness, respectively. Direct control on the flexibility (and stiffness) of the extracted optimal topology is still possible by varying the design variables on a global scale. Since the structural stiffness matrix, **K** is linear in design variables, increasing the latter by a factor, γ ($\gamma > 1$) increases the stiffness to $\gamma\,\mathbf{K}$. From (3)–(6), the altered values of mutual potential and strain energies (MPE_n and SE_n, respectively) can be computed as

$$MPE_n = \frac{1}{\gamma} MPE_o\,, \quad SE_n = \frac{1}{\gamma} SE_o\,, \tag{20}$$

where MPE_o and SE_o are the (old) optimal values. Values of factor, γ smaller than unity results in an increase in flexibility and decrease in stiffness. Likewise, the optimal topology can be made stiffer (and less compliant) for γ greater than one. The parameter γ may or may not alter the resulting values of the multicriteria objectives. Depending on whether the value of the objective for (P1) at optimum is positive or negative, decreasing γ will be respectively detrimental or favourable. However, a decrease in γ does not alter the optimum objective value in (P2) while the latter in (P3) is further minimized. Hence, the increasing functions corresponding to the measures of flexibility and stiffness, and the multicriteria formulation should be chosen appropriately to account for this behaviour. Furthermore, γ can be weakly associated with the stress levels in the continuum. For instance, if more compliance is desired, γ can be decreased to a value for which a region of the extracted optimal topology is stressed to the maximum permissible limit.

6 Design parameterization and implementation

The optimality criteria approach is implemented using two-dimensional linear frame elements and four noded bilinear plane stress elements for domain discretization. In both cases, linear finite element analysis is employed for function evaluation. This section describes the design parameterization using the two element types. Some numerical difficulties in the finite element discretization, namely checkerboard patterns and mesh dependencies

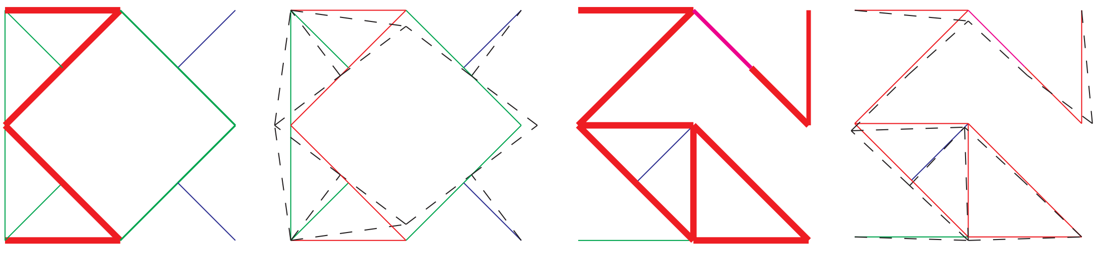


Fig. 5 A synthesis example of a displacement inverter using sequential quadratic programming. (a) Optimal topology I of displacement inverter, (b) deformed profile, (c) optimal topology II of displacement inverter obtained using asymmetric initial guess, (d) deformed profile

