B. A. Salamon
Dow Chemical Company
Midland, MI 48667

A. Midha
Department of Mechanical & Aerospace Engineering & Engineering Mechanics
University of Missouri-Rolla
Rolla, MO 65409

An Introduction to Mechanical Advantage in Compliant Mechanisms

An energy approach is utilized to determine mechanical advantage in compliant mechanisms by duly accounting for lost work due to deformation. Three mechanical advantage types are then defined which examine the isolated influences of various parameters. Finally, a case study is investigated to exemplify these definitions and demonstrate resulting trends in mechanical advantage.

Introduction

The mechanical advantage of single-input and single-output port, rigid-link mechanisms is well understood and readily evaluated. There are numerous references, e.g., Shigley and Uicker (1980) and Erdman and Sandor (1991), which discuss the mechanical advantage of conventional single-input and single-output port mechanisms. Midha et al. (1984) presented a discussion of mechanical advantage concepts for a more general case of single-input and multiple-output port, rigid-link mechanisms. More recently, Howell and Midha (1995) considered the effects of a compliant workpiece on the input and output characteristics of rigid-link toggle mechanisms. A more recent treatise on compliant mechanisms may be found in Howell (1993).

In general, for rigid-link mechanisms, e.g., the slider-crank mechanism shown in Fig. 1, the links are assumed to be infinitely rigid, and if friction and inertia forces are neglected, work (or power) will be conserved between the input and output ports. The mechanical advantage of rigid-link mechanisms can be shown to be a function of the geometry of the given position of the mechanism. For example, using the instant center method, the mechanical advantage (MA) of the mechanism in Fig. 1 is given as

$$\mathrm{MA} = \frac{I_{14}I_{34}}{I_{13}I_{34}}\frac{d_i}{d_o} \tag{1}$$

where I_{ij} is the instant center of rotation of link j about link i, and d_i and d_o are the perpendicular distances to the input and output forces (F_i and F_o) from the instant centers I_{13} and I_{14}, respectively. For this single-degree-of-freedom mechanism, it is then simple to plot the variation in mechanical advantage with position.

Generalized Mechanical Advantage

In the case of compliant mechanisms, due to member compliance, energy is absorbed with deformation, and thus may not be assumed to be conserved between the input and output ports. Not only does member deformation lessen the available energy at the output, it also affects the kinematics by varying effective link lengths. The dependence of mobility on applied forces and their locations is discussed by Her (1986). Considering all these factors, to quantify mechanical advantage in compliant mechanisms is a rather complex procedure. Using the energy method then, general relations for mechanical advantage of single-input and single-output port mechanisms are developed.

Contributed by the Design Automation Committee for publication in the JOURNAL OF MECHANICAL DESIGN. Manuscript received Mar. 1998. Revised Apr. 1998. Associate Technical Editor: David A. Hoeltzel.

For any structural system, the total energy (Π) of the system in any given state can be expressed by the following relationship:

$$\Pi = U + V \tag{2}$$

where U is the strain energy of the system and V the potential energy with respect to the zero potential energy reference. The potential energy of the system is also equal to the negative of the work (W) done on the system by the external forces. Thus,

$$V = -W \tag{3}$$

For the system to be in equilibrium, the energy function must assume a stationary value. This occurs when

$$\delta\Pi = 0 \tag{4}$$

Using Eqs. (2), (3), and (4) yields the following expression

$$0 = \delta U - \delta W \tag{5}$$

Equation (5) states that the differential change in work δW is equal to the differential change in the strain energy δU. This equation holds for any incremental change in the system from one equilibrium condition to another nearby equilibrium condition. Equation (5) is general, and is applicable to any structural system, including compliant and rigid-body mechanisms. For the degenerate case of a rigid-body mechanism, the differential strain energy is assumed to be zero, and thus the differential external work is conserved. As stated earlier, this is not true for compliant mechanisms.

