ME 254

Non-dimensional portrayal of nonlinear elastic response of beams

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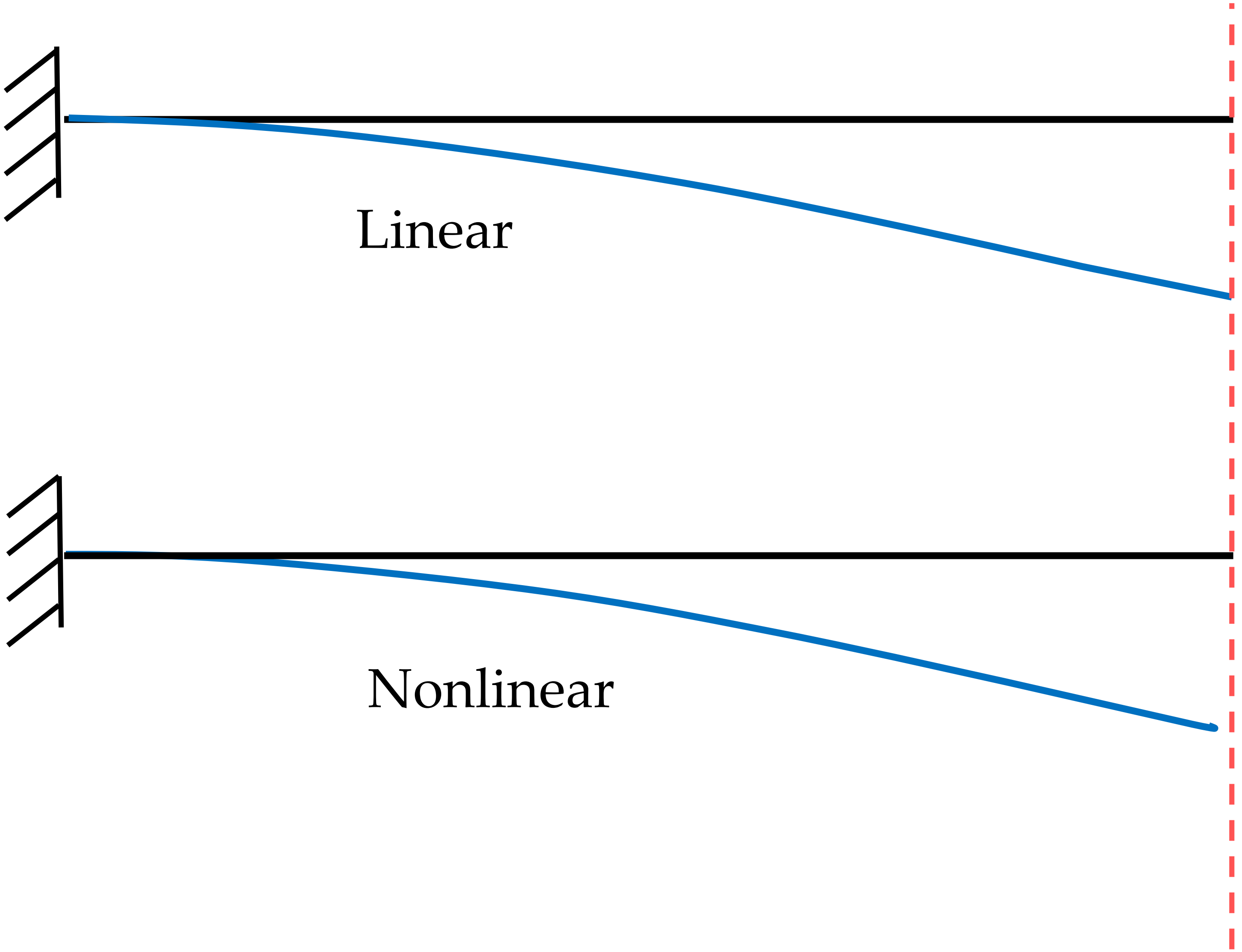
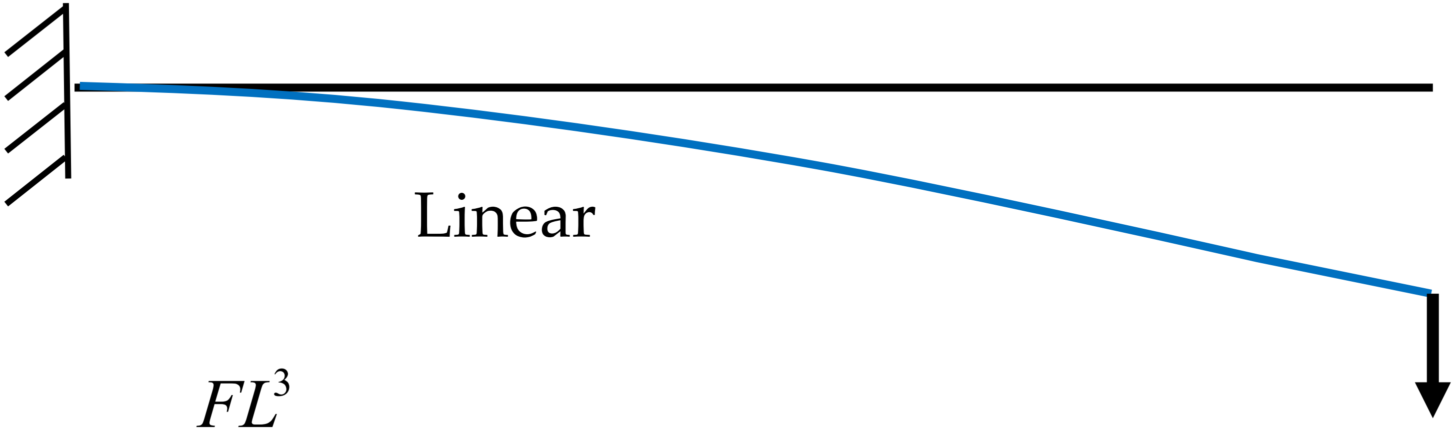
Deformation of a cantilever beam



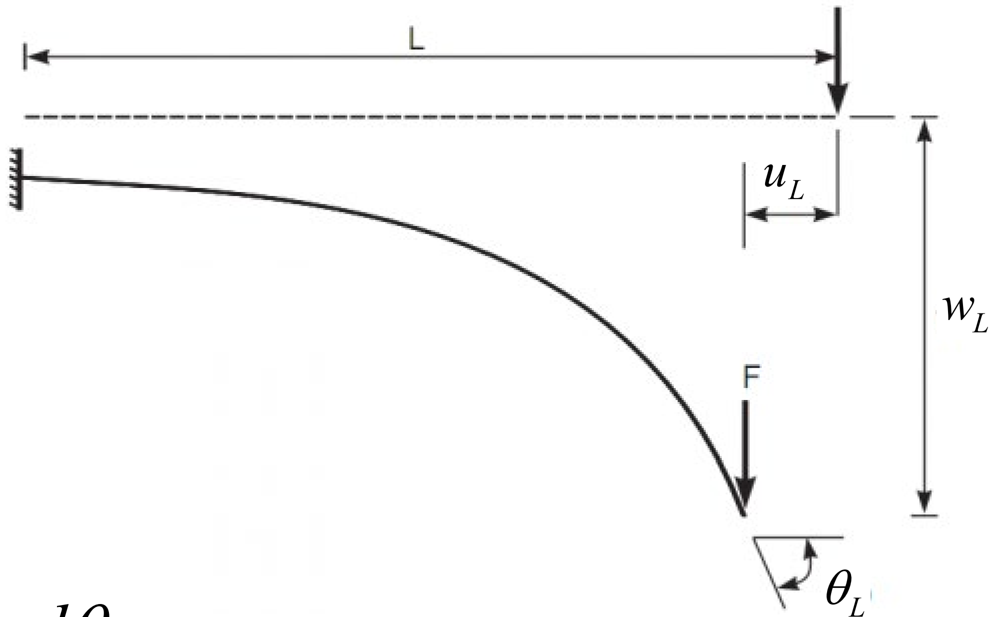
Non-dimensionality in the linear case



$$w_L = \frac{FL^3}{3EI}$$

$$\frac{w_L}{L} = \frac{FL^2}{3EI} = \frac{\eta}{3}$$

Large displacements analysis of a cantilever beam with a tip-load



$$EI\frac{d\theta}{ds} = F(L - x - u_L)$$ Large displacements

$$EI\frac{d^2w(x)}{dx^2} = F(L - x)$$ Small displacements

Elastica equation

$$EI\frac{d\theta}{ds} = F(L - x - u_L)$$

Differentiate to get

$$\frac{d^2\theta}{ds^2} = -\frac{F}{EI}\frac{dx}{ds} = -\frac{F}{EI}\cos\theta$$

Kirchhoff's pendulum analogy

$$\frac{d^2\theta}{dt} + \frac{g}{l}\sin\theta = 0$$

Elastica equation

$$EI\frac{d\theta}{ds} = F(L - x - u_L)$$

Differentiate to get

$$\frac{d^2\theta}{ds^2} = -\frac{F}{EI}\frac{dx}{ds} = -\frac{F}{EI}\cos\theta$$

(a little manipulation)

$$\frac{d^2\theta}{ds^2}\frac{d\theta}{ds} = -\frac{F}{EI}\cos\theta\frac{d\theta}{ds}$$

Integrate to get

$$\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 = -\frac{F}{EI}\sin\theta + C$$

(Curvature is zero at the tip)

$$C = \frac{F}{EI}\sin\theta_L$$

Elastica equation (contd.)

Differential equation now:

$$\frac{d\theta}{ds} = \sqrt{\frac{2F}{EI}\left(\sin\theta_L - \sin\theta\right)}$$

Assumption of no stretching.

$$\int_0^L ds = L = \int_0^{\theta_L} \frac{d\theta}{\sqrt{\frac{2F}{EI}\left(\sin\theta_L - \sin\theta\right)}}$$

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{\left(\sin\theta_L - \sin\theta\right)}} = \sqrt{2}\sqrt{\frac{FL^2}{EI}} = \sqrt{2\eta}$$

$$\eta = \frac{FL^2}{EI}$$

Change of variable

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{\left(\sin\theta_L - \sin\theta\right)}} = \sqrt{2\eta}$$

$$\sin\theta = 2p^2 \sin^2\phi - 1$$

$$p^2 = \frac{1+\sin\theta_L}{2}$$

$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\pi/2} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} = \sqrt{\eta}$$

Let see how…

Change of variable

$$\theta \rightarrow \phi$$

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{\left(\sin\theta_L - \sin\theta\right)}} = \sqrt{2\eta}$$

$$\sin\theta = 2p^2\sin^2\phi - 1$$

$$p^2 = \frac{1+\sin\theta_L}{2}$$

$$\sin\theta = 2p^2\sin^2\phi - 1$$

$$2p^2 - 1 = \sin\theta_L$$

Limits:

Change of variable

$$\theta \to \phi$$

$$\sin\theta = 2p^2\sin^2\phi - 1$$

$$p^2 = \frac{1+\sin\theta_L}{2}$$

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{(\sin\theta_L - \sin\theta)}} = \sqrt{2\eta}$$

$$\sin\theta_L = 2p^2 - 1$$

$$\sin\theta = 2p^2\sin^2\phi - 1$$

$$\sqrt{\sin\theta_L - \sin\theta} = \sqrt{2}\,p\cos\phi$$

$$\sin\theta = 2p^2\sin^2\phi - 1$$

$$\cos\theta\,d\theta = 4p^2\sin\phi\,\cos\phi\,d\phi$$

$$\sqrt{1-\sin^2\theta}\,d\theta = 4p^2\sin\phi\,\cos\phi\,d\phi$$

$$\sqrt{1-\left(2p^2\sin^2\phi - 1\right)^2}\,d\theta = 4p^2\sin\phi\,\cos\phi\,d\phi$$

$$2p\sin\phi\sqrt{1-p^2\sin^2\phi}\;d\theta = 4p^2\sin\phi\,\cos\phi\,d\phi$$

$$d\theta = \frac{2p\cos\phi}{\sqrt{1-p^2\sin^2\phi}}\,d\phi$$

Change of variable $\theta \to \phi$

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{(\sin\theta_L - \sin\theta)}} = \sqrt{2\eta}$$

$$d\theta = \frac{2p\cos\phi}{\sqrt{1-p^2\sin^2\phi}}\,d\phi$$

$$\sqrt{\sin\theta_L - \sin\theta} = \sqrt{2}\,p\cos\phi$$

$$\sin\theta = 2p^2\sin^2\phi - 1$$

$$p^2 = \frac{1+\sin\theta_L}{2}$$

$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\pi/2} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} = \sqrt{\eta}$$

Elliptic integrals

I kind (incomplete)

$$\int_0^{\phi^*} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} = \mathbf{F}(p,\phi^*)$$

I kind (complete)

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} = \mathbf{F}(p,\pi/2)$$

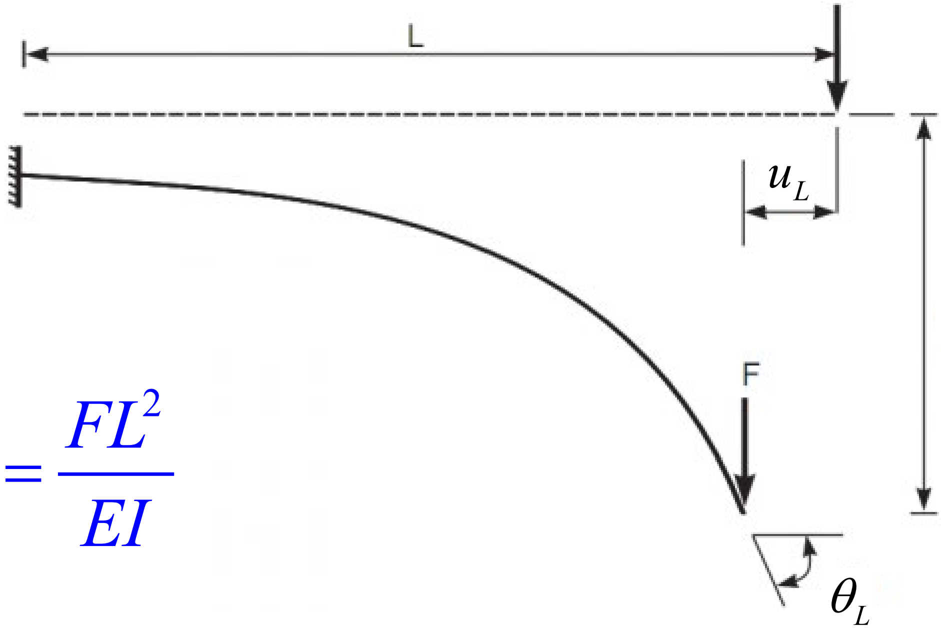
II kind (incomplete)

$$\int_0^{\phi^*} \sqrt{1-p^2\sin^2\phi}\;d\phi = \mathbf{E}(p,\phi^*)$$

II kind (complete)

$$\int_0^{\pi/2} \sqrt{1-p^2\sin^2\phi}\;d\phi = \mathbf{E}(p,\pi/2)$$

Finally, the solution!