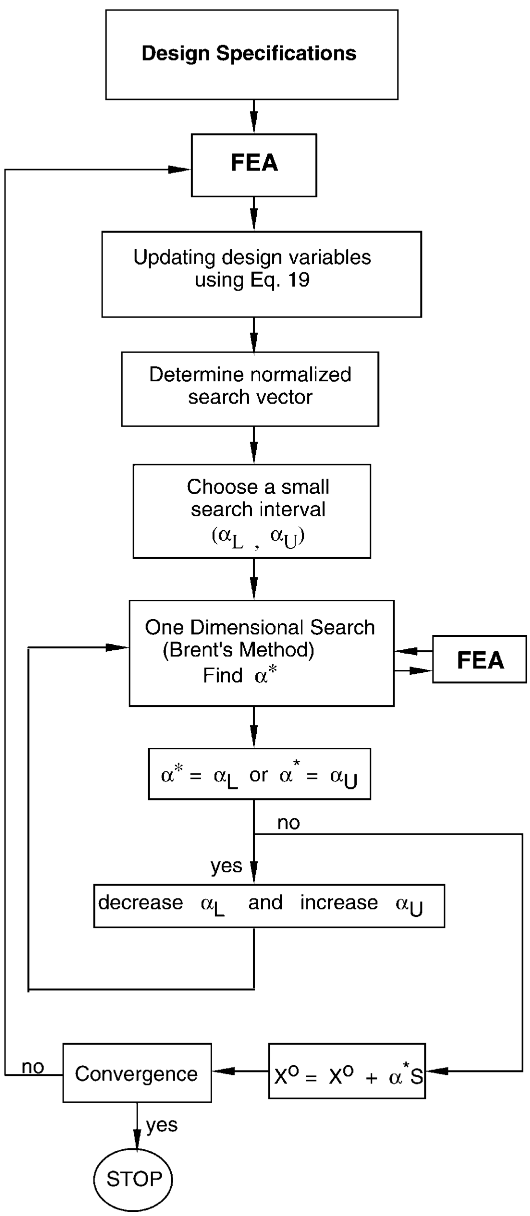


Fig. 6 Schematic of the optimality criteria approach

are also addressed in the context of continuum modeling with four noded bilinear plane stress elements.

6.1 Frame elements

Frame elements have the capability to deform both in the longitudinal and transverse directions and therefore can incorporate bending modes. Since it is required that the element stiffness should be linear in the corresponding design variable, the respective widths of the elements are chosen as the design variables. A full ground structure is recommended for the topology synthesis using truss, beam or frame elements. A ground structure is a set of elements in a grid of points where each point is connected to every other point. A full frame ground structure is impractical to use as it involves very long and overlapping elements in the continuum. Instead, the ground structure shown in Fig. 7a is used.

6.2 Bilinear quadrilateral plane stress elements

Following the idea of the standard topology optimization procedures, the material *density*, x_i in each element is considered to be the design variable (Bendsøe 1995). This conforms with the requirement of the domain stiffness to be modelled as a linear function of the design variables. The material densities are allowed to acquire any value between x_ℓ and 1 where x_ℓ is a small positive quantity; x_ℓ is not allowed to be zero due to the numerical problems associated with the finite element stiffness matrix. The element for which the density value, x_i reaches a minimum is regarded as absent from the topology. Similarly, a value of 1 for x_i represents a *solid* element. Material properties of the continuum can be modelled by relating the Young's modulus, E of the element with its density in the following manner:

$$E = x_i\, E^o\,, \tag{21}$$

where E^o is the Young's modulus of the solid material. Some of the most frequently encountered difficulties in topology optimization with four noded bilinear finite elements are the checkerboard patterns and the mesh dependency problems. A checkerboard patch is typically a pattern of alternating solid and void elements in an optimal topology which corresponds to a region of artificially high (numerical) stiffness (Jog and Haber 1996; Díaz and Sigmund 1995). The appearance of such patterns is common in topology optimization and can be attributed to poor numerical modelling and the use of lower order finite elements.

Another difficulty is the problem of mesh dependency or the nonexistence of solutions wherein the optimal solutions do not converge with mesh refinement. Mesh refinement should ideally yield the same topology as for a coarse mesh but with better boundary definitions. This problem can be circumvented using a computationally efficient, yet simple, method proposed by Sigmund (1996). Implementation of the heuristic approach called the *filtering scheme* does not allow rapid variation of the density gradients in the design space.