The general force-deflection characteristics of a compliant mechanism over its total range of operation are nonlinear. For an incremental change in position, however, the mechanism force-deflection behavior may be approximated as linear. If then, for a given state of the mechanism, the input force F_i is increased by an amount δF_i, the output force F_o will increase by an amount δF_o. Assuming that these incremental changes in the forces occur linearly with respect to the corresponding displacements, the incremental work at the input and output ports, δW_i and δW_o, respectively, are given as

$$\delta W_i = (F_i + \tfrac{1}{2}\delta F_i)\delta d_i$$
$$\delta W_o = (F_o + \tfrac{1}{2}\delta F_o)\delta d_o \tag{6}$$

where δd_i and δd_o are incremental displacements of the input and output ports in the directions of the input and output forces, respectively. Neglecting the higher-order terms in Eq. (6) gives

$$\delta W_i = F_i\delta d_i$$
$$\delta W_o = F_o\delta d_o \tag{7}$$

For a single-input and single-output port compliant mechanism, the externally applied input force and a reactive output force are assumed to be the only forces that do work on the system. All other forces are assumed to be reaction forces which correspond to displacement boundary conditions. The differential external work δW done on the system is then given as

$$\delta W = \delta W_i - \delta W_o \tag{8}$$

The minus sign associated with the differential work at the output (δW_o) indicates that the mechanism is doing work on a workpiece, or the workpiece is doing negative work on the mechanism.

Strain energy is usually written in one of two forms. For a system with a finite number of discrete compliances, strain energy (U) takes the form

$$U = \sum \tfrac{1}{2} k x^2 \tag{9}$$

where k represents the stiffness, or the reciprocal of the value of the discrete compliance, and x the amount of deformation associated with the given compliance. Equation (9) cannot describe the strain energy for a compliant mechanism, however, since the corresponding compliance is distributed rather than discrete.

It is possible, when the compliance distribution is known, to represent the strain energy as an integral of the distributed strain energy of the internal forces over the entire continuum. However, to do so requires the equilibrium geometry of the continuum to be known. For the case of small deflections, the final equilibrium geometry is approximated by the original undeformed geometry. For a compliant mechanism which may experience large deflections, the final equilibrium geometry is not known, and thus this method of denoting strain energy lacks applicability.

In general, the strain energy can be considered as the summation of the individual strain energies of a finite number of segments which idealize the continuum of the compliant mechanism. Incremental strain energy may also be represented as

$$\delta U = \sum_{i=1}^{N} \delta U_i \tag{10}$$

where δU_i is the incremental change in the strain energy of the i^{th} segment, and N the total number of segments representing the mechanism. Each δU_i can be considered as resulting from either a discrete or distributed compliance.

Combining the results in Eqs. (5), (7) and (8) gives the relation

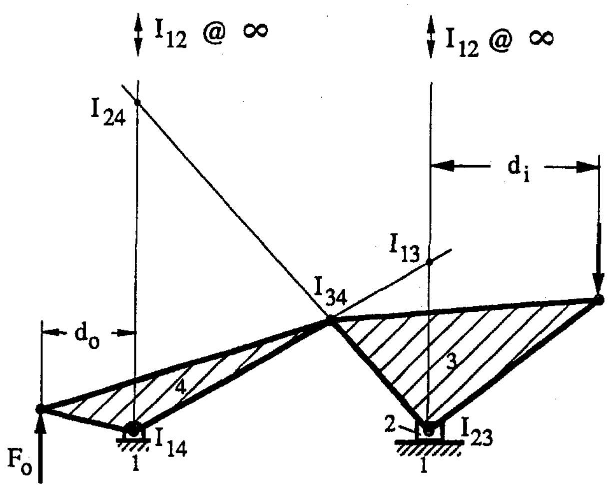


Fig. 1 Rigid-body slider crank mechanism

$$0 = \delta U - F_i \delta d_i + F_o \delta d_o \tag{11}$$

Defining mechanical advantage MA as the instantaneous ratio of the output force (F_o) to the input force (F_i), equation (11) is rearranged to give

$$\text{MA} = \frac{F_o}{F_i} = \frac{1}{\delta d_o}\left(\delta d_i - \frac{\delta U}{F_i}\right) \tag{12}$$

This general relation is valid for any mechanism, compliant or otherwise, provided it has a single-input and a single-output port (Midha et al., 1984). For example, if δU is zero, as for a rigid-body mechanism, Eq. (12) becomes

$$\text{MA} = \frac{F_o}{F_i} = \frac{\delta d_i}{\delta d_o} \tag{13}$$

Eq. (12) may be used to develop an insight into the mechanical advantage characteristics of compliant mechanisms.