$$2p^2 - 1 = \sin\theta_L$$

$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\pi/2} \frac{d\phi}{\sqrt{1 - p^2 \sin^2\phi}} = \sqrt{\eta}$$

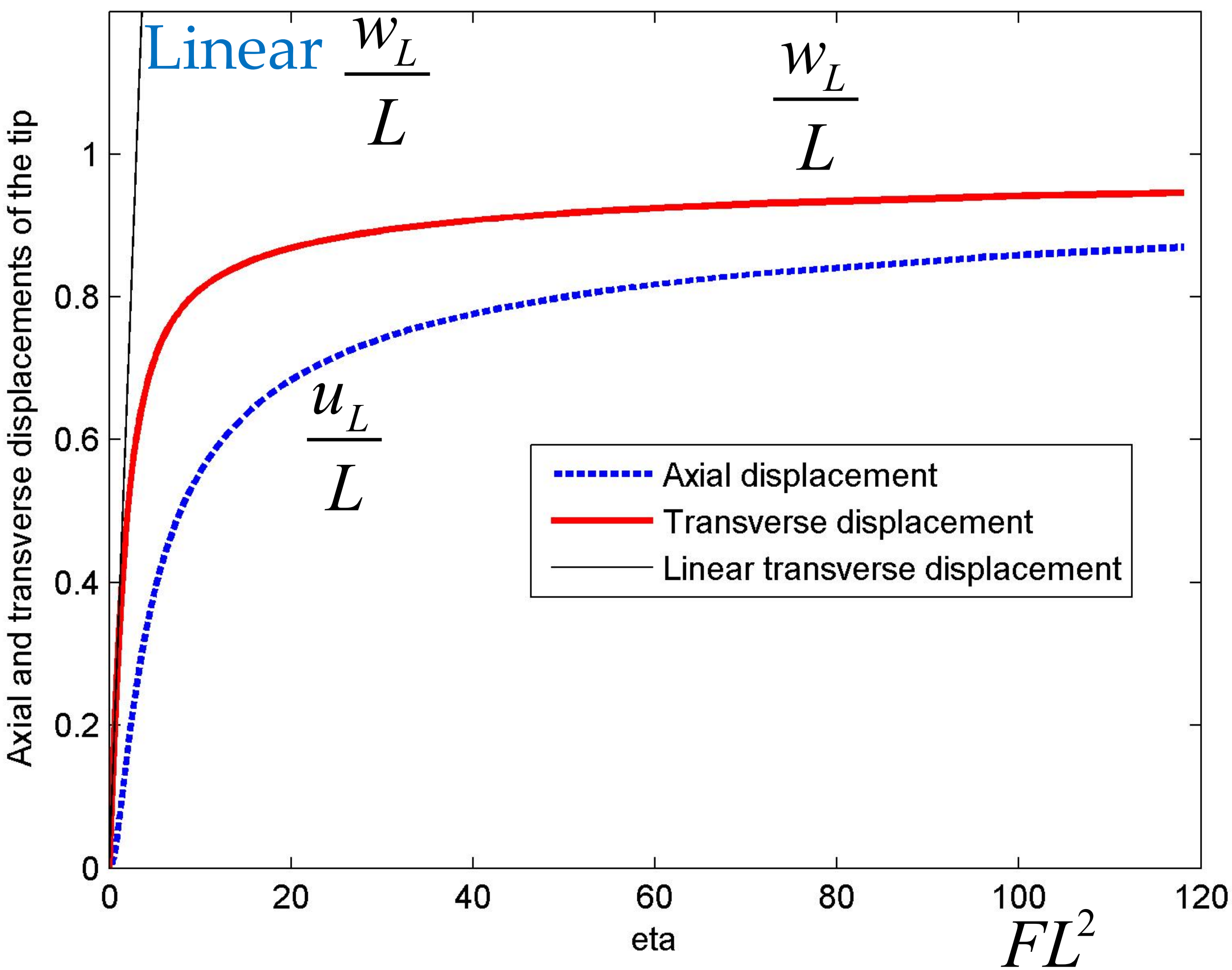
$$\eta = \frac{FL^2}{EI}$$

$$\phi_0 = \sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)$$

$$\frac{w_L}{L} = \sqrt{\frac{1}{\eta}}\left\{\mathbf{F}(p,\pi/2) - \mathbf{F}(p,\phi_0) - 2\mathbf{E}(p,\pi/2) + 2\mathbf{E}(p,\phi_0)\right\}$$

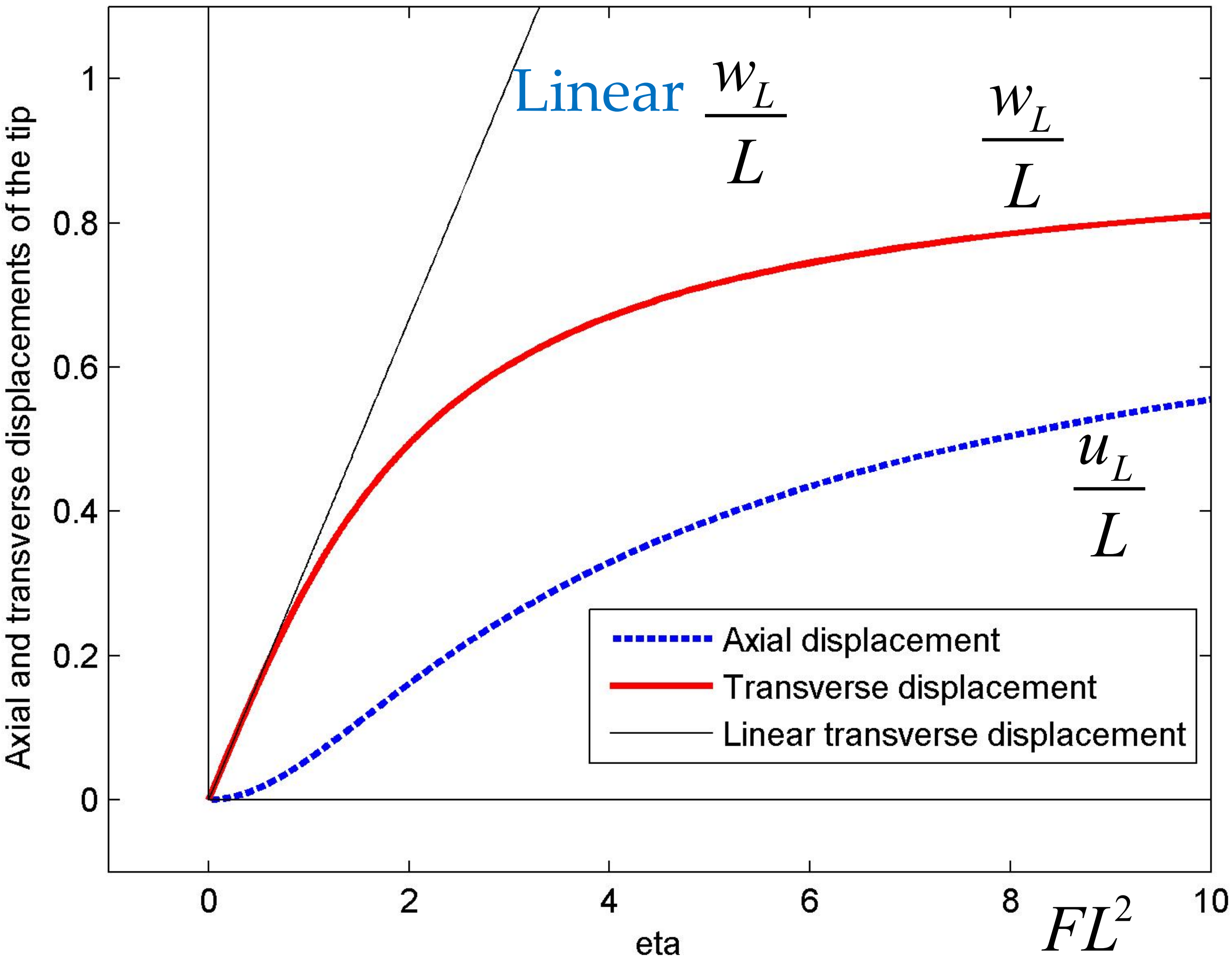
$$\frac{u_L}{L} = 1 - \sqrt{\frac{2EI}{FL^2}\left(2p^2 - 1\right)} = 1 - \sqrt{\frac{2}{\eta}\left(2p^2 - 1\right)}$$

Transverse and axial displacements of the loaded tip



$$\eta = \frac{FL^2}{EI}$$

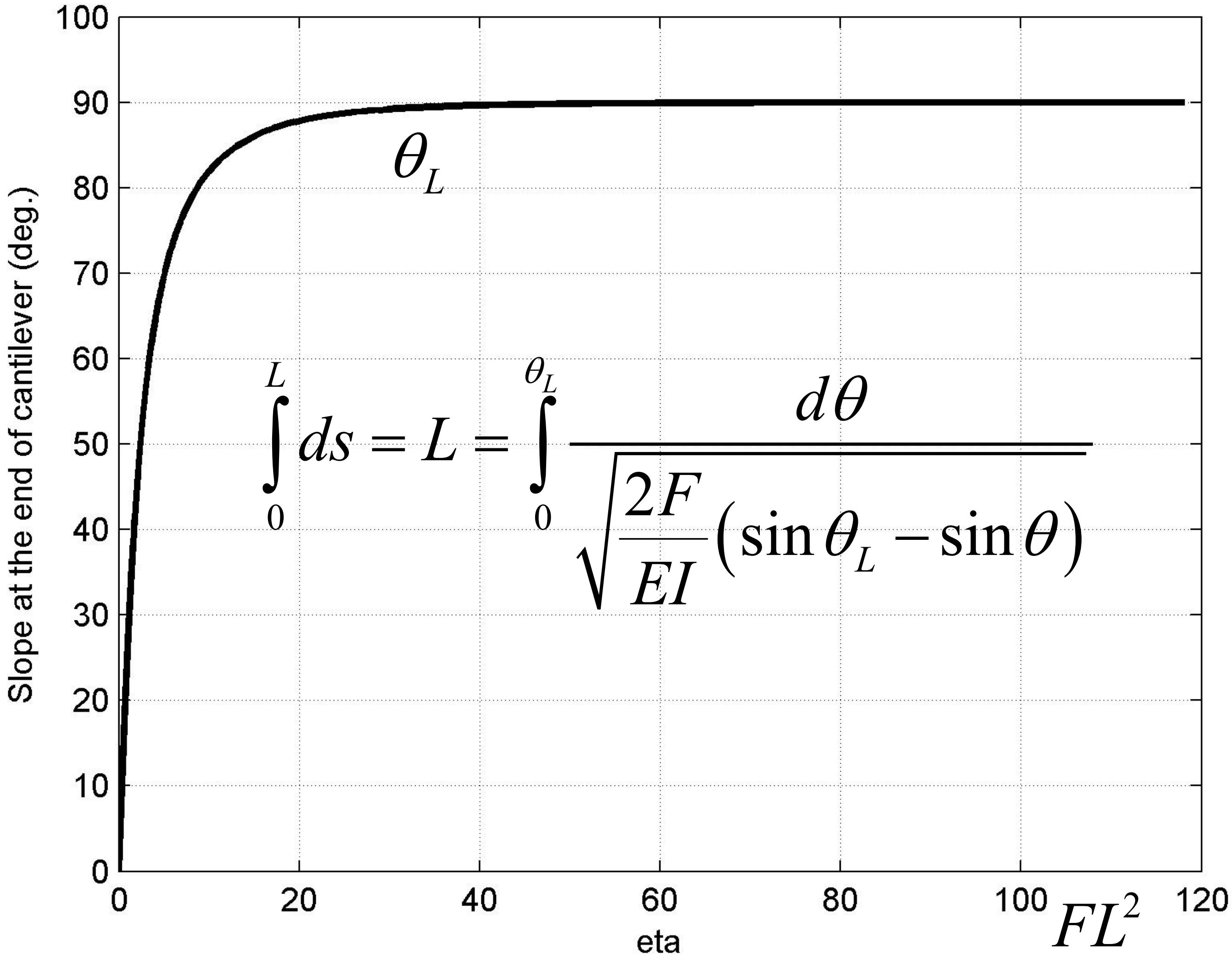
A close-up view



$$\eta = \frac{FL^2}{EI}$$

Index of bending

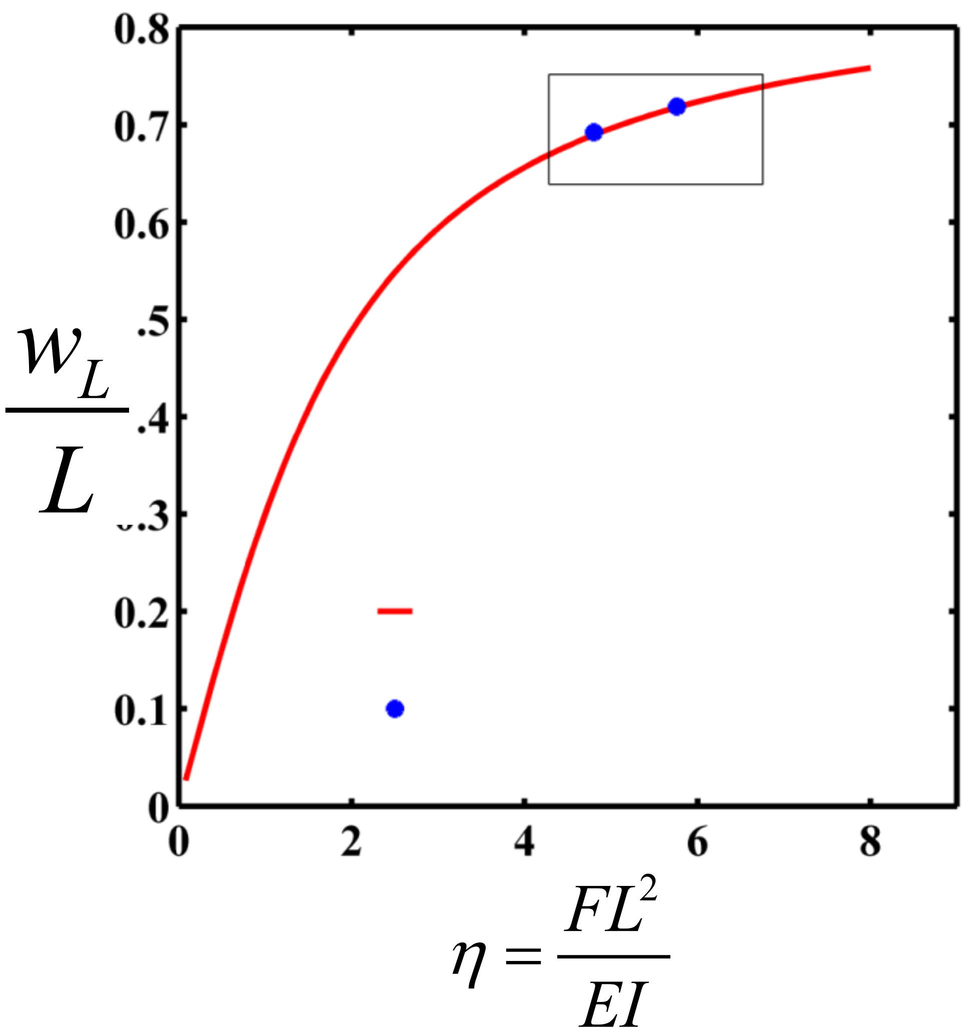
$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{(\sin\theta_L - \sin\theta)}} = \sqrt{2\eta}$$

