ME 254

Elastic similarity in large displacement analysis of beams

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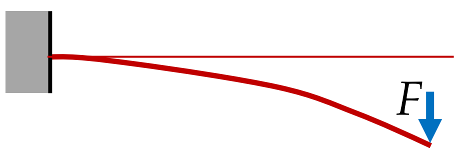
Important equations

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{2\frac{F}{EI}\left(\sin\theta_L - \sin\theta\right)}} = \int_0^L ds = L$$

$$\theta \rightarrow \phi$$

$$\sin\theta = 2p^2\sin^2\phi - 1$$

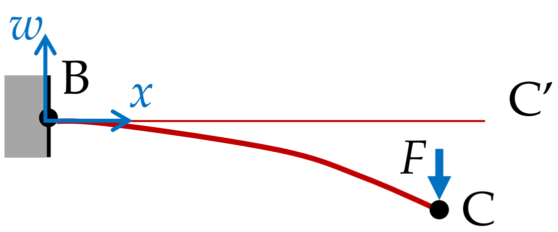
$$p^2 = \frac{1+\sin\theta_L}{2}$$



$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\pi/2} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} = \sqrt{\eta} = \sqrt{\frac{FL^2}{EI}}$$

$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\pi/2} \frac{d\phi}{\sqrt{\frac{F}{EI}}\sqrt{1-p^2\sin^2\phi}} = L$$

Displacements at the loaded tip



$$\eta = \sqrt{\frac{FL^2}{EI}}$$

$$\frac{w_L}{L} = \sqrt{\frac{1}{\eta}}\left\{\mathbf{F}(p,\pi/2) - \mathbf{F}(p,\phi_0) - 2\mathbf{E}(p,\pi/2) + 2\mathbf{E}(p,\phi_0)\right\}$$

$$\frac{x_L}{L} = \sqrt{\frac{2EI}{FL^2}\left(2p^2 - 1\right)} = \frac{2p}{\sqrt{\eta}}\cos\phi_B$$

Examine the limits

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{2\frac{F}{EI}\left(\sin\theta_L - \sin\theta\right)}} = \int_0^L ds = L$$

$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\pi/2} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} = \sqrt{\eta} = \sqrt{\frac{FL^2}{EI}}$$

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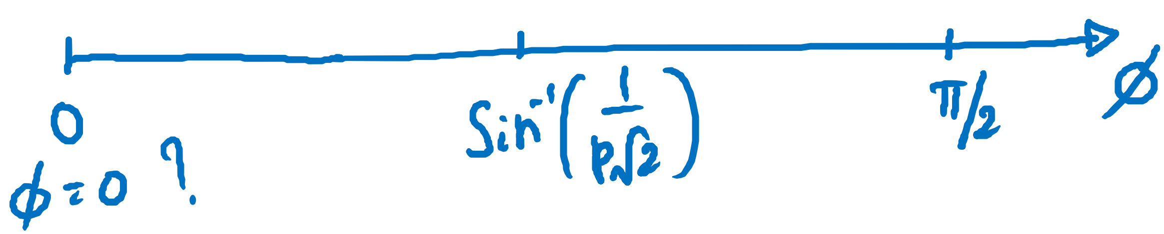