Consider the following rearrangement:

$$\text{MA} = \frac{\delta d_i}{\delta d_o} - \frac{\delta U}{\delta d_o F_i} = \text{MA}_r - \text{MA}_c \tag{14}$$

The first term in Eq. (14) takes the form of a rigid-body mechanical advantage. This term would result if an instant center analysis for mechanical advantage (Shigley and Uicker, 1980) could be applied to the compliant mechanism in any instantaneous position. It would be a function of several parameters including those defining the original mechanism geometry as well as the externally applied loads. The effective link lengths thus change with the load, and the "rigid-body" mechanical advantage of the compliant mechanism (MA_r) cannot be represented by a single rigid-body counterpart for the entire range of operation of the compliant mechanism.

The second term in Eq. (14) also resembles a mechanical advantage term. It is referred to as the compliant component of the mechanical advantage (MA_c), and it accounts for the energy stored in the mechanism. The single-input and single-output port compliant mechanism may be considered to have two output ports, the actual physical output port and an internal port which performs work by elastically deforming the mechanism members. The mechanical advantage is thus maximized at a given instant when the compliant component of mechanical advantage (MA_c) becomes zero. When this occurs, the compliant mechanism behaves identically as a representative rigid-body mechanism.

Another useful form of Eq. (12) is given as

$$\text{MA} = \frac{\delta d_i}{\delta d_o}\left(1 - \frac{\delta U}{\delta d_i F_i}\right) = \text{MA}_r\left(1 - \frac{F_c}{F_i}\right) \tag{15}$$

where F_c is the compliant component of the input force (compliance force), or that part of the input force which is needed just to deform the mechanism members. Thus, the actual mechanical advantage is some fraction of the rigid-body mechanical advantage (MA_r) associated with a given mechanism position. Again, if the work of elastic deformation is minimized, the mechanical advantage is maximized.

Note that when δd_o is zero, Eq. (12) is still valid but the mechanical advantage is not necessarily infinite since it is also true in this instance that

$$\delta d_i - \frac{\delta U}{F_i} = 0 \tag{16}$$

This is obtained by letting the last term in Eq. (11) be zero. Equation (16) can also be expressed as a mechanical advantage by introducing F_o and rearranging to give

$$\text{MA} = \frac{F_o}{F_i} = \frac{F_o \delta d_i}{\delta U} \tag{17}$$

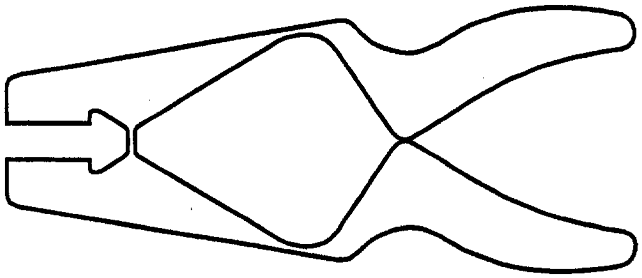
For this case, since the output displacement is fixed, i.e., $\delta d_o = 0$, the output force may be considered to be a reaction force and is a nonlinear function of the input force.

Defining Mechanical Advantage Types

Because mechanical advantage of a rigid-body mechanism is a function of the linkage position only, a plot of its variation over the mobility range of the mechanism is readily constructed. As stated earlier, the mobility of a compliant mechanism is also a function of the applied forces. It would therefore be not possible to construct one single, two-dimensional plot describing the variation of mechanical advantage for a compliant mechanism. Three mechanical advantage types are defined herein, which in turn also help alleviate this problem.

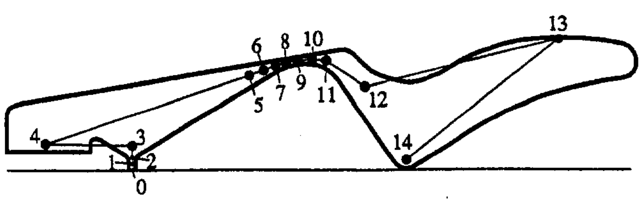
These definitions are based on the assumption that there is only one input force, and that no applied loads other than the input force are changing. All forces that change as a result of changes in the input force are considered as reaction forces (including the output force) which correspond to given displacement boundary conditions. Only one of these reaction forces is treated as the output force. Thus, the following definitions of the mechanical advantage types (Types 1, 2 and 3) are forwarded for single-input and single-output port compliant mechanisms.