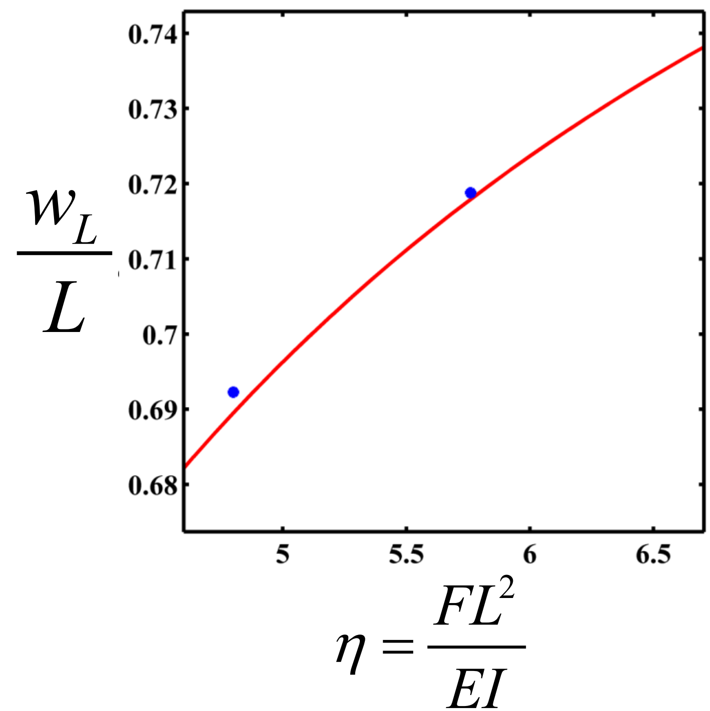


$$\int_0^L ds = L = \int_0^{\theta_L} \frac{d\theta}{\sqrt{\frac{2F}{EI}(\sin\theta_L - \sin\theta)}}$$

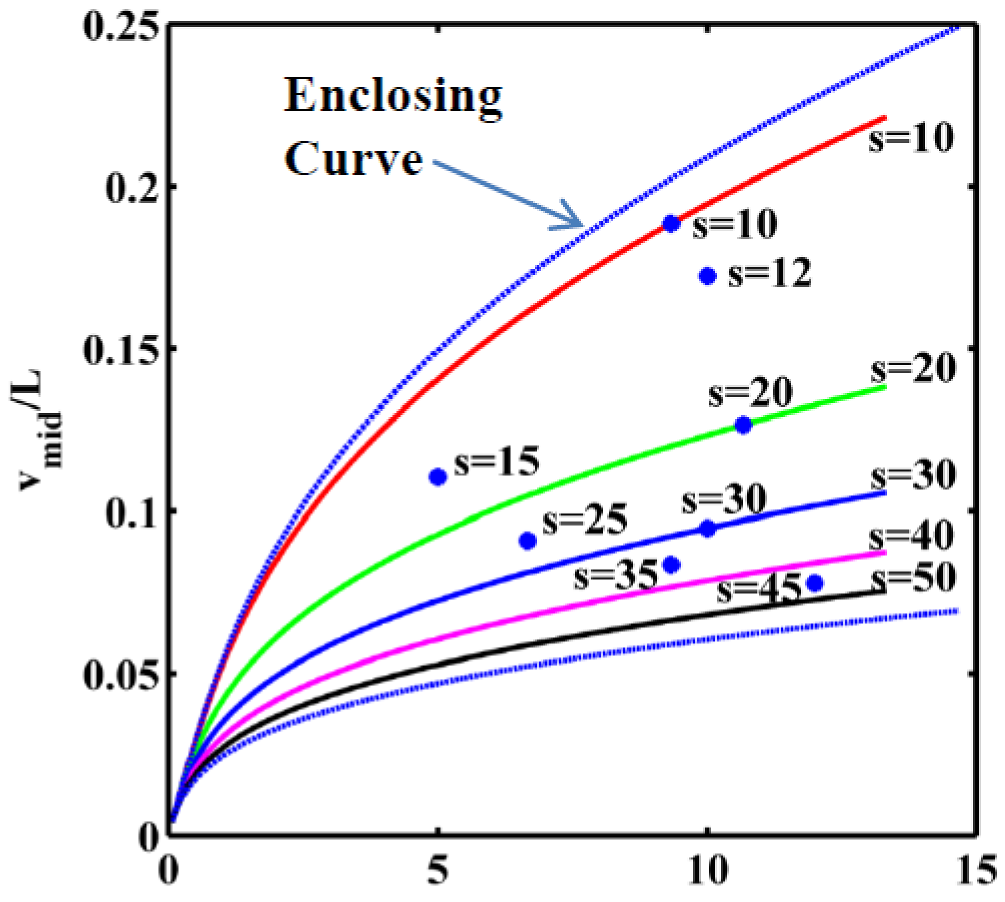
$$\eta = \frac{FL^2}{EI}$$

Comparing with geometrically nonlinear finite element analysis

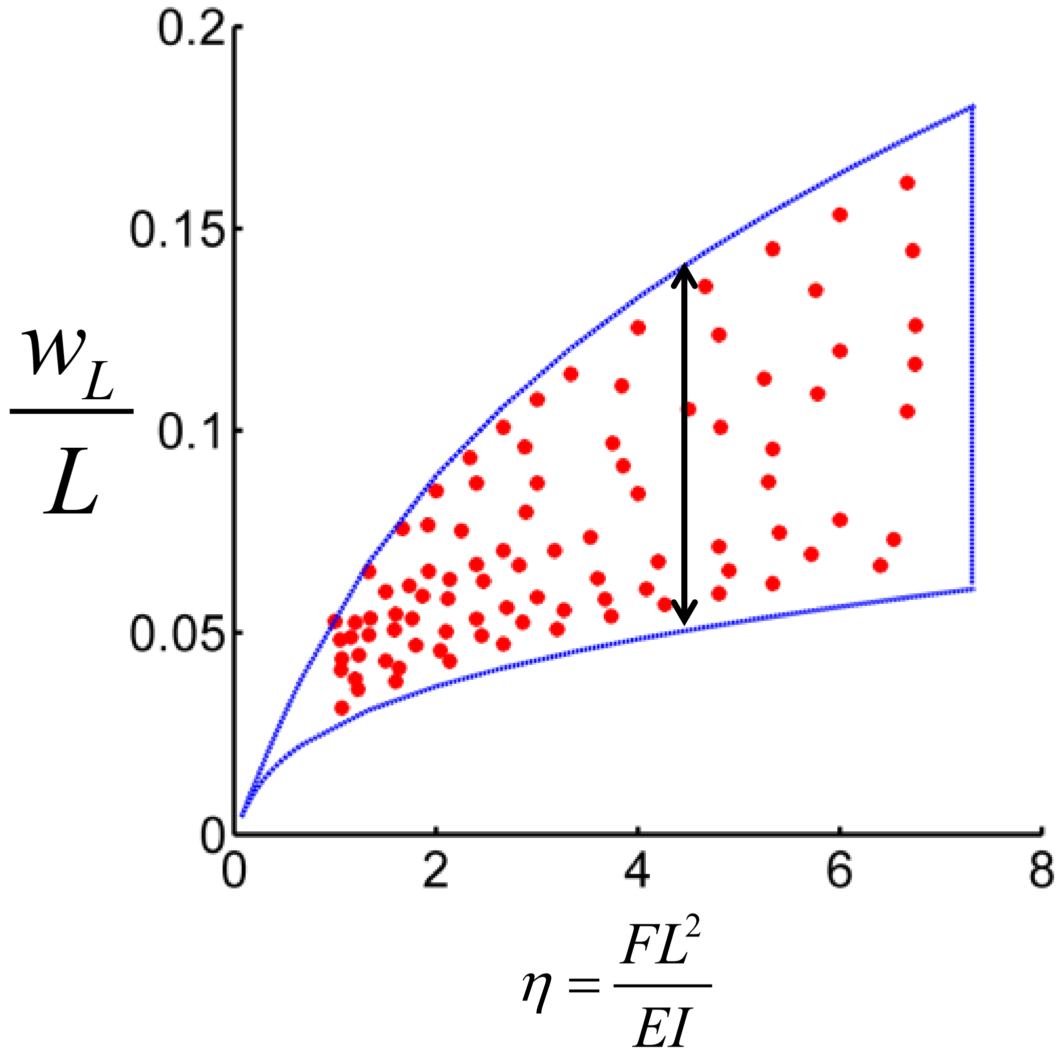




A fixed-fixed beam



For a fixed-fixed beam



There may be more than one non-dimensional parameter!

"Beam" finite element stiffness matrix

$$\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Substitute for A and I for the rectangular cross-section

$$
\begin{bmatrix}
\frac{Ebd}{L} & 0 & 0 & -\frac{Ebd}{L} & 0 & 0 \\
0 & \frac{Ebd^3}{L^3} & \frac{Ebd^3}{2L^2} & 0 & \frac{-Ebd^3}{L^3} & \frac{Ebd^3}{2L^2} \\
0 & \frac{Ebd^3}{2L^2} & \frac{Ebd^3}{3L} & 0 & \frac{-Ebd^3}{2L^2} & \frac{Ebd^3}{6L} \\
-\frac{Ebd}{L} & 0 & 0 & \frac{Ebd}{L} & 0 & 0 \\
0 & \frac{-Ebd^3}{L^3} & \frac{-Ebd^3}{2L^2} & 0 & \frac{Ebd^3}{L^3} & \frac{-Ebd^3}{2L^2} \\
0 & \frac{Ebd^3}{2L^2} & \frac{Ebd^3}{6L} & 0 & \frac{-Ebd^3}{2L^2} & \frac{Ebd^3}{3L}
\end{bmatrix}
$$

Factor out something.

$$\frac{Ebd^3}{L^2}\begin{bmatrix} \frac{L}{d^2} & 0 & 0 & -\frac{L}{d^2} & 0 & 0 \\ 0 & \frac{1}{L} & \frac{1}{2} & 0 & -\frac{1}{L} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{L}{3} & 0 & -\frac{1}{2} & \frac{L}{6} \\ -\frac{L}{d^2} & 0 & 0 & \frac{L}{d^2} & 0 & 0 \\ 0 & -\frac{1}{L} & -\frac{1}{2} & 0 & \frac{1}{L} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{L}{6} & 0 & -\frac{1}{2} & \frac{L}{3} \end{bmatrix}$$

Define "slenderness" ratio.

$$\frac{Ebd}{s^2}\begin{bmatrix} \frac{s^2}{L} & 0 & 0 & -\frac{s^2}{L} & 0 & 0 \\ 0 & \frac{1}{L} & \frac{1}{2} & 0 & -\frac{1}{L} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{L}{3} & 0 & -\frac{1}{2} & \frac{L}{6} \\ -\frac{s^2}{L} & 0 & 0 & \frac{s^2}{L} & 0 & 0 \\ 0 & -\frac{1}{L} & -\frac{1}{2} & 0 & \frac{1}{L} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{L}{6} & 0 & -\frac{1}{2} & \frac{L}{3} \end{bmatrix}$$

$$s = \frac{L}{d}$$

Factor in something using proportions.

$$\frac{Ebd}{s^2}\begin{bmatrix} \frac{s^2}{L} & 0 & 0 & -\frac{s^2}{L} & 0 & 0 \\ 0 & \frac{1}{L} & \frac{1}{2} & 0 & -\frac{1}{L} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{L}{3} & 0 & -\frac{1}{2} & \frac{L}{6} \\ -\frac{s^2}{L} & 0 & 0 & \frac{s^2}{L} & 0 & 0 \\ 0 & -\frac{1}{L} & -\frac{1}{2} & 0 & \frac{1}{L} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{L}{6} & 0 & -\frac{1}{2} & \frac{L}{3} \end{bmatrix}$$

$$b = \alpha_b \bar{b}$$

$$d = \alpha_d \bar{d}$$

$$E = \alpha_E \bar{E}$$

$$L = \alpha_L \bar{L}$$

$$s = \bar{s}\frac{\alpha_L}{\alpha_d}$$

Proportions used.

$$s = \bar{s}\frac{\alpha_L}{\alpha_d}$$

$$b = \alpha_b \bar{b}$$

$$d = \alpha_d \bar{d}$$

$$E = \alpha_E \bar{E}$$

$$L = \alpha_L \bar{L}$$

$$\frac{\bar{E}\bar{b}\bar{d}}{\bar{s}^2}\frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^2}\begin{bmatrix} \frac{\bar{s}^2}{\bar{L}}\frac{\alpha_L}{\alpha_d^2} & 0 & 0 & -\frac{\bar{s}^2}{\bar{L}}\frac{\alpha_L}{\alpha_d^2} & 0 & 0 \\ 0 & \frac{1}{\alpha_L\bar{L}} & \frac{1}{2} & 0 & -\frac{1}{\alpha_L\bar{L}} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\alpha_L\bar{L}}{3} & 0 & -\frac{1}{2} & \frac{\alpha_L\bar{L}}{6} \\ -\frac{\bar{s}^2}{\bar{L}}\frac{\alpha_L}{\alpha_d^2} & 0 & 0 & \frac{\bar{s}^2}{\bar{L}}\frac{\alpha_L}{\alpha_d^2} & 0 & 0 \\ 0 & -\frac{1}{\alpha_L\bar{L}} & -\frac{1}{2} & 0 & \frac{1}{\alpha_L\bar{L}} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\alpha_L\bar{L}}{6} & 0 & -\frac{1}{2} & \frac{\alpha_L\bar{L}}{3} \end{bmatrix}$$