Such a filtering scheme is used to prevent rapid variation in the mutual potential energies, MPE_i and strain energies, SE_i of the elements in the domain. For an element, i, neighboring elements are first identified within a circular region of radius, $r_{\min}$ about the considered element. The convolution operator is then written as (Sigmund 1996)

$$H^j = r_{\min} - \mathrm{dist}(i,j)\,,\ \{i \,\varepsilon\, N \mid \mathrm{dist}(i,j) \le r_{\min}\}\,,$$
$$i = 1, \dots, N\,, \tag{22}$$

where $\mathrm{dist}(i,j)$ is defined as the distance between the center of the neighbouring element, j and the considered element, i. The filter factor, H^j for the element, j decays linearly with its distance from the element, i. The smoothened value of MPE_i is obtained as

$$MPE_i^f = \frac{1}{x_i \sum_{k=1}^{N} (H^j)} \sum_{k=1}^{N} H^j MPE_j\,(x_j)\,, \tag{23}$$

where MPE_i^f are the filtered values of the mutual potential energies of the elements and $MPE_j, j = 1, \dots, N$ are the true values. Filtered values of strain energies, SE_i^f are computed in a similar manner.

7 Synthesis examples

Numerous synthesis examples using the multicriteria formulations previously mentioned are presented both for macro and micro scale applications using the proposed resizing scheme. The first example is of a pliers mechanism the design specifications of which are given in Fig. 7a. Figure 7b shows the optimal topology of the mechanism synthesized using the formulation in (P2). The variation in the element sizes shown in the figure actually represents their relative out-of-plane widths. The convergence history is shown in Fig. 7c wherein it can be noticed that the output displacement is not maximized individually. After reaching a maximum, there is a drop in *MPE* accompanied by a steep fall in the value of the strain energy. At convergence, the output deformation is 0.016 cm and the strain energy stored is 0.0053 N-cm for which the objective function has an optimal value of −3. Even though the objective function is minimized, the stiffness requirement in this example dominates over the flexibility requirement and as a result, the mechanism is relatively stiff. However, the output port does move along the desired direction, thus fulfilling the functional requirement. The deformed configuration of the mechanism shown in Fig. 7d is obtained using plane stress finite element analysis in ABAQUS (Hibbit, Karlsson & Sorensen Inc. 1986) for an input force of 10 N. For manufacturing reasons, the minimum width of the elements significant to the topology is chosen as 0.2 cm. The output deformation along the prescribed direction is 0.08 cm while that along the direction perpendicular to it is 0.02 cm which suggests that the output port moves primarily along the direction specified. The final fabricated model of the pliers mechanism is shown in Fig. 7e.

The example of compliant pliers is also solved using the energy based approach in formulation (P3). Figure 8a shows the optimal topology of the mechanism and the convergence history is shown in Fig. 8b. The values of *MPE* and *SE* at convergence are 0.0393 N-cm and 0.014 N-cm, respectively, for which the function attains a minimum of −0.11. Comparison of the values of *MPE* and *SE* with those in the previous example from the frame finite element model and the fact that the output deformation reaches a maximum in this case, both suggest that the structure in Fig. 8a is more flexible than the one in Fig. 7b. This is because the flexibility requirement has more weightage than the stiffness requirement in the energy based multicriteria objective. Deformation results of the finite element analysis of the structure with plane stress elements shown in Fig. 8c conform with the design specifications.

The second example is the synthesis of a compliant gripper using four noded bilinear plane stress elements. The symmetric half of the design domain is shown in Fig. 9a. The input and output specifications along with boundary conditions are shown in the figure with the input force, F_{in} of 10 N. A finite element mesh of 35×35 is used in this example. Mesh densities of the elements are considered as the design variables. Optimal topology of the half section of the gripper mechanism obtained using formulation (P2) is shown in Fig. 9b and the mechanism is shown in Fig. 9c. At convergence, an output deformation of 1.55 cm is obtained while the strain energy stored is 66.35 N-cm. The multicriteria function is minimized to −0.0235. The grippers example is also solved using the energy based formulation (P3) for which the full section topology is shown in Fig. 9d. Convergence in this