Type 1 (or input-force-dependent) mechanical advantage is measured by fixing the output port displacement at a given constant value. The output force then varies with the input force.



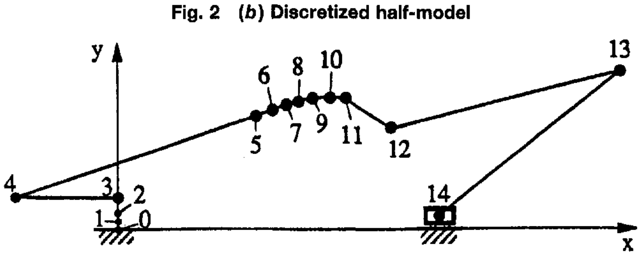
(a)

Fig. 2 (*a*) A compliant crimping mechanism



(b)

Fig. 2 (*b*) Discretized half-model



(c)

Fig. 2 (*c*) Finite element model

Table 1 Geometric properties of the compliant crimping mechanism in Fig. 2

i	x_i (in)	y_i (in)	I_i (in^4)
1	0.000	0.080	1.350×10^{-5}
2	0.000	0.160	1.350×10^{-5}
3	0.000	0.260	7.813×10^{-3}
4	-1.000	0.400	6.250×10^{-2}
5	1.340	1.100	6.250×10^{-2}
6	1.510	1.160	5.788×10^{-4}
7	1.660	1.210	1.725×10^{-4}
8	1.780	1.250	4.556×10^{-5}
9	1.920	1.270	2.637×10^{-5}
10	2.080	1.285	2.637×10^{-5}
11	2.240	1.280	6.250×10^{-5}
12	2.710	1.000	6.250×10^{-2}
13	5.000	1.570	6.250×10^{-2}
14	3.200	0.150	6.250×10^{-2}

Type 2 (or output-port-displacement-dependent) mechanical advantage is measured when the input force is held constant. The output force then varies as a function of the output port displacement.

Type 3 mechanical advantage is a result of an interaction between the mechanism and the workpiece. It may appropriately be termed as workpiece-dependent mechanical advantage. For this type, the input force is determined based on the requirements at the output port. These are requirements of both force and displacement and result from the force-displacement characteristics of the workpiece.

Types 1 and 2 mechanical advantages are more easily constructed and give more direct insight to mechanical advantage of compliant mechanisms than does Type 3. Type 3 mechanical advantage, however, is expected to be the most useful and prevalent of the three types in evaluating the overall performance of a compliant mechanism.

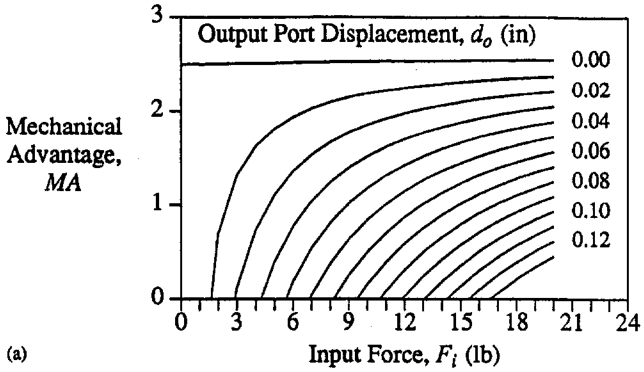
A Compliant Mechanism Case Study

To illustrate the definitions in the previous section, various mechanical advantage plots for a compliant mechanism are presented. The specific mechanism considered (Midha 1983) is shown in Fig. 2*a*. Due to its symmetry, only one-half of the mechanism is analyzed. Figures 2*b* and 2*c* show the nodal distribution of a simply discretized model used for this example. The corresponding geometric properties are listed in Table 1. The flexural modulus of elasticity is 0.9×10^6 psi, and the chain algorithm with a shooting method is employed, as a method of large-deflection analysis described in Her (1986), using 10 load increments.