Let us consider force-displacement relationship

$$\frac{\overline{E}\overline{bd}}{\overline{s}^2}\frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^2}\begin{bmatrix} \frac{\overline{s}^2}{\overline{L}}\frac{\alpha_L}{\alpha_d^2} & 0 & 0 & -\frac{\overline{s}^2}{\overline{L}}\frac{\alpha_L}{\alpha_d^2} & 0 & 0 \\ 0 & \frac{1}{\alpha_L\overline{L}} & \frac{1}{2} & 0 & -\frac{1}{\alpha_L\overline{L}} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\alpha_L\overline{L}}{3} & 0 & -\frac{1}{2} & \frac{\alpha_L\overline{L}}{6} \\ -\frac{\overline{s}^2}{\overline{L}}\frac{\alpha_L}{\alpha_d^2} & 0 & 0 & \frac{\overline{s}^2}{\overline{L}}\frac{\alpha_L}{\alpha_d^2} & 0 & 0 \\ 0 & -\frac{1}{\alpha_L\overline{L}} & -\frac{1}{2} & 0 & \frac{1}{\alpha_L\overline{L}} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\alpha_L\overline{L}}{6} & 0 & -\frac{1}{2} & \frac{\alpha_L\overline{L}}{3} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ M_{z1} \\ F_{x2} \\ F_{y2} \\ M_{z2} \end{Bmatrix}$$

Non-dimensionalize the displacements.

$$\frac{\overline{E}\overline{bd}}{\overline{s}^2}\frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^2}\begin{bmatrix} \overline{s}^2\frac{\alpha_L}{\alpha_d^2} & 0 & 0 & -\overline{s}^2\frac{\alpha_L}{\alpha_d^2} & 0 & 0 \\ 0 & \frac{1}{\alpha_L} & \frac{1}{2} & 0 & -\frac{1}{\alpha_L} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\alpha_L\overline{L}}{3} & 0 & -\frac{1}{2} & \frac{\alpha_L\overline{L}}{6} \\ -\overline{s}^2\frac{\alpha_L}{\alpha_d^2} & 0 & 0 & \overline{s}^2\frac{\alpha_L}{\alpha_d^2} & 0 & 0 \\ 0 & -\frac{1}{\alpha_L} & -\frac{1}{2} & 0 & \frac{1}{\alpha_L} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\alpha_L\overline{L}}{6} & 0 & -\frac{1}{2} & \frac{\alpha_L\overline{L}}{3} \end{bmatrix}\begin{Bmatrix} u_1/\overline{L} \\ v_1/\overline{L} \\ \theta_1 \\ u_2/\overline{L} \\ v_2/\overline{L} \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ M_{z1} \\ F_{x2} \\ F_{y2} \\ M_{z2} \end{Bmatrix}$$

Consistent units for the forces/moments.

$$\frac{\overline{E}\overline{bd}}{\overline{s}^2}\frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^2}\begin{bmatrix} \overline{s}^2\frac{\alpha_L}{\alpha_d^2} & 0 & 0 & -\overline{s}^2\frac{\alpha_L}{\alpha_d^2} & 0 & 0 \\ 0 & \frac{1}{\alpha_L} & \frac{1}{2} & 0 & -\frac{1}{\alpha_L} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\alpha_L}{3} & 0 & -\frac{1}{2} & \frac{\alpha_L}{6} \\ -\overline{s}^2\frac{\alpha_L}{\alpha_d^2} & 0 & 0 & \overline{s}^2\frac{\alpha_L}{\alpha_d^2} & 0 & 0 \\ 0 & -\frac{1}{\alpha_L} & -\frac{1}{2} & 0 & \frac{1}{\alpha_L} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\alpha_L}{6} & 0 & -\frac{1}{2} & \frac{\alpha_L}{3} \end{bmatrix}\begin{Bmatrix} u_1/\overline{L} \\ v_1/\overline{L} \\ \theta_1 \\ u_2/\overline{L} \\ v_2/\overline{L} \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ M_{z1}/\overline{L} \\ F_{x2} \\ F_{y2} \\ M_{z2}/\overline{L} \end{Bmatrix}$$

A slight re-arrangement

$$
\frac{\overline{E}\overline{b}\overline{d}}{\overline{s}^2}
\begin{bmatrix}
\overline{s}^2 \frac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 & -\overline{s}^2 \frac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 \\
0 & \frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & \frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & 0 & -\frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & \frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} \\
0 & \frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \frac{\alpha_E\alpha_b\alpha_d^3}{3\alpha_L} & 0 & -\frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \frac{\alpha_E\alpha_b\alpha_d^3}{6\alpha_L} \\
-\overline{s}^2 \frac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 & \overline{s}^2 \frac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 \\
0 & -\frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & -\frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & 0 & \frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & -\frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} \\
0 & \frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \frac{\alpha_E\alpha_b\alpha_d^3}{6\alpha_L} & 0 & -\frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \frac{\alpha_E\alpha_b\alpha_d^3}{3\alpha_L}
\end{bmatrix}
\begin{Bmatrix} u_1/\overline{L} \\ v_1/\overline{L} \\ \theta_1 \\ u_2/\overline{L} \\ v_2/\overline{L} \\ \theta_2 \end{Bmatrix}
=
\begin{Bmatrix} F_{x1} \\ F_{y1} \\ M_{z1}/\overline{L} \\ F_{x2} \\ F_{y2} \\ M_{z2}/\overline{L} \end{Bmatrix}
$$

What does this mean?

- Non-dimensional analysis holds for multi-beam ensemble.
- For large-displacement analysis too.

See what emerges…

$$\eta = \frac{Fs^2}{Ebd}$$

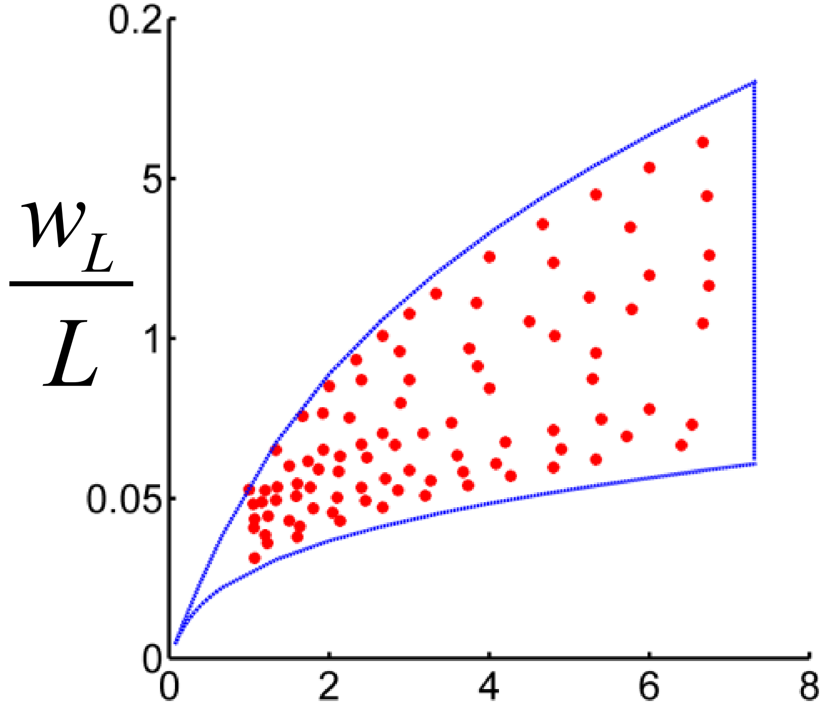
$$\begin{bmatrix}
\bar{s}^2 \frac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 & -\bar{s}^2 \frac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 \\
0 & \frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & \frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & 0 & -\frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & \frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} \\
0 & \frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \frac{\alpha_E\alpha_b\alpha_d^3}{3\alpha_L} & 0 & -\frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \frac{\alpha_E\alpha_b\alpha_d^3}{6\alpha_L} \\
-\bar{s}^2 \frac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 & \bar{s}^2 \frac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 \\
0 & -\frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & -\frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & 0 & \frac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & -\frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} \\
0 & \frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \frac{\alpha_E\alpha_b\alpha_d^3}{6\alpha_L} & 0 & -\frac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \frac{\alpha_E\alpha_b\alpha_d^3}{3\alpha_L}
\end{bmatrix}
\begin{Bmatrix} u_1/\bar{L} \\ v_1/\bar{L} \\ \theta_1 \\ u_2/\bar{L} \\ v_2/\bar{L} \\ \theta_2 \end{Bmatrix}
= \frac{\bar{s}^2}{\overline{Ebd}}
\begin{Bmatrix} F_{x1} \\ F_{y1} \\ M_{z1}/\bar{L} \\ F_{x2} \\ F_{y2} \\ M_{z2}/\bar{L} \end{Bmatrix}$$

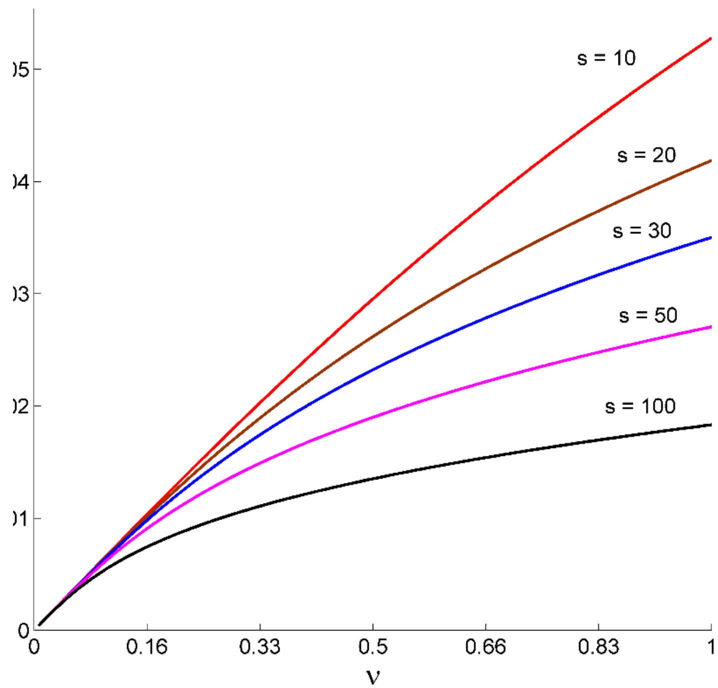
See what emerges… $\eta = \dfrac{Fs^2}{Ebd}$

$$
\begin{bmatrix}
\bar{s}^2\dfrac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 & -\bar{s}^2\dfrac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 \\
0 & \dfrac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & \dfrac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & 0 & -\dfrac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & \dfrac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} \\
0 & \dfrac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \dfrac{\alpha_E\alpha_b\alpha_d^3}{3\alpha_L} & 0 & -\dfrac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \dfrac{\alpha_E\alpha_b\alpha_d^3}{6\alpha_L} \\
-\bar{s}^2\dfrac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 & \bar{s}^2\dfrac{\alpha_E\alpha_b\alpha_d}{\alpha_L} & 0 & 0 \\
0 & -\dfrac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & -\dfrac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & 0 & \dfrac{\alpha_E\alpha_b\alpha_d^3}{\alpha_L^3} & -\dfrac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} \\
0 & \dfrac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \dfrac{\alpha_E\alpha_b\alpha_d^3}{6\alpha_L} & 0 & -\dfrac{\alpha_E\alpha_b\alpha_d^3}{2\alpha_L^2} & \dfrac{\alpha_E\alpha_b\alpha_d^3}{3\alpha_L}
\end{bmatrix}
\begin{Bmatrix} u_1/\bar{L} \\ v_1/\bar{L} \\ \theta_1 \\ u_2/\bar{L} \\ v_2/\bar{L} \\ \theta_2 \end{Bmatrix}
= \frac{\bar{s}^2}{\overline{Ebd}}
\begin{Bmatrix} F_{x1} \\ F_{y1} \\ M_{z1}/\bar{L} \\ F_{x2} \\ F_{y2} \\ M_{z2}/\bar{L} \end{Bmatrix}
$$

Parameterization with slenderness ratio, *s*

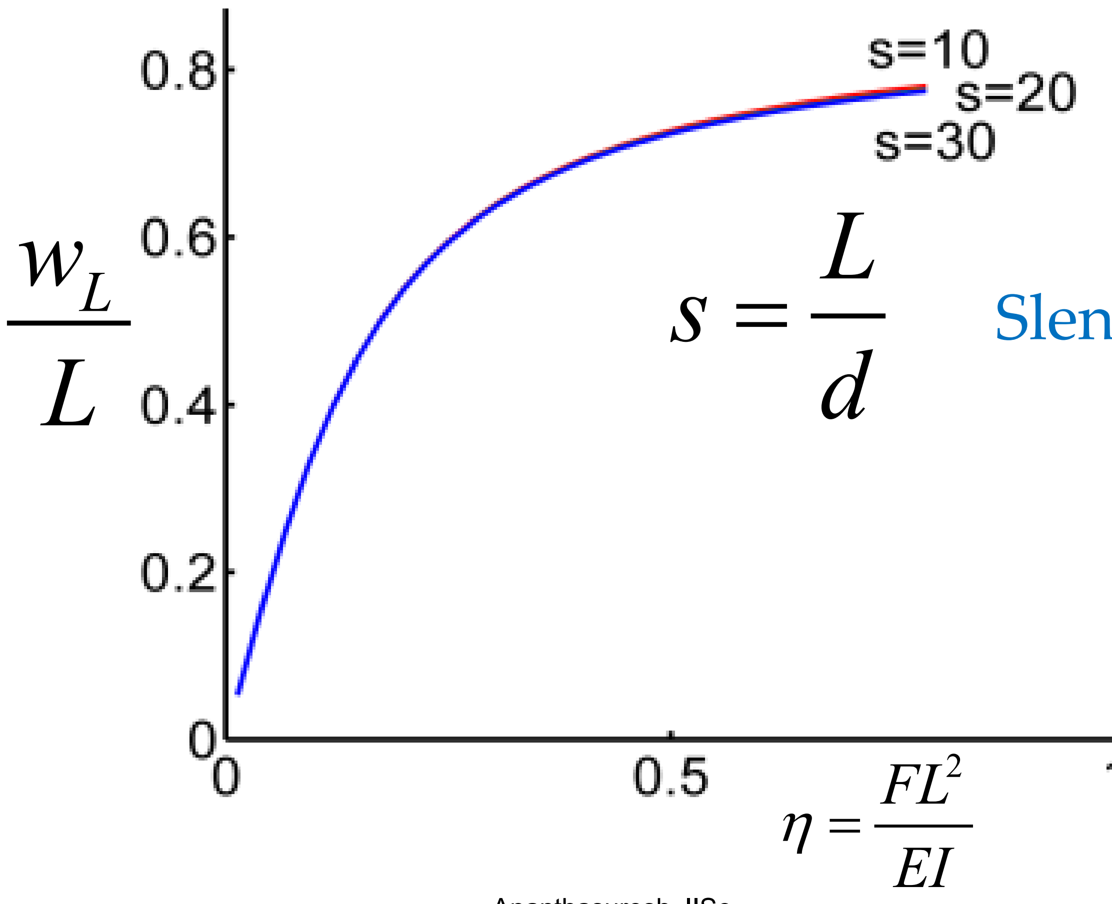
Slenderness ratio = $s = \dfrac{L}{d}$





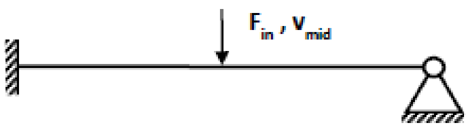
$$\eta = \frac{FL^2}{EI} \Rightarrow \eta = \frac{F\,\overline{s}^2}{\overline{E}\overline{b}\overline{d}}$$