Geometric interpretation of $\theta \to \phi$

$$\sin\theta = 2p^2 \sin^2\phi - 1$$

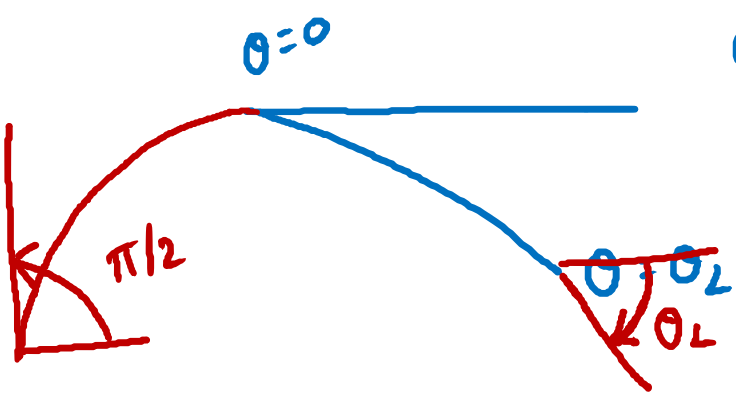
$$p^2 = \frac{1+\sin\theta_L}{2}$$

$\phi = 0 \Rightarrow \sin\theta_0 = 0 - 1$

$@\ \phi = 0$

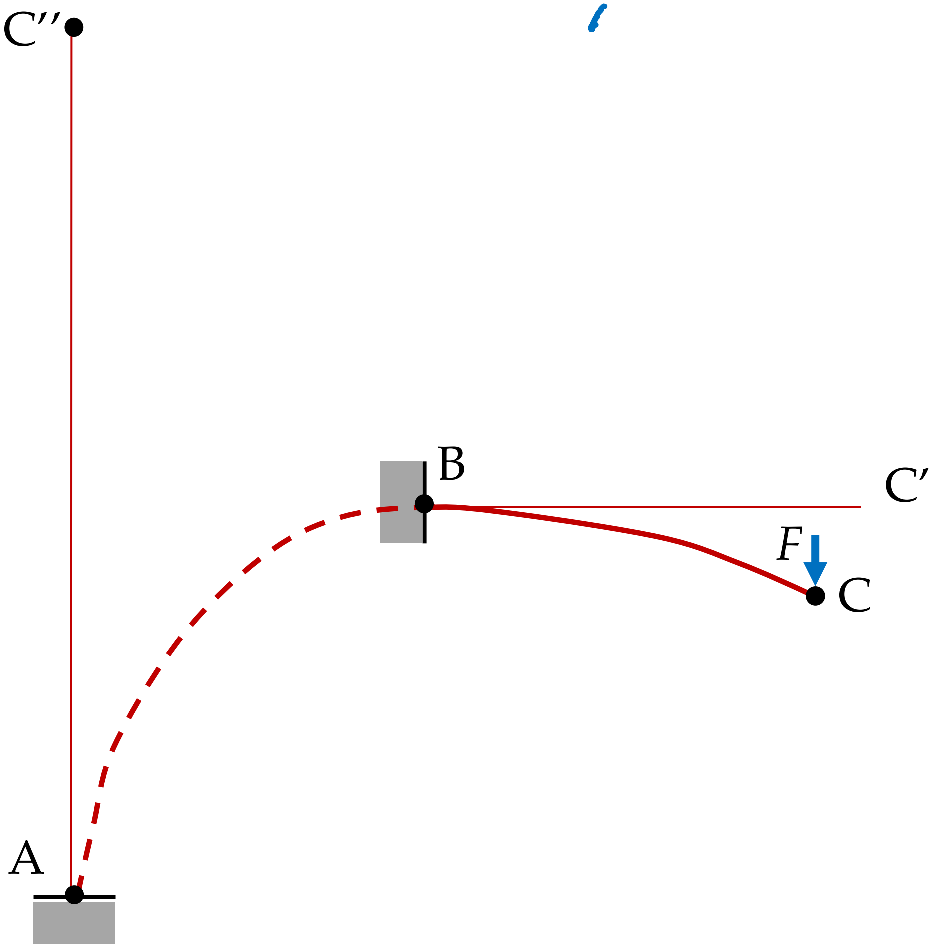