For this mechanism, the input port is at node 13 and the output port at node 4. The input force acts in the negative y-direction and the output force in the positive y-direction. Node 14 is attached to a slider (Fig. 2*c*) which does not permit a y-direction displacement.

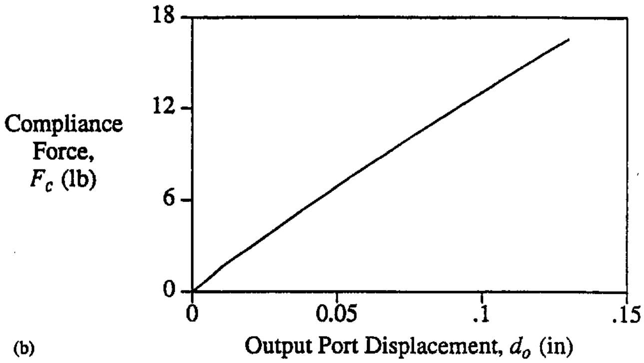
Type 1 mechanical advantage curves for this mechanism are shown in Fig. 3*a*. Each curve corresponds to an output port displacement d_o between 0.00 in. and 0.13 in., in increments of 0.01 in. The input force (F_i) is varied between 1 and 20 lb in 1-lb steps.

For the range of the data shown, the Type 1 curves are bounded above (Fig. 3*a*) by the curve corresponding to zero



(a)

Fig. 3 (a) Type 1 mechanical advantage plot



(b)

Fig. 3 (b) Compliance force (F_c) variation with output port displacement (d_o)

output port displacement. The value of the mechanical advantage for this curve is nearly constant at 2.55. All Type 1 curves shown in Fig. 3*a* may be approximated as

$$\text{MA} = \text{MA}_s\left(1 - \frac{F_c}{F_i}\right) \tag{17}$$

where MA_s is the mechanical advantage associated with the bounding curve, and it corresponds to the rigid-body mechanical advantage (MA_r) of the initial mechanism position. F_c is the input force required to displace the output port a distance d_o without generating an output force; contact is then made with the workpiece. In other words, this is the input force required to overcome compliance in moving the mechanism to a given position, and it is therefore called the compliance force. Having reached this position of constant output port displacement (d_o), further increasing the input force will yield useful output and the mechanical advantage increases. Equation (17) takes the same form as Eq. (15).

The value of F_c for a given Type 1 curve is easily obtained. It is the input force value that corresponds to an output port displacement d_o and zero mechanical advantage. Figure 3*b* shows the variation of F_c with d_o for the mechanism under consideration. This curve illustrates the input force versus output port deflection characteristic of the mechanism when there is no output force present. The area under this curve represents the energy stored in the mechanism.

The Type 2 mechanical advantage curves are shown in Fig. 4*a*. In this figure, each curve corresponds to a constant value of input force. Each of these curves is nearly linear. They show that as output port displacement (d_o) increases, more energy is stored in the mechanism and less force is available at the output. This is evidenced by the decreasing mechanical advantage (MA).

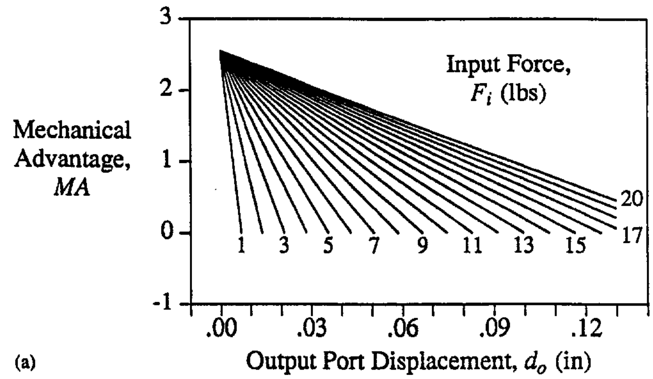
Because the mechanical advantage of single-input and single-output port compliant mechanisms can be suitably described as a function of two variables, i.e., the input force (F_i) and the output port displacement (d_o), it is appropriate to construct a three-dimensional surface plot of the mechanical advantage for this mechanism. This plot is shown in Fig. 4*b*, and it fully describes the mechanical advantage characteristics of this mechanism. Note that the Type 1 and Type 2 mechanical advantage curves are the intersection of this surface and planes parallel to the MA $-$ F_i and MA $-$ d_o planes, respectively. The F_c versus d_o curve discussed earlier and shown in Fig. 3*b* is found as the intersection of the mechanical advantage surface with the F_i $-$ d_o plane.