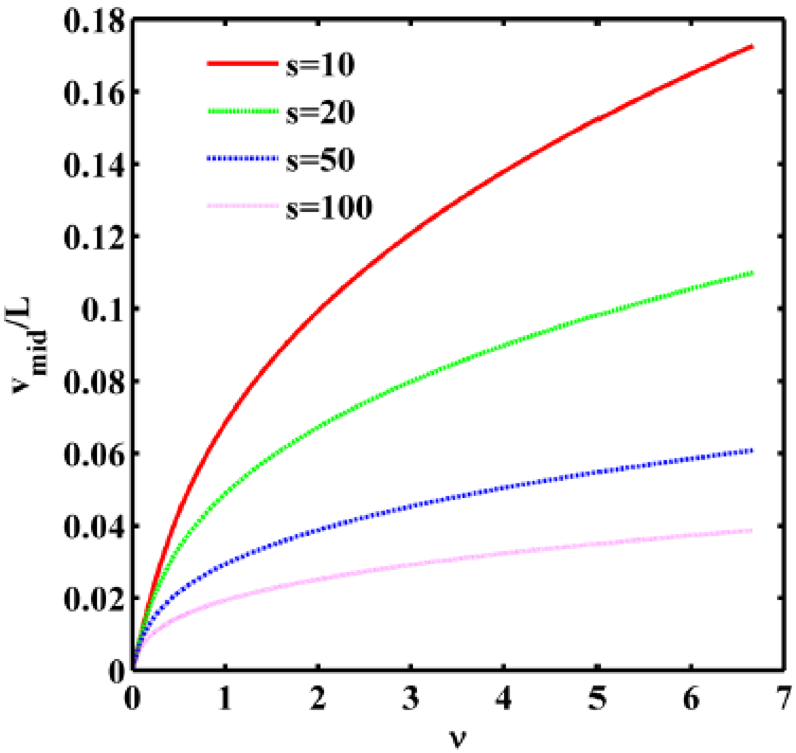
Comparing with geometrically nonlinear finite element analysis



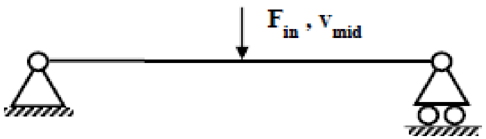
Slenderness ratio

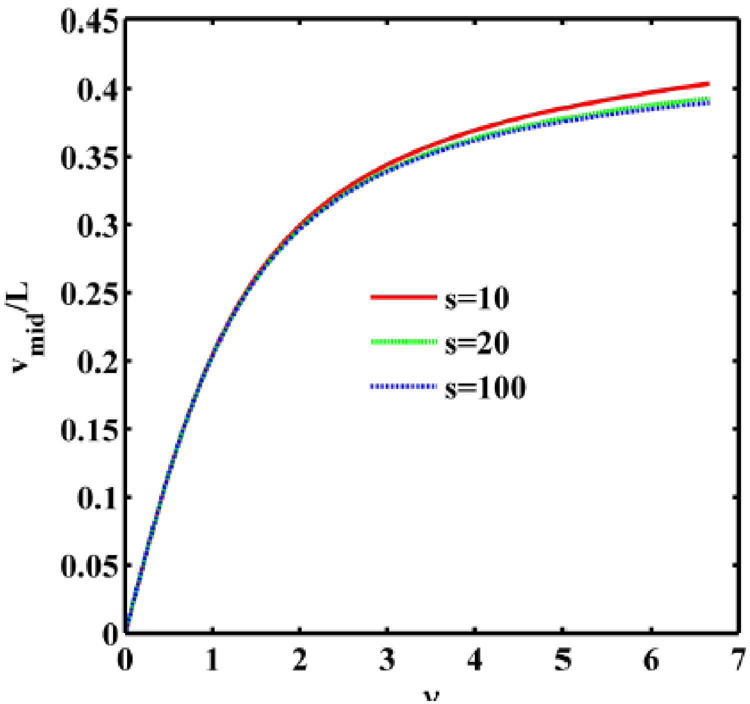
A fixed-pinned beam





A simply supported beam





Buckingham pi theorem

If there is a physically meaningful relationship $f(p_1, p_2, p_3, \ldots, p_n)$ in n physical parameters, then there is an equivalent relationship $\eta(\pi_1, \pi_2, \cdots, \pi_m)$ in terms of the non-dimensional parameters where $m = n - r$, r being the rank of the dimensional matrix.

$$
\begin{array}{c|cccccc|}
 & w & F & L & E & b & d \\
M & 0 & 1 & 0 & 1 & 0 & 0 \\
L & 1 & 1 & 1 & -1 & 1 & 1 \\
T & 0 & -2 & 0 & -2 & 0 & 0
\end{array}
$$

$$m = 6 - 2 = 4$$

w / L

$s = L / d$

$$\eta = \frac{Fs^2}{Ebd}$$

$s_b = L / b$ or $a = \dfrac{b}{d}$

Four non-dimensional parameters

Nullspace gives possible non-dimensional parameters

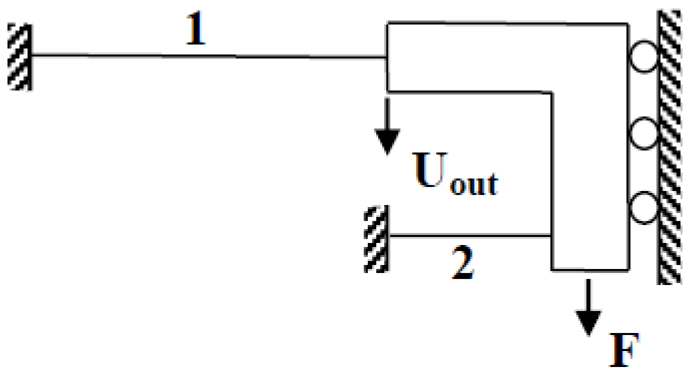
$$\begin{array}{c} \\ M \\ L \\ T \end{array}\begin{array}{c} \begin{array}{cccccc} w & F & L & E & b & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 0 & -2 & 0 & -2 & 0 & 0 \end{bmatrix} \end{array} \begin{Bmatrix} \alpha_w \\ \alpha_F \\ \alpha_L \\ \alpha_E \\ \alpha_b \\ \alpha_d \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

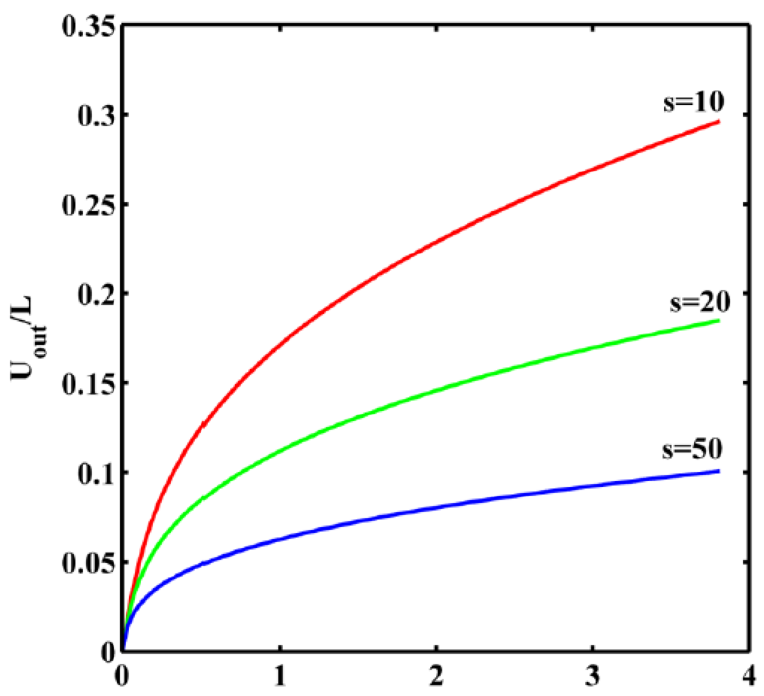
$$w^{\alpha_w} \; F^{\alpha_F} \; L^{\alpha_L} \; E^{\alpha_E} \; b^{\alpha_b} \; d^{\alpha_d} = \left[M^0 L^0 T^0 \right]$$

How about a compliant mechanism?

- Will this hold?

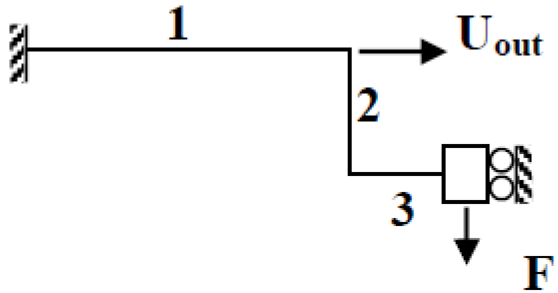
Two beams taken together

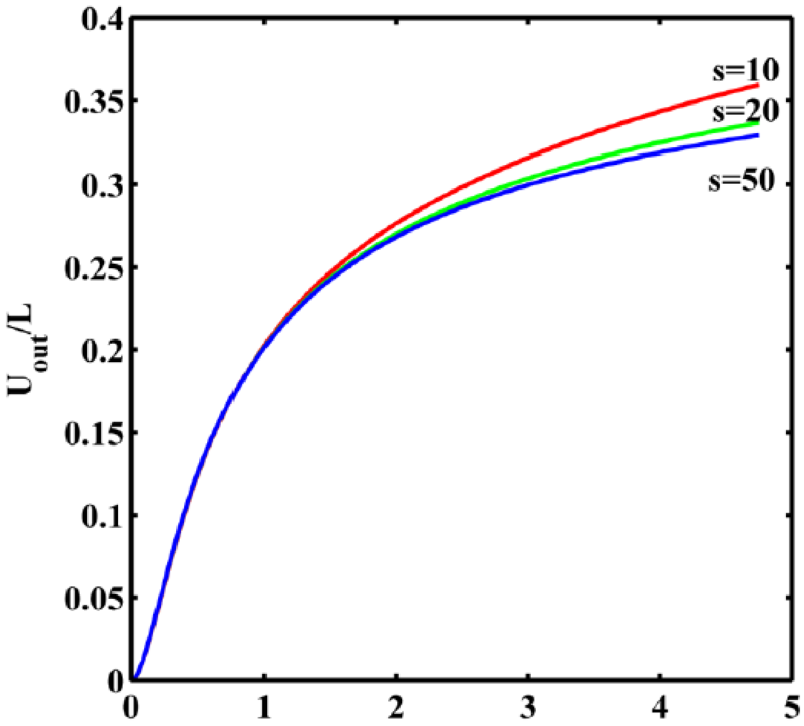




$$\eta = \frac{Fs^2}{Ebd}$$

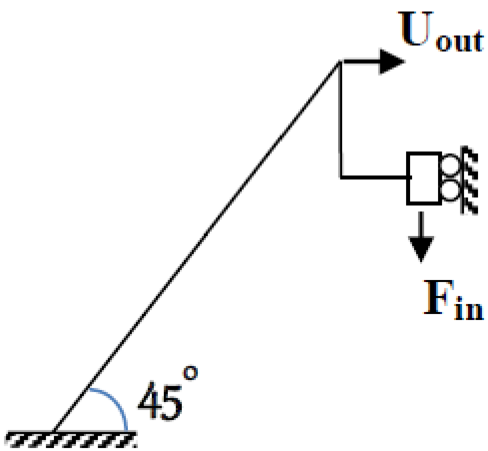
A stepped-beam

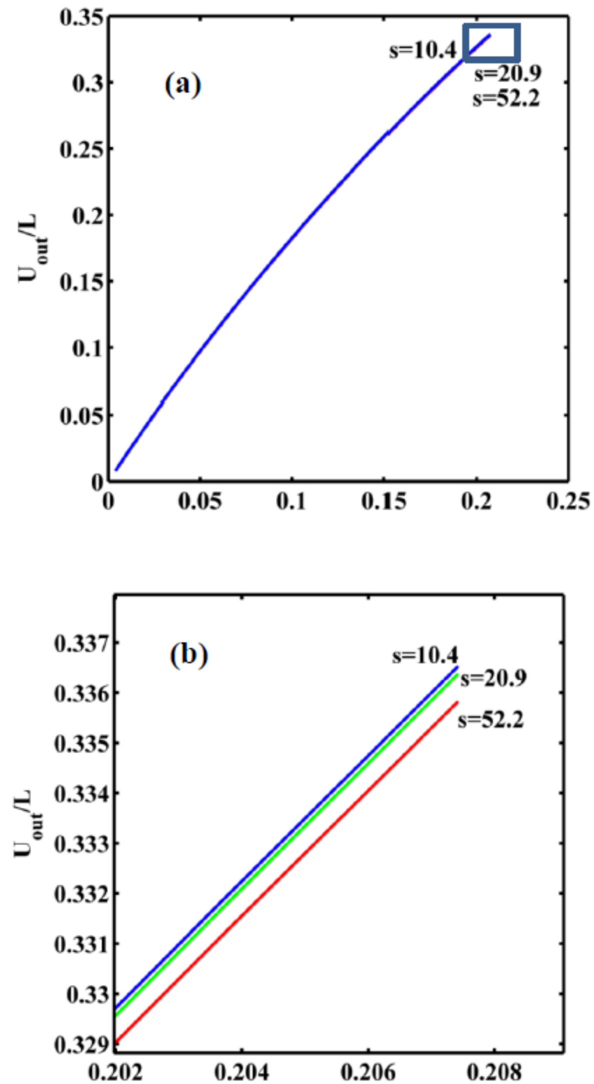




$$\eta = \frac{Fs^2}{Ebd}$$

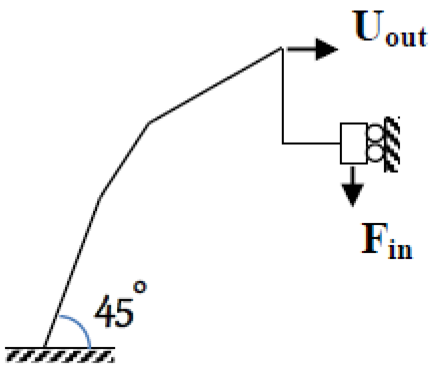
Approaching a compliant mechanism

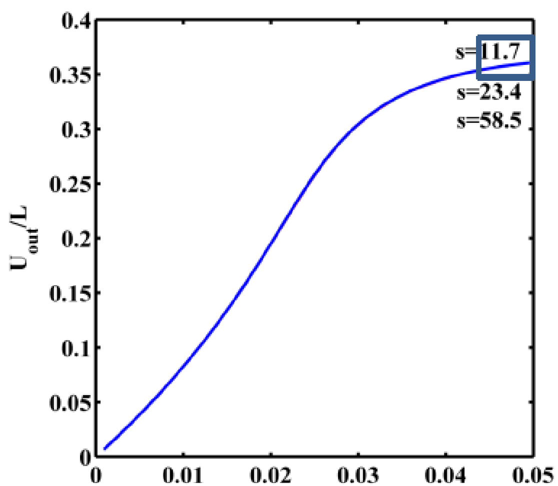




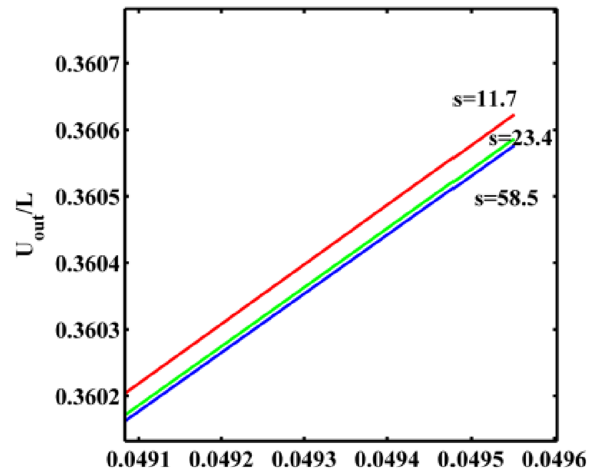
$$\eta = \frac{Fs^2}{Ebd}$$

A compliant mechanism!