$\Rightarrow \phi = 0$

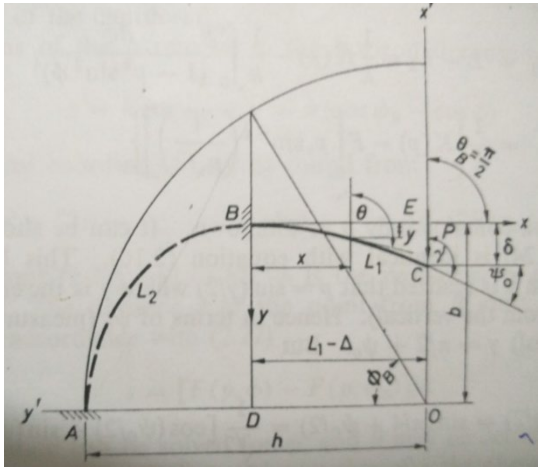
$\theta_{\phi=0} = -\pi/2$



Elastic similarity

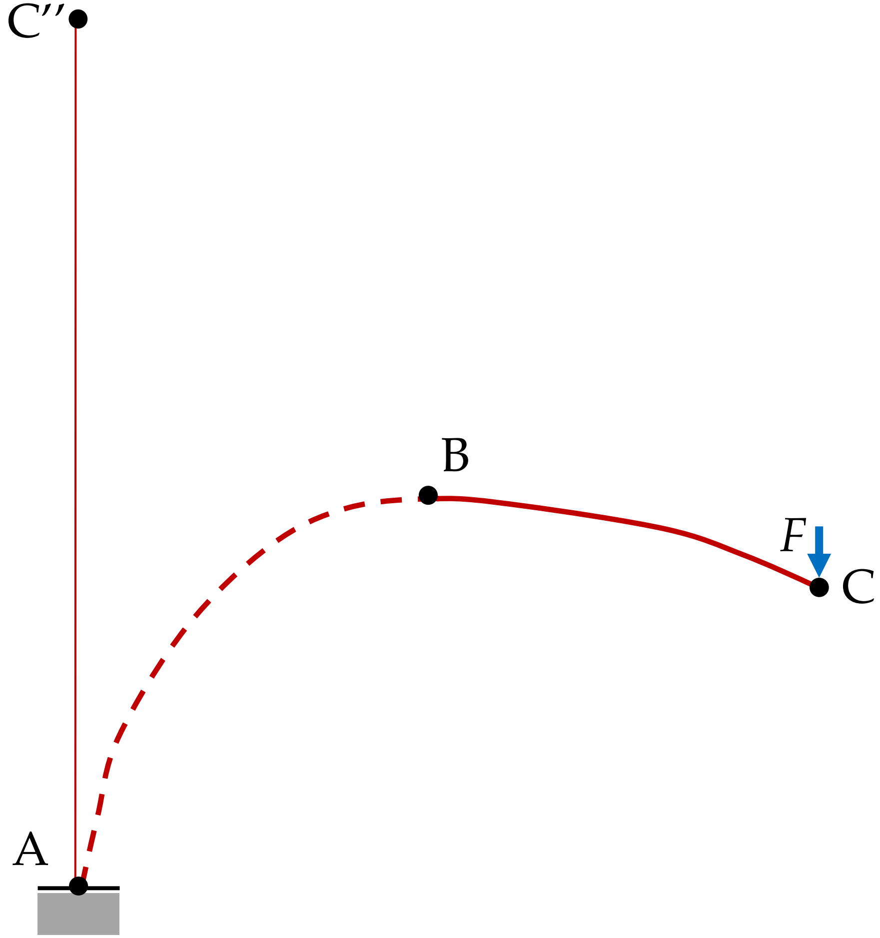
Frisch-Fay, R., *Flexible Bars*, 1962.

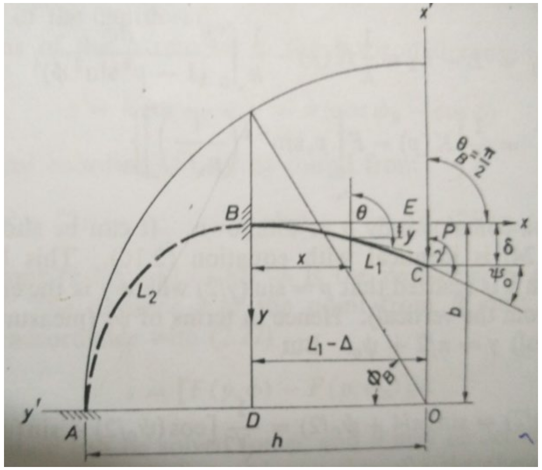




Frisch-Fay, R., *Flexible Bars*, 1962.