The Type 3 mechanical advantage curves are plotted in the MA $-$ F_i plane in Fig. 5*a*. The curves shown assume a workpiece having a linear force-deflection relation. Each curve corresponds to a different stiffness value. Points on these curves are determined numerically by applying the output force to the mechanism, and then finding the input force that will provide the corresponding output port displacement as per the force-deflection behavior of the workpiece.

For the mechanism under consideration, the mechanical advantage increases slightly (Fig. 5*a*) with the input force for a constant stiffness workpiece. The performance of this mechanism increases with increased workpiece stiffness as shown in Fig. 5*b*. Assuming the constant stiffness curves to have constant MA, this relation may be shown to take the general form

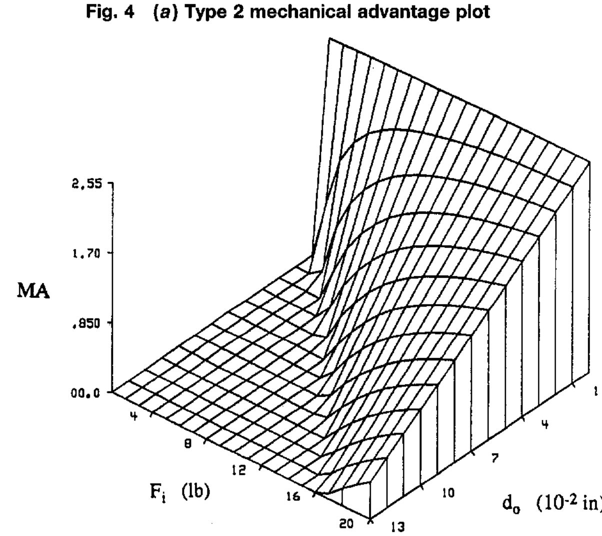
$$\text{MA} = \text{MA}_s\left(\frac{k_w}{\sigma + k_w}\right) \tag{18}$$

where MA_s is the bounding mechanical advantage as discussed with regard to Eq. (17), and σ is the sensitivity index which



(a)

Fig. 4 (a) Type 2 mechanical advantage plot



(b)

Fig. 4 (b) Mechanical advantage surface plot

defines the sensitivity of mechanism performance to the stiffness of the workpiece. The sensitivity index (σ) is minimized by minimizing the energy stored in the mechanism. This results in a mechanism with performance having little dependence upon the stiffness of the workpiece. This, of course, becomes an important parameter in the design of compliant mechanisms.

Another useful plot, shown in Fig. 6, depicts constant output force curves in the MA $-$ d_o plane. The ease with which the force-deflection properties of the workpiece are coordinated with this plot leads to its utility. These curves can be thought of as a transformed coordinate grid on which the force-deflection relation of any workpiece may be plotted. Thus, by having this type of a plot for a mechanism, it is simple to manually construct a Type 3 curve, corresponding to a given workpiece stiffness, rather than determining it numerically.

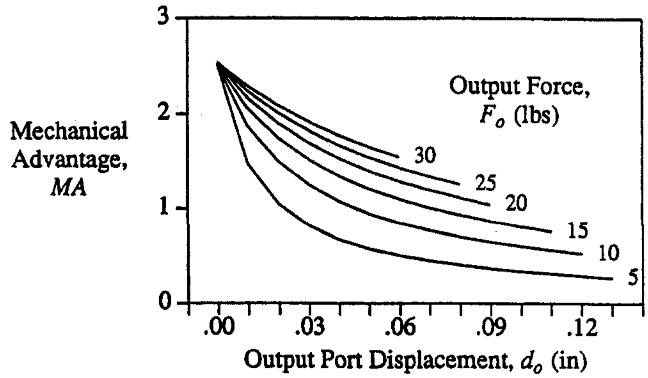


Fig. 6 Output port characteristics

Conclusions

Generalized equations for mechanical advantage in compliant mechanisms, which duly account for energy stored with mechanism deformation, have been derived. Also forwarded are the concepts of the rigid-body and compliant components of mechanical advantage, and the idea of compliance force. Mechanical advantage has been shown to be maximized as the elastic deformation is minimized.