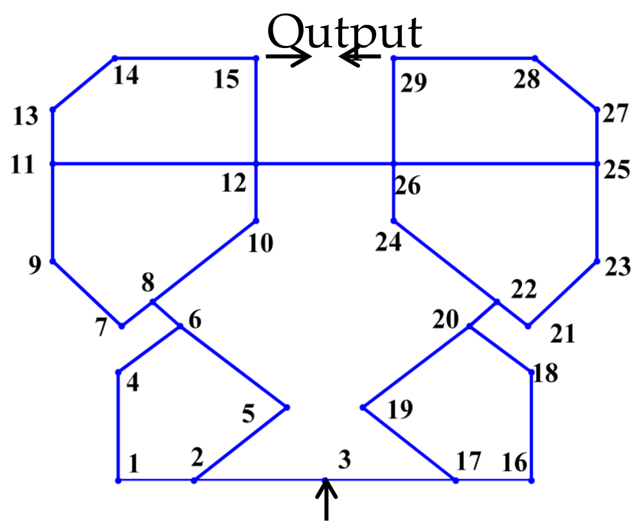


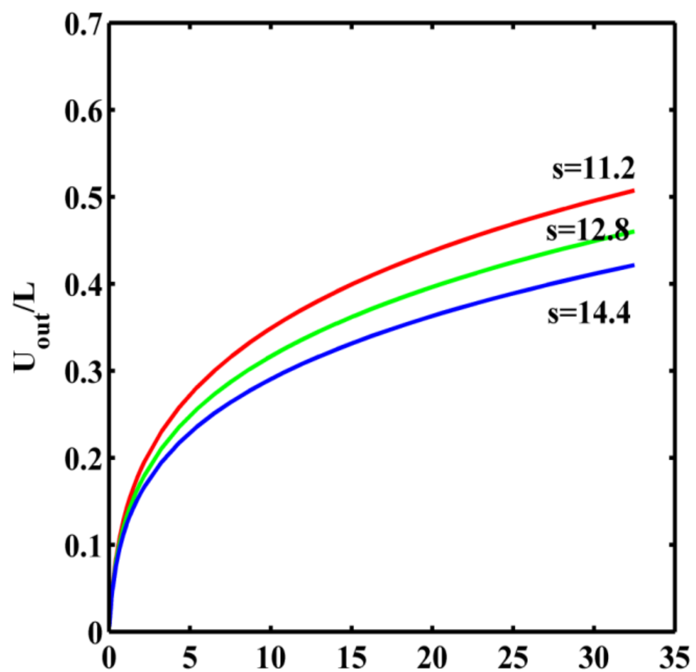
$$\eta = \frac{Fs^2}{Ebd}$$



$$\eta = \frac{Fs^2}{Ebd}$$

A compliant mechanism





$$s = \frac{\sum_{i=1}^{N} s_i}{N_s}; d = \frac{\sum_{i=1}^{N} d_i}{N}; b = \frac{\sum_{i=1}^{N} b_i}{N}$$

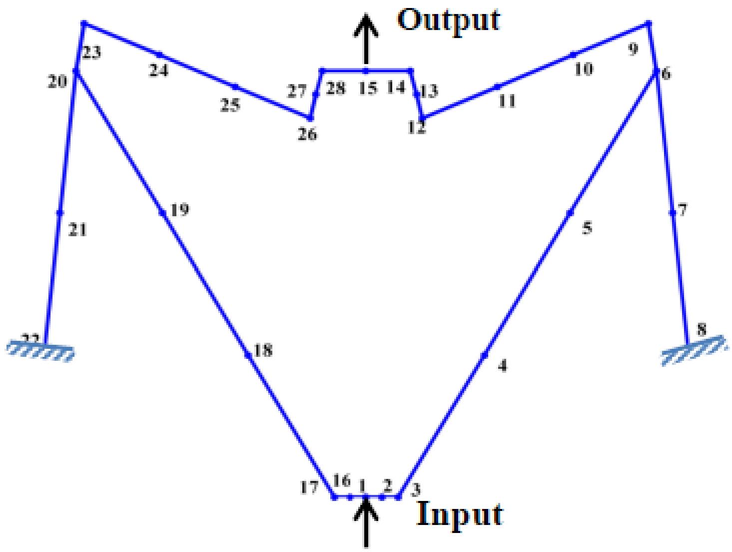
$$\eta = \frac{\overline{F}\,\overline{s}^2}{\overline{E}\,\overline{b}\,\overline{d}}$$