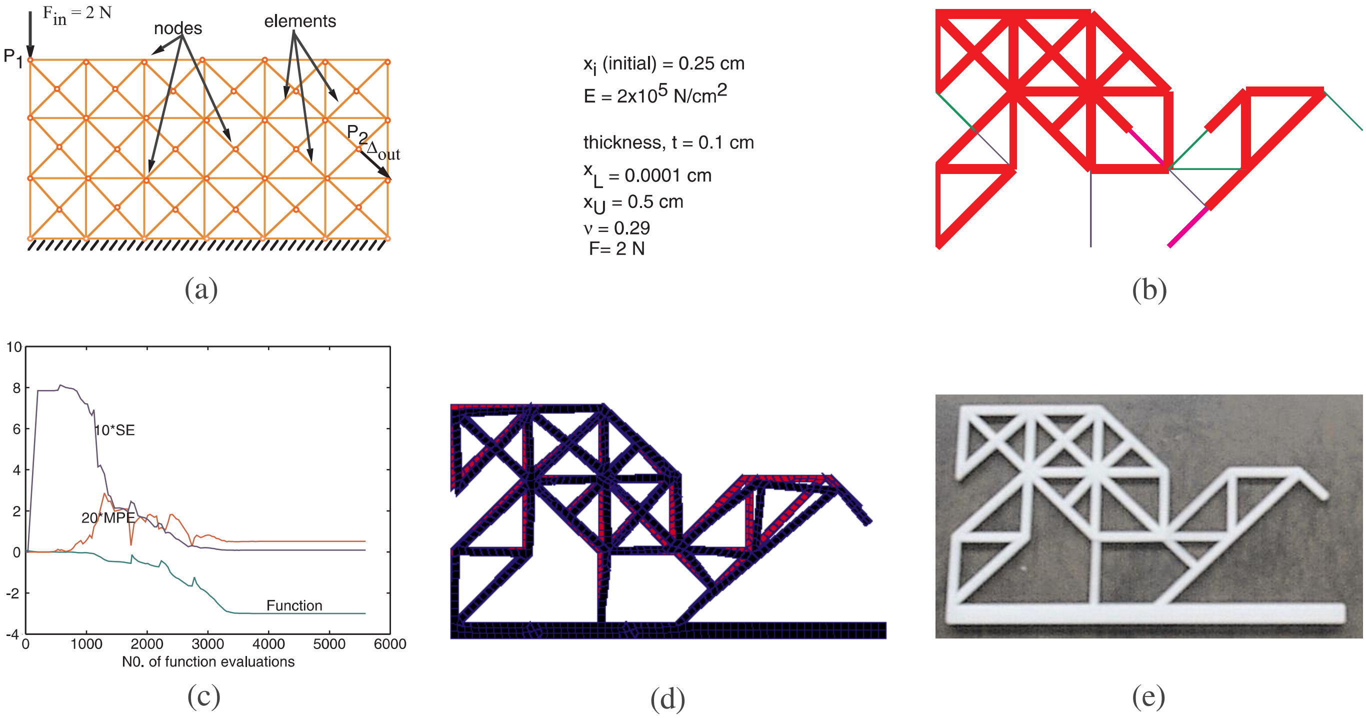


Fig. 7 Synthesis example of a compliant pliers using formulation (P2). (a) Problem specification on the ground structure, (b) optimal topology, (c) convergence history, (d) finite element analysis with plane stress elements, (e) realized modified mechanism manufactured using ABS plastic