Mechanical advantage types, Types 1, 2 and 3, have been defined which address the typically encountered boundary conditions of force and displacement. These also aid in simplifying the understanding of the mechanical advantage property in compliant mechanisms. A case study has been presented to exemplify these definitions. Type 1 mechanical advantage curves are found to be of a form similar to the generalized mechanical advantage equation. An elucidative mechanical advantage surface plot has been introduced that incorporates the behavior of mechanical advantage Types 1 and 2. When acting on a compliant workpiece, the compliant mechanism examined has been shown to maintain a nearly constant mechanical advantage over the range of input force considered. In addition, the concept of a sensitivity index in a compliant mechanism has been introduced to show the reliance of its mechanical advantage on the workpiece stiffness.

(a)

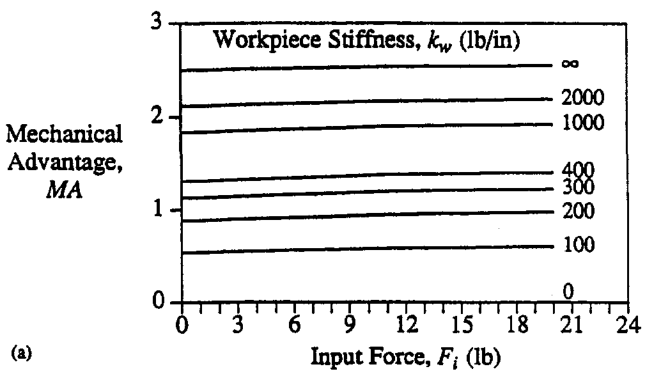


Fig. 5 (*a*) Type 3 mechanical advantage plot

(b)

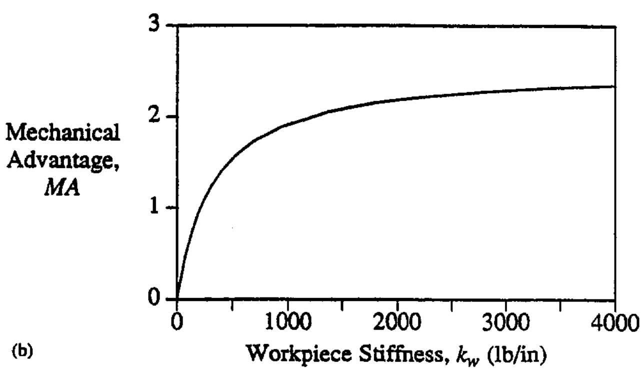


Fig. 5 (*b*) Mechanical advantage variation with workpiece stiffness

Acknowledgments

The authors would like to acknowledge the support of the School of Mechanical Engineering and the use of its facilities. The support of National Science Foundation for this research, through NSF Grant No. MSS-8902777, is gratefully acknowledged.

References

Erdman, A. G., and Sandor, G. N., *Mechanism Design: Analysis and Synthesis*, Vol. 1, Second Edition, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1991.

Her, I., "Methodology for Compliant Mechanisms Design," Ph.D. Dissertation, Purdue University, 1986.

Howell, L. L., "A Generalized Loop-Closure Theory for the Analysis and Synthesis of Compliant Mechanisms," Ph.D. Dissertation, Purdue University, 1993.

Howell, L. L., and Midha, A., "The Effects of a Compliant Workpiece on the Input/Output Characteristics of Rigid-Link Toggle Mechanisms," *Mechanism and Machine Theory*, Vol. 30, No. 6, 1995, pp. 801–810.

Midha, A., "Class Notes for ME 597: Special Topics in Compliant Mechanisms," The Pennsylvania State University, May 1983.

Midha, A., Hall, Jr., A. S., Her, I., and Bubel, G. M., "Mechanical Advantage of Single-Input and Multiple-Output Ports Mechanical Device," ASME *Journal of Mechanisms, Transmissions, and Automation in Design*, Vol. 106, No. 4, 1984, pp. 462–469.

Shigley, J. E., and Uicker, Jr., J. J., *Theory of Machines and Mechanisms*, McGraw-Hill Book Company, New York, New York, 1980.