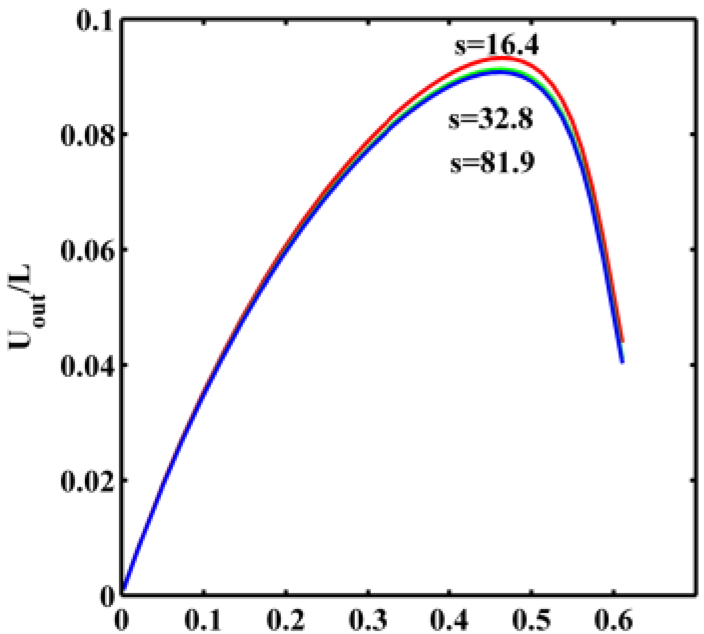
N – number of beam elements

N_s- number of beam segments

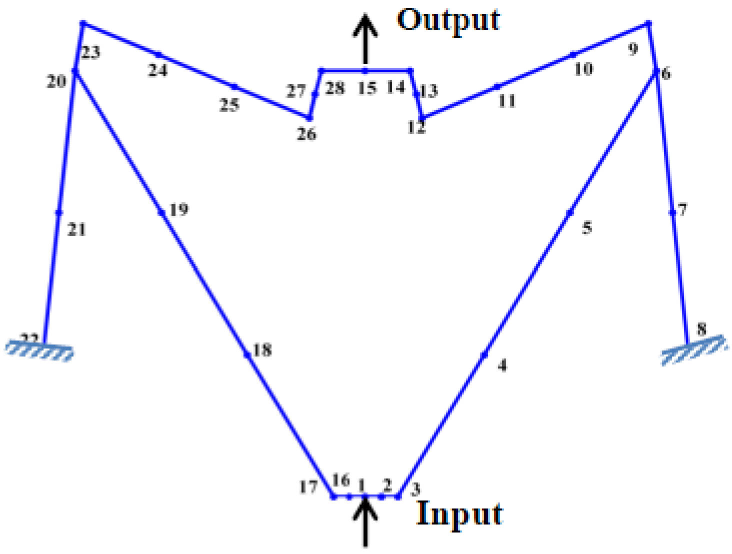
$$\eta = \frac{FL^2}{Ebd^3}$$

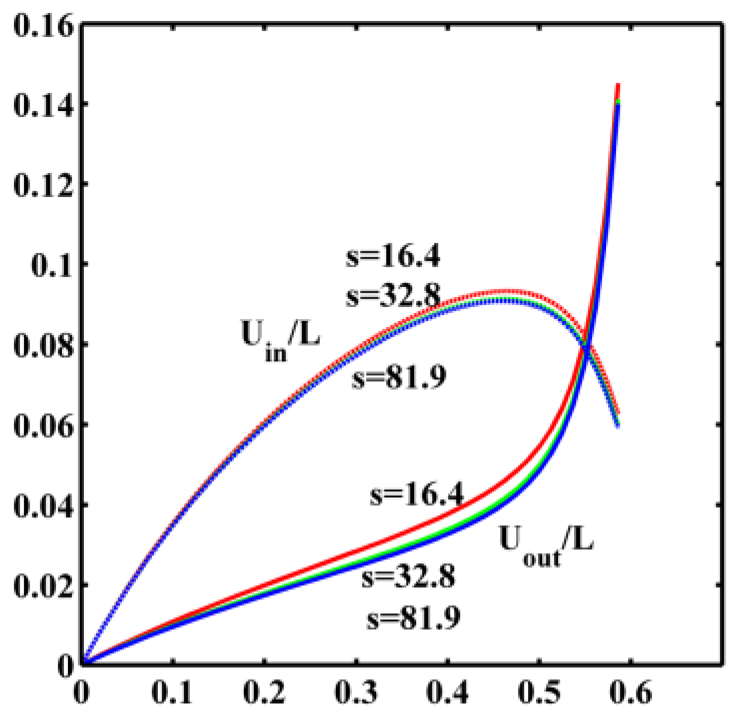
Another example





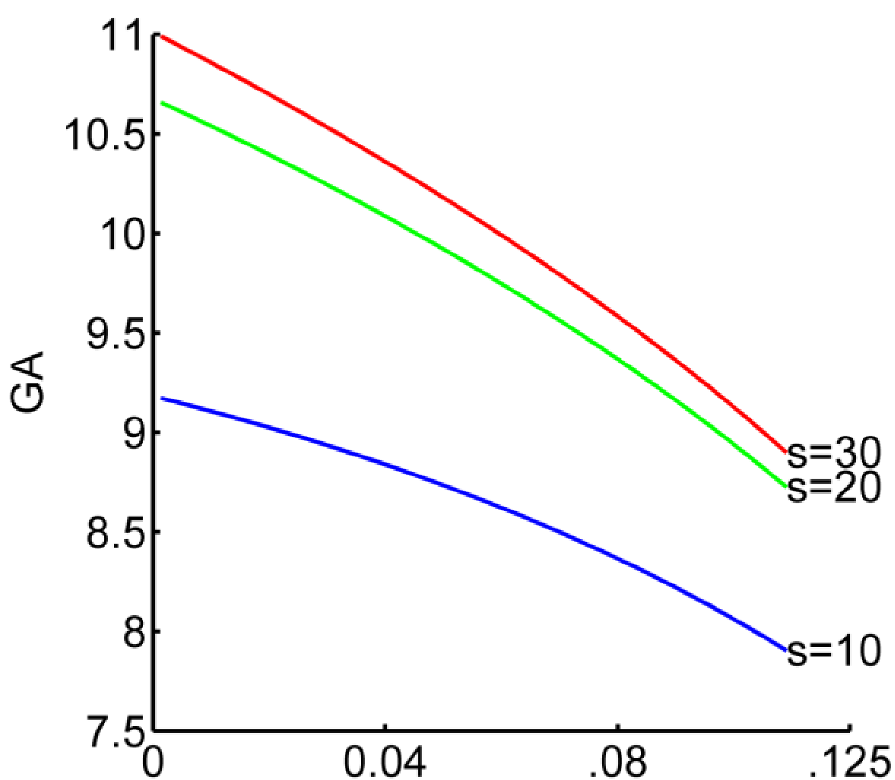
Input and out displacements



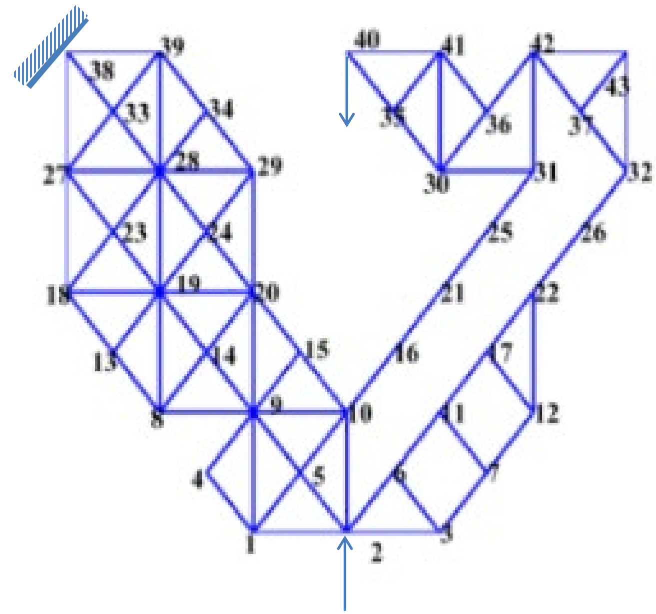


A general compliant mechanism

GA = geometric advantage



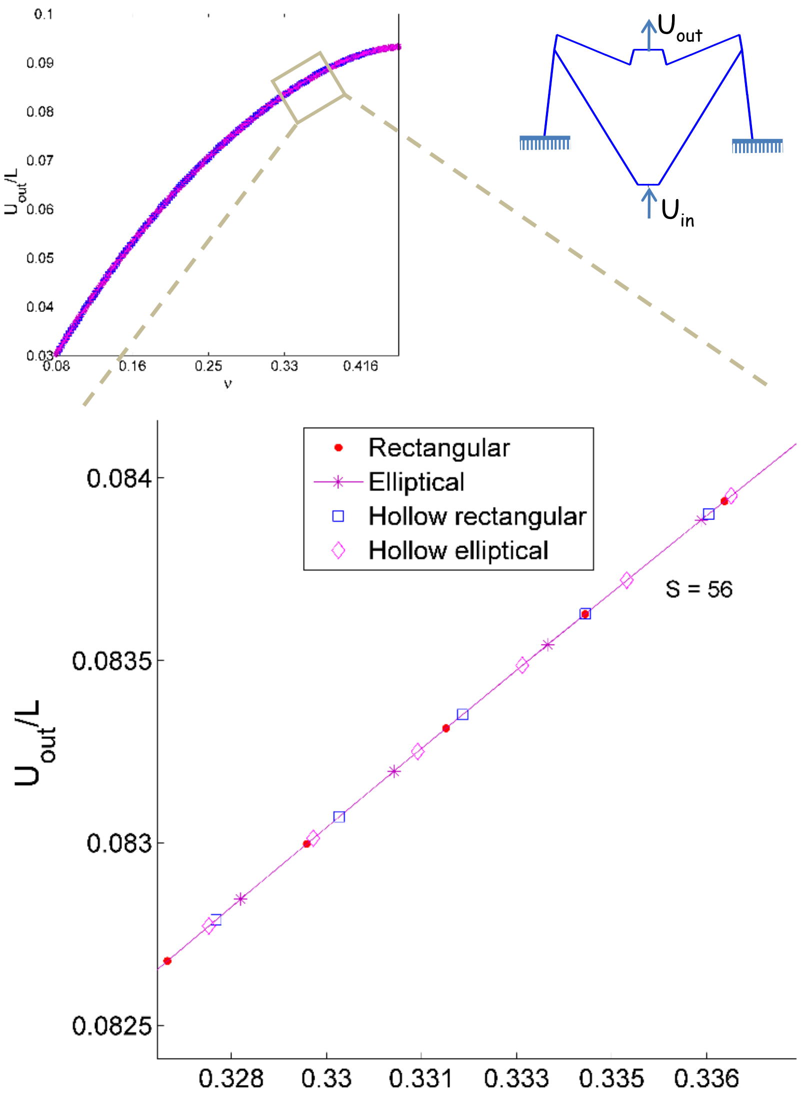
$$\eta = \frac{FL^2}{EI}$$



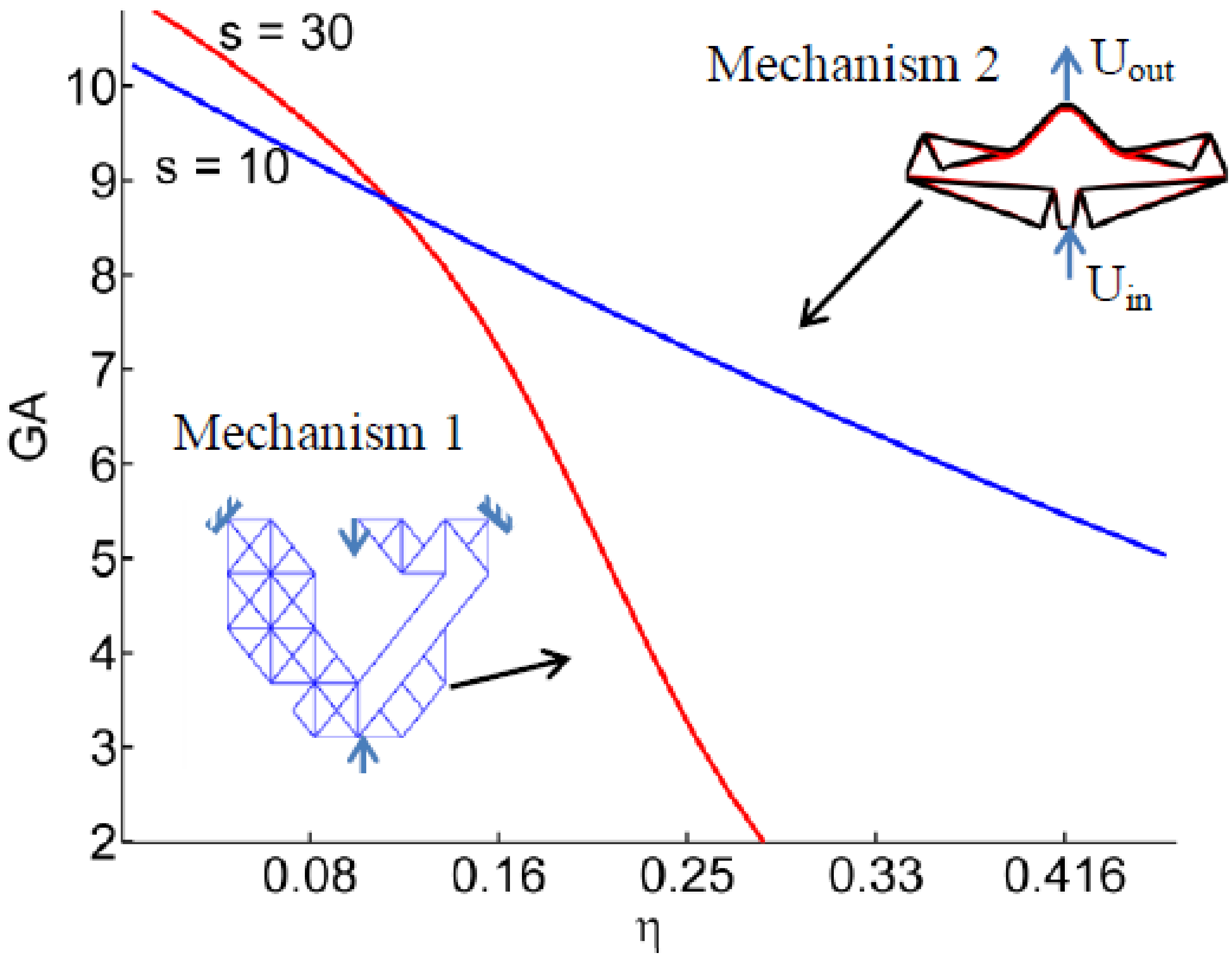
$$\eta = \frac{\bar{F}\,\bar{s}^2}{\bar{E}\,\bar{b}\,\bar{d}}$$

Quantities with over-bars are representative of all the beam segments for preserving the proportions.

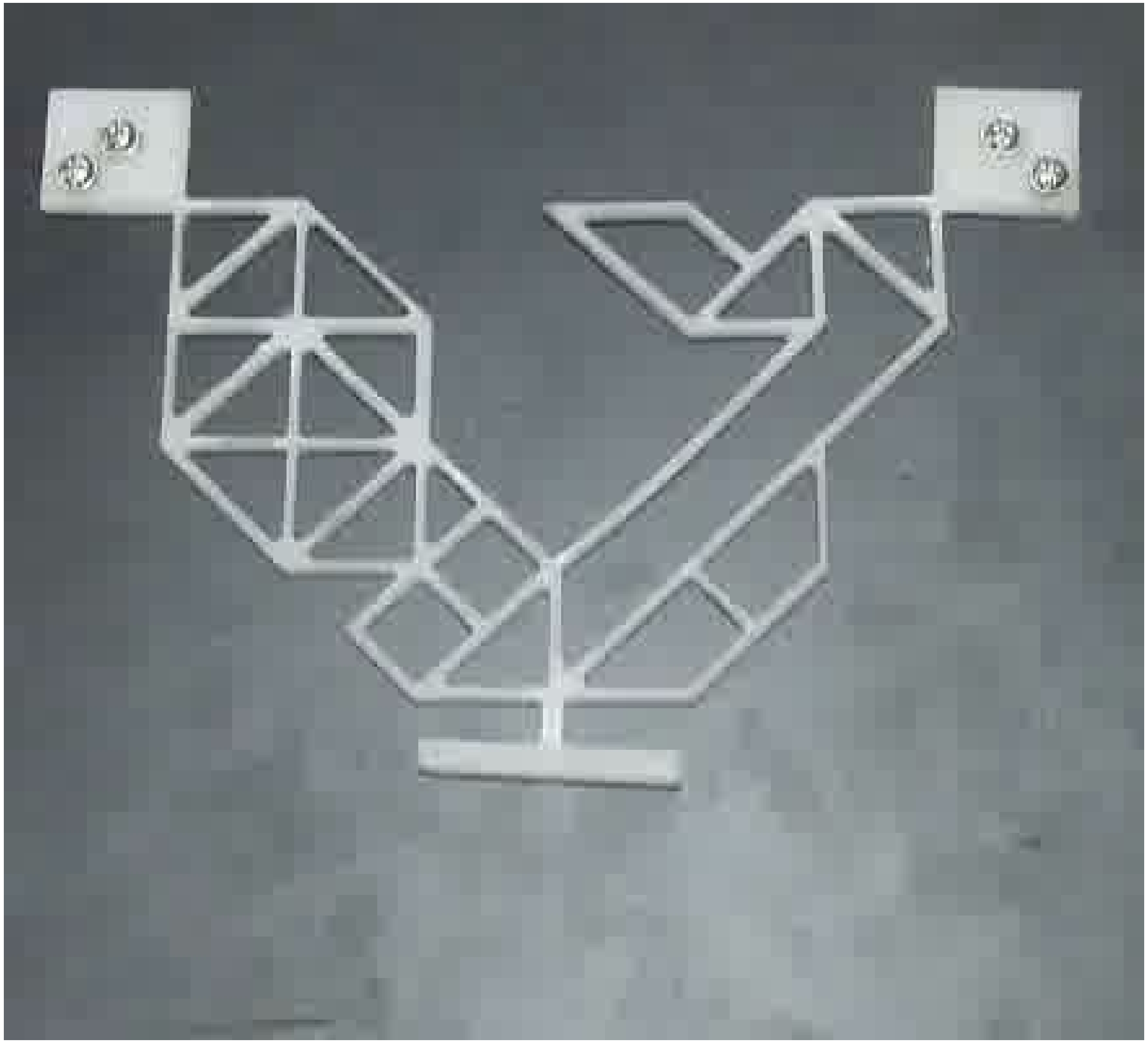
Cross-section shape does not matter.



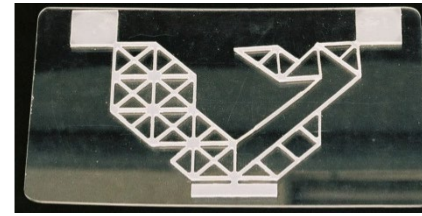
Comparing two topologies

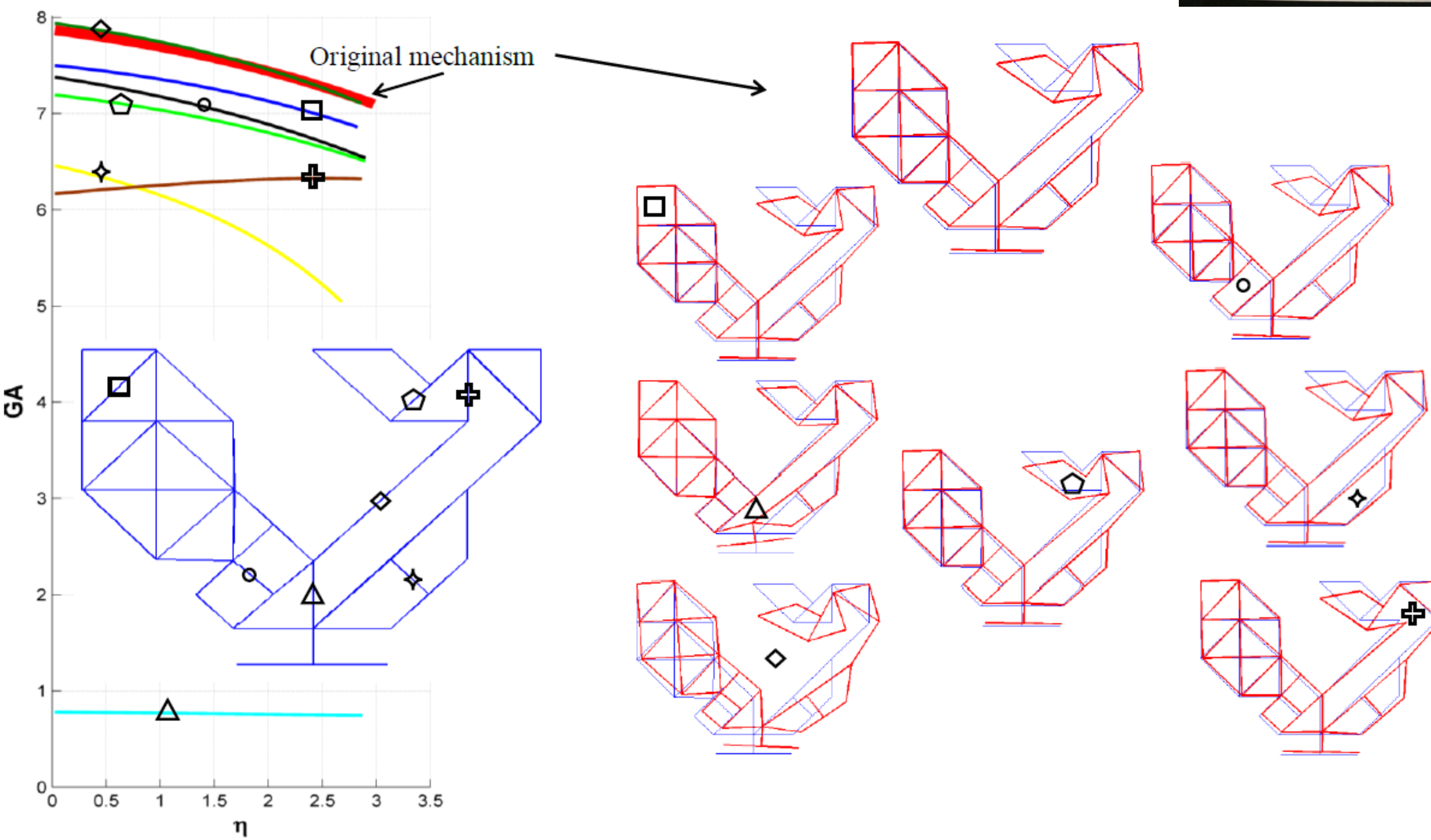


Which segment is the most crucial in this?