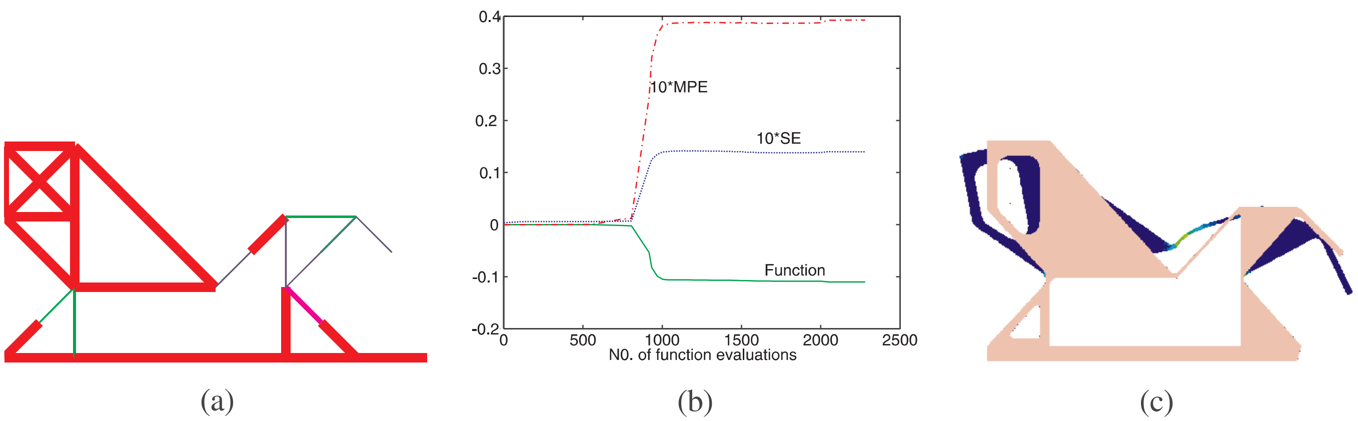


Fig. 8 Synthesis of a compliant pliers using the energy based objective. (a) Optimal topology, (b) convergence history, (c) finite element analysis with plane stress elements

case is obtained with optimal values of *MPE* and *SE* as 2 N-cm and 115.27 N-cm, respectively. The optimal value of the energy based multicriteria objective is obtained as −0.034.

As an application in the micro-scale for Micro Electro Mechanical Systems (MEMS), the synthesis example of a micro-compliant amplifier is presented next. A micro-compliant amplifier finds an application in micro-accelerometers. The intent here is to enhance the sensitivity of the device by amplifying the capacitively measured output displacement. This requires the mechanism to possess a large geometric advantage in addition to being compliant. The rectangular design domain of dimensions $200 \times 100\,\mu m^2$ with the required specifications is shown in Fig. 10a. The domain is discretized into linear frame elements with the input force and the direction of the intended deformation as specified. The input force of 500 μN is applied to simulate the acceleration to be sensed by the device.

The proposed resizing algorithm with formulation (P2) yields an optimal topology shown in Fig. 10b wherein the in-plane thicknesses represent the relative out-of-plane widths of the elements. The convergence history is shown in Fig. 10c. At optimum, the output displacement is 0.29 μm and the strain energy is 0.1 N-μm. The multicriteria objective in (P2) is minimized to −2.79 requiring about 900 function evaluations. The geometric advantage of the resulting mechanism is about 7. A macro-

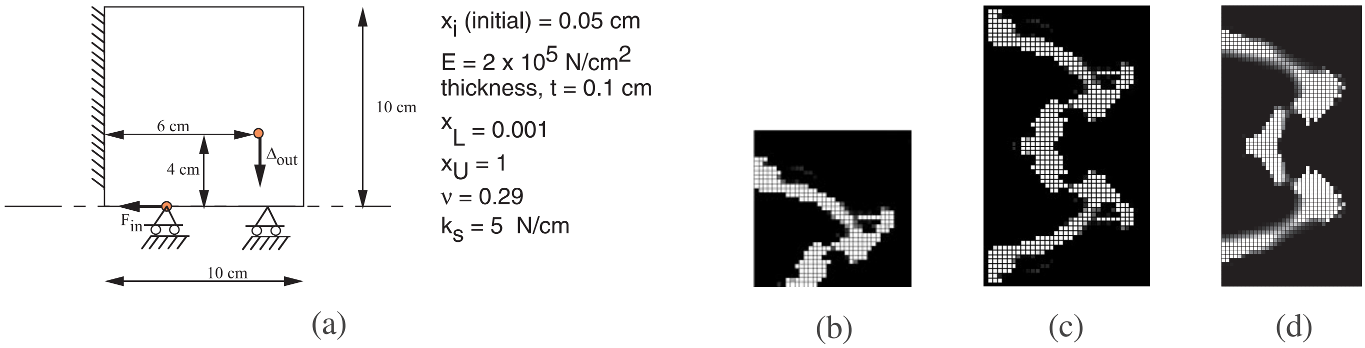


Fig. 9 Synthesis of compliant grippers using bilinear plane stress elements. (a) Design domain for the symmetric half, (b) optimal topology for the symmetric half of compliant grippers using formulation (P2), (c) optimal topology for compliant grippers, (d) optimal topology for grippers using the energy based approach of formulation (P3)