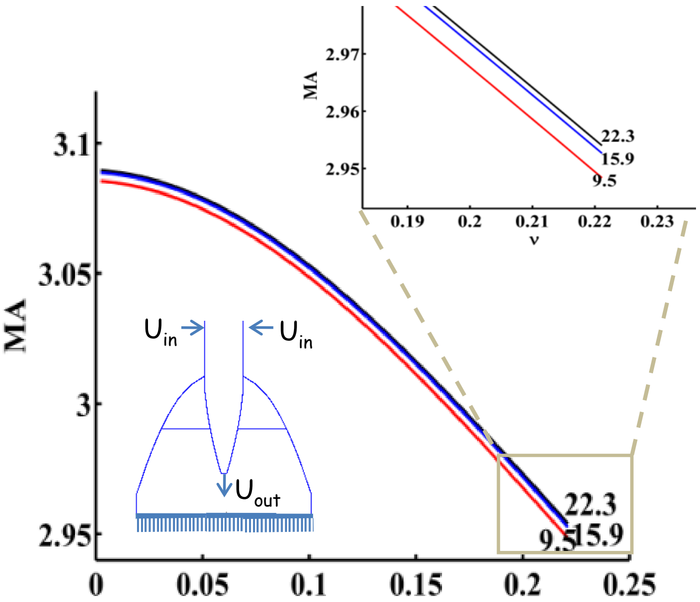


Which beam segment is the most crucial?

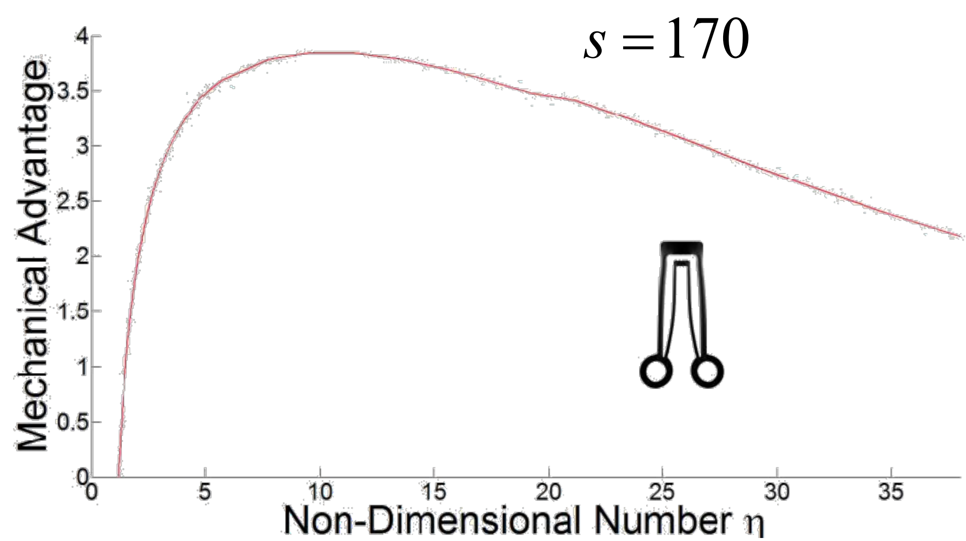




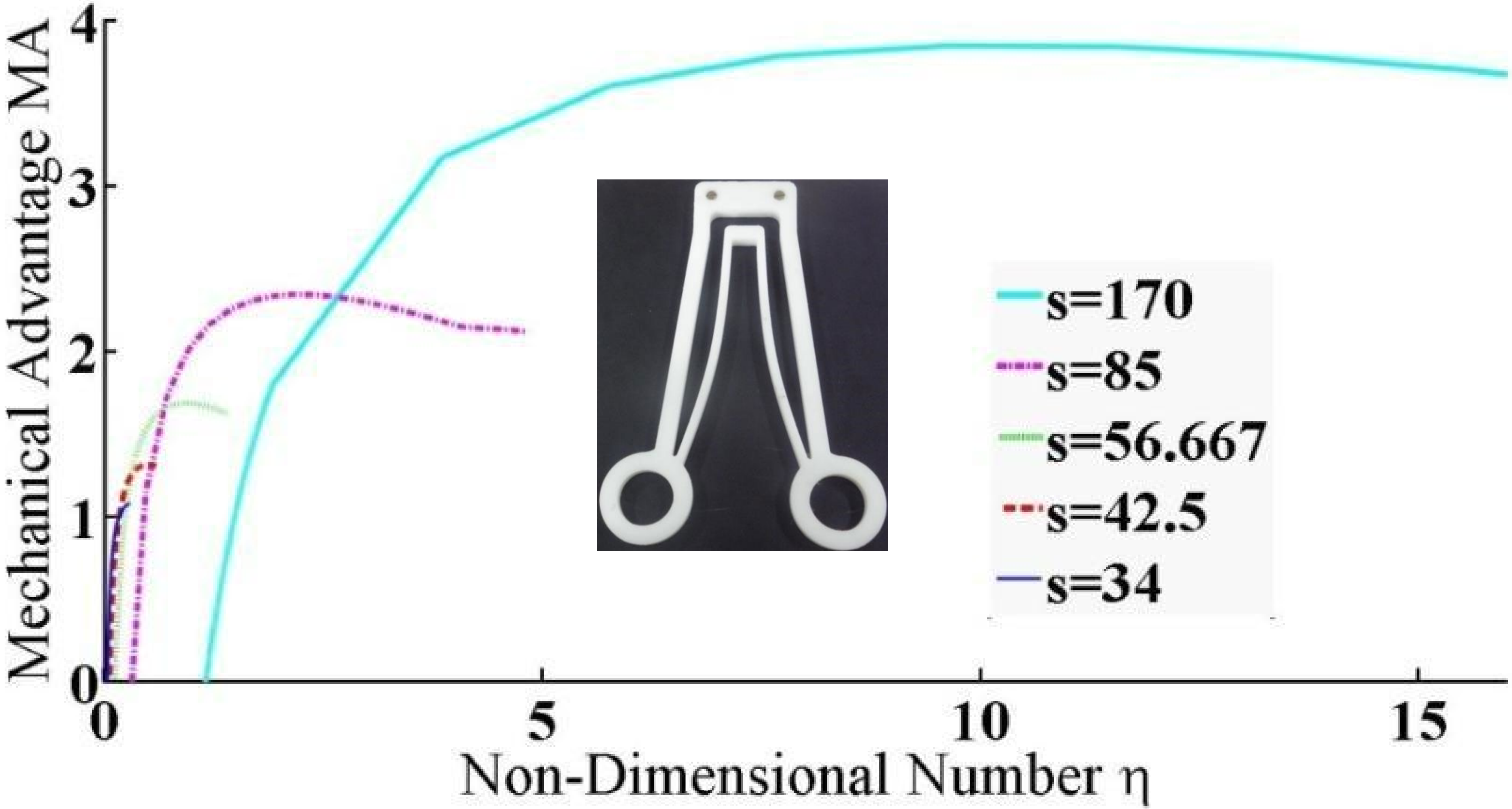
Mechanical advantage (MA)



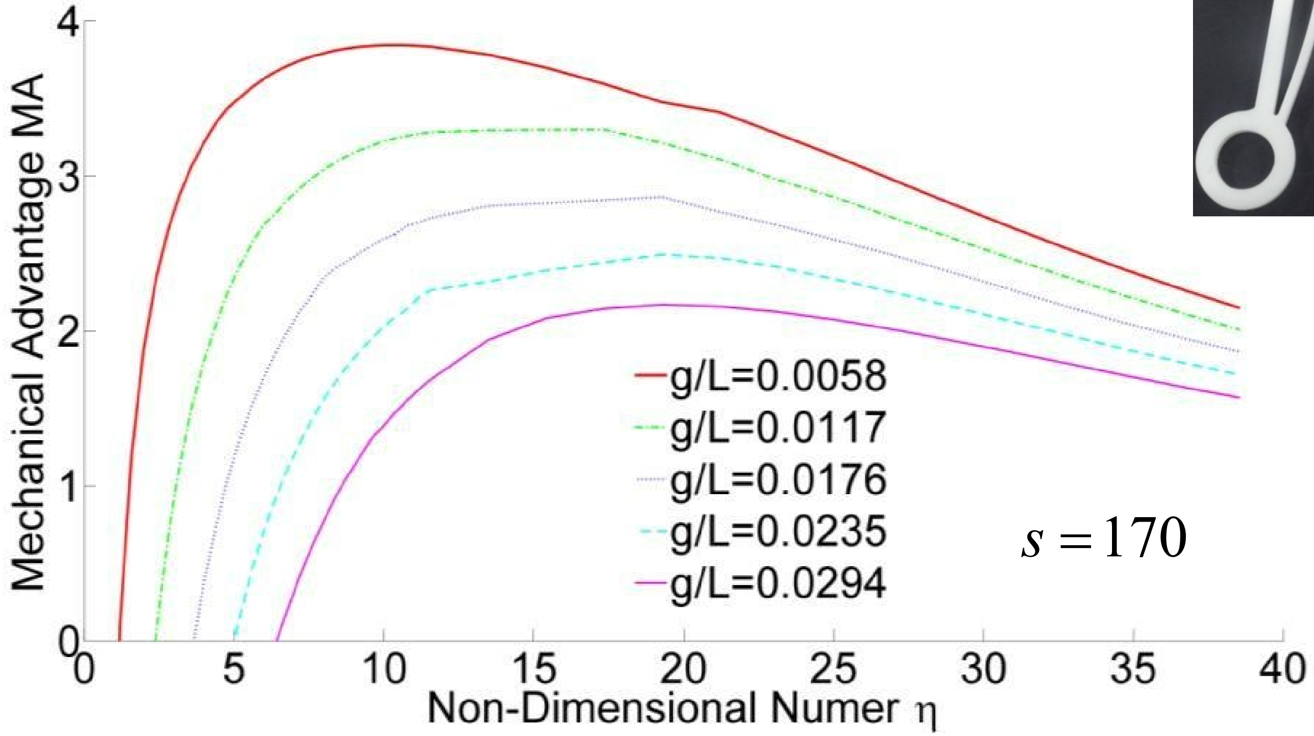
$$\eta = \frac{\bar{\bar{F}}\ \bar{s}^2}{\bar{E}\,\bar{b}\,\bar{d}}$$



MA with slenderness ratio

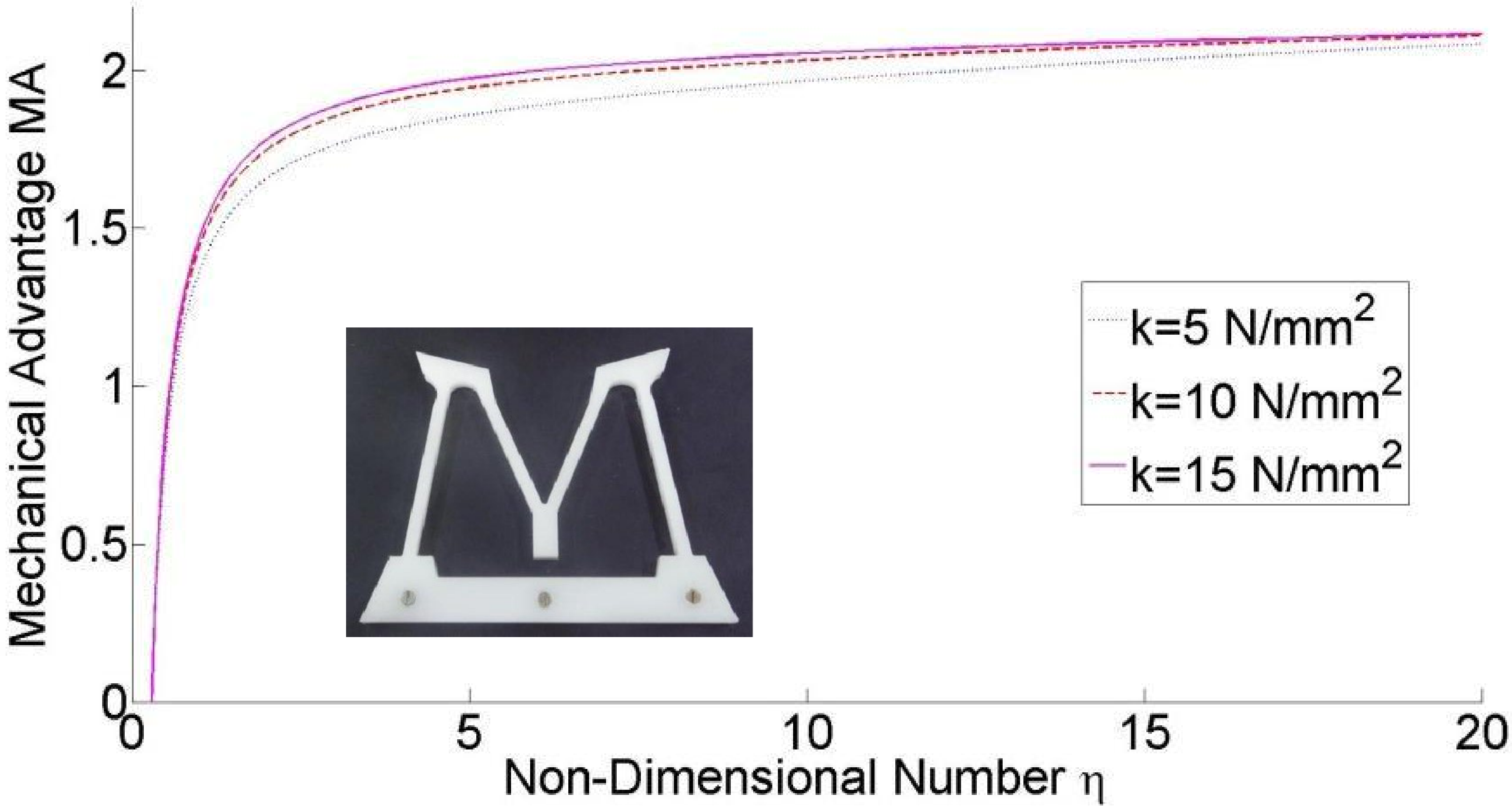


MA with non-dimensional gap



$s = 170$

Sensitivity of MA with workpiece stiffness



Further reading

Bhargav, S. D. B., Varma, H. I., and Ananthasuresh, G. K., "Non-dimensional Kinetoelastostatic Maps for Compliant Mechanisms," Proc. ASME 2013 International Design Engineering Technical Conferences, August 4-7, 2013, Portland, Oregon, USA, Paper no. DETC2013-12178.

"Feasible and Intrinsic Kinetoelastostatic Maps for Designing Compliant Mechanisms," Harish V. Indukuri, MSc (Eng.) thesis, IISc, 2013.