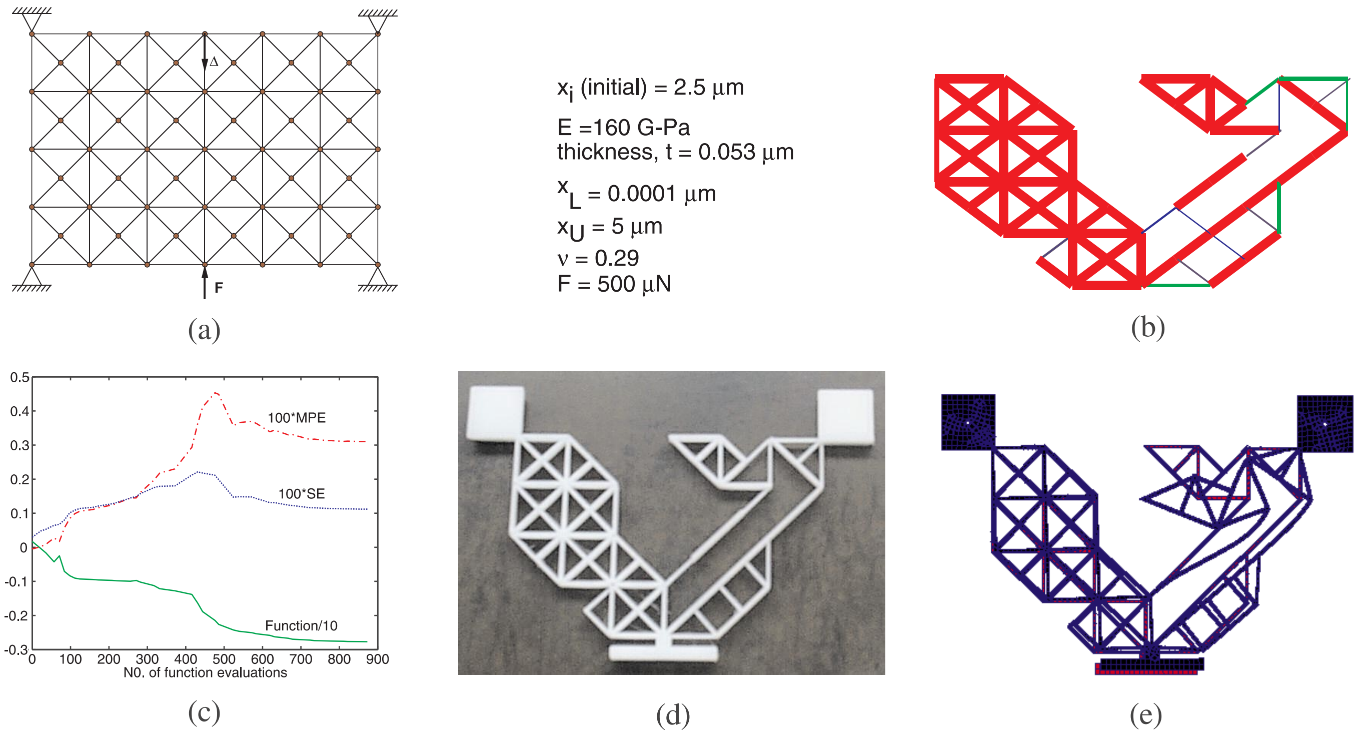


Fig. 10 Synthesis example of micro compliant amplifier. (a) Problem specification on the ground structure, (b) optimal topology (obtained using asymmetric initial guess), (c) convergence history, (d) realized design with ABS plastic, (e) finite element analysis with plane stress elements

prototype for the displacement amplifier of length 12 cm and width 6 cm is manufactured using the ABS plastic (Young's modulus = 2.1 GPa) which is shown in Fig. 10d. For manufacturing reasons, the out-of-plane widths of the significant elements are scaled between 0.2 cm and 0.5 cm, respectively. Linear plane stress finite element analysis is performed using ABAQUS (Hibbit, Karlsson & Sorensen Inc. 1986) for the input force of 2 N (Fig. 10e). The output displacement along the specified direction is 1.15 cm while that along the transverse direction is 0.04 cm demonstrating that the output port moves primarily along the direction specified. The input displacement is 0.1 cm and thus the geometric advantage obtained for the continuum model is about 10. It should be noted that the inherent compliant behaviour of the mechanism is valid for macro and micro scales.

Another example in microscale application is that of a noncontact micro compliant **AND** logic gate. The design specifications require two forces, F_1 and F_2 shown in Fig. 11a together to produce the intended output deformation. Here, mechanical forces are used as signals to operate the **AND** gate. A structure such as this one can be micro-fabricated in silicon. The energy based formulation (P3) is employed for the synthesis. The resultant topology for the **AND** logic gate is shown in Fig. 11b and the deformed configurations of the continuum model are shown in Figs. 11c and d, which explain the working principle of the mechanism. With no actuation, no

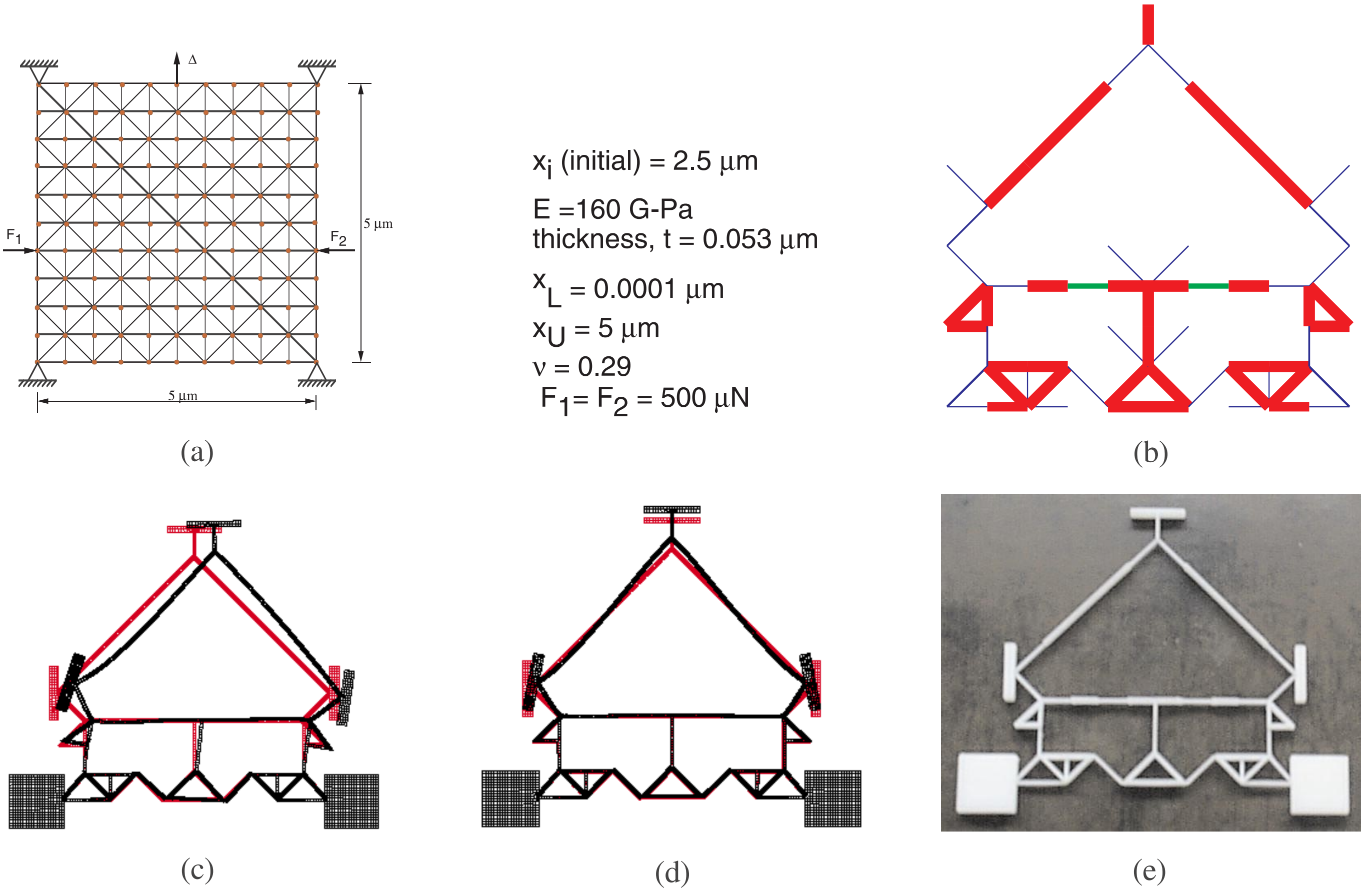


Fig. 11 Synthesis example of a micro **AND** logic gate. (a) Design specifications, (b) optimal topology of the AND gate, (c) deformed configuration with a single load, (d) deformed configuration with both loads, (e) prototype of the AND gate fabricated with the ABS plastic

displacement is registered at the output port. This can be treated as the $\mathbf{0}+\mathbf{0}=\mathbf{0}$ case in the gate. When only one of the forces, F_1 or F_2 is used for actuation (Fig. 11c represents both cases since the design is symmetric), the deformation at the output port is not significant in the vertical upward direction. This can be considered as the $\mathbf{1}+\mathbf{0}=\mathbf{0}$ case. It is only when both forces are acting on the continuum (Fig. 11d) that the change in the output capacitance is positive and detectable ($\mathbf{1}+\mathbf{1}=\mathbf{1}$ case).

8 Conclusions

The multicriteria formulations for the topology synthesis of compliant mechanisms are based on the intuitive notion that these mechanisms should simultaneously satisfy flexibility and stiffness requirements. However, no further physical insight is made available when implementing a mathematical programming search technique in seeking an optimal compliant topology. In this paper, previously reported multicriteria formulations are broadened, unified and categorized into two groups. A physically insightful property for optimal compliant topologies is rigorously derived for these formulations using the first-order necessary condition. It is later shown using numerical examples that the multicriteria formulations are non-convex and can yield multiple optimal solutions for the desired objective. Based on this observation, an efficient and reliable synthesis algorithm is developed for compliant mechanisms which iteratively employs the optimality property in conjunction with a robust one-variable search technique. The synthesis algorithm is implemented in numerous examples with the frame finite element ground structure that appropriately accounts for the bending behaviour in the continuum. This, and the variable density plane stress bilinear elements are employed to solve many examples to illustrate the efficacy of the synthesis method.

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