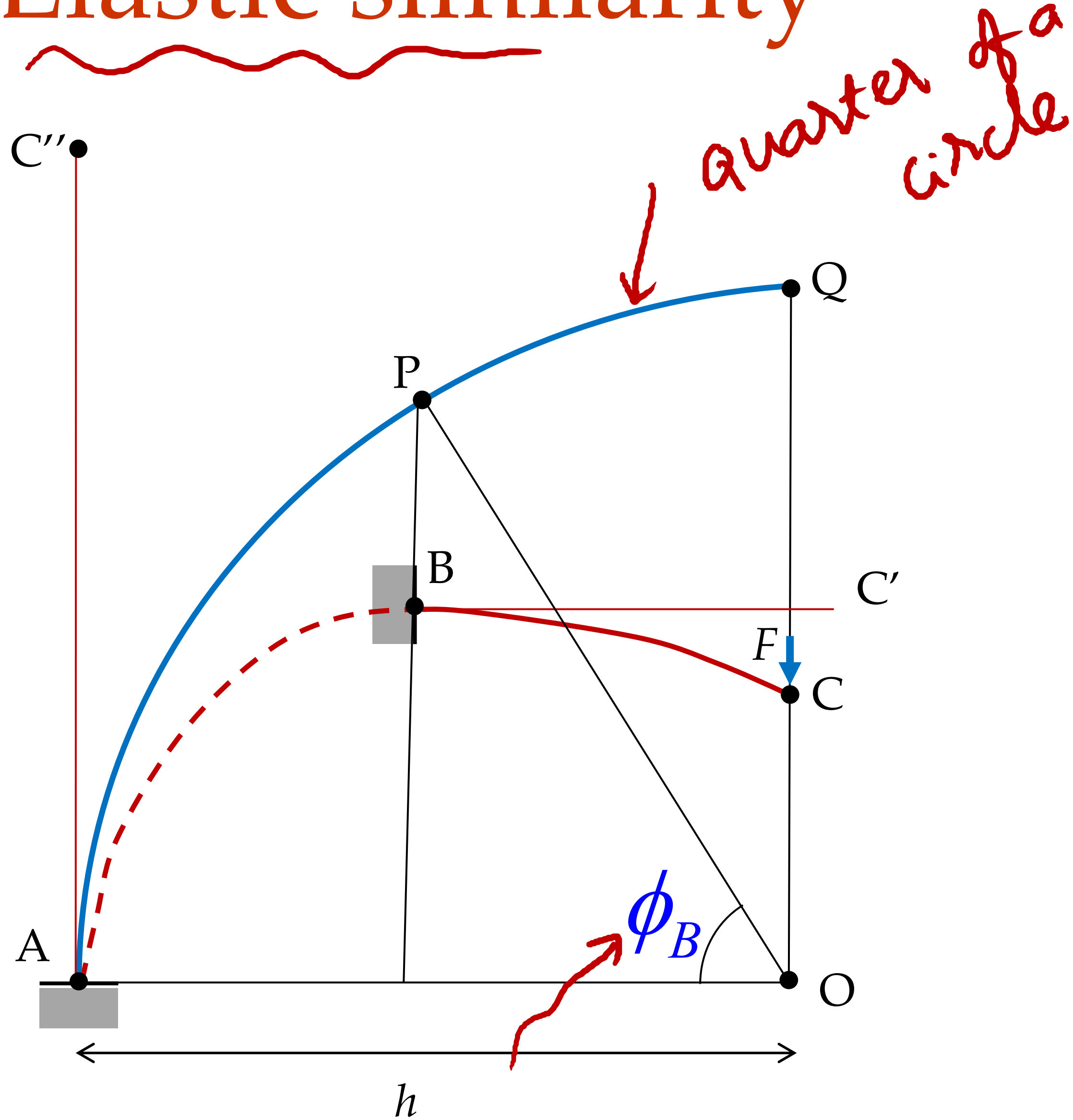
Elastic similarity

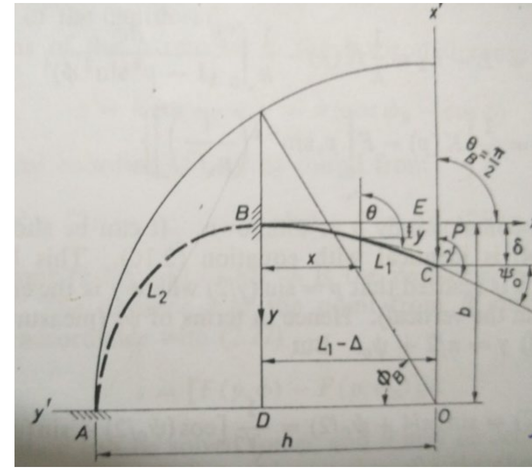




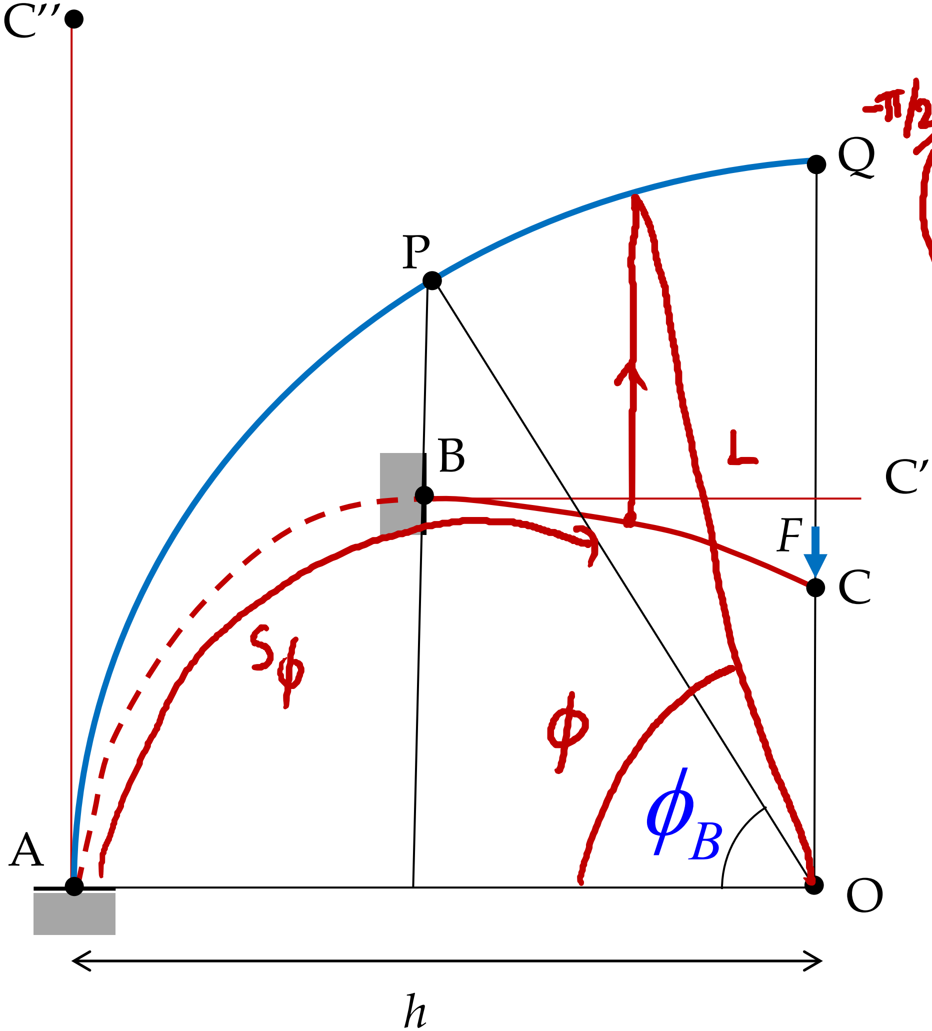
Elastic similarity

Frisch-Fay, R., *Flexible Bars*, 1962.





Arc-length interpretation



$$\int_{-\pi/2}^{\theta} \frac{d\theta}{\sqrt{2\frac{F}{EI}\left(\sin\theta_L - \sin\theta\right)}} = \int_0^{s_\theta} ds = s_\theta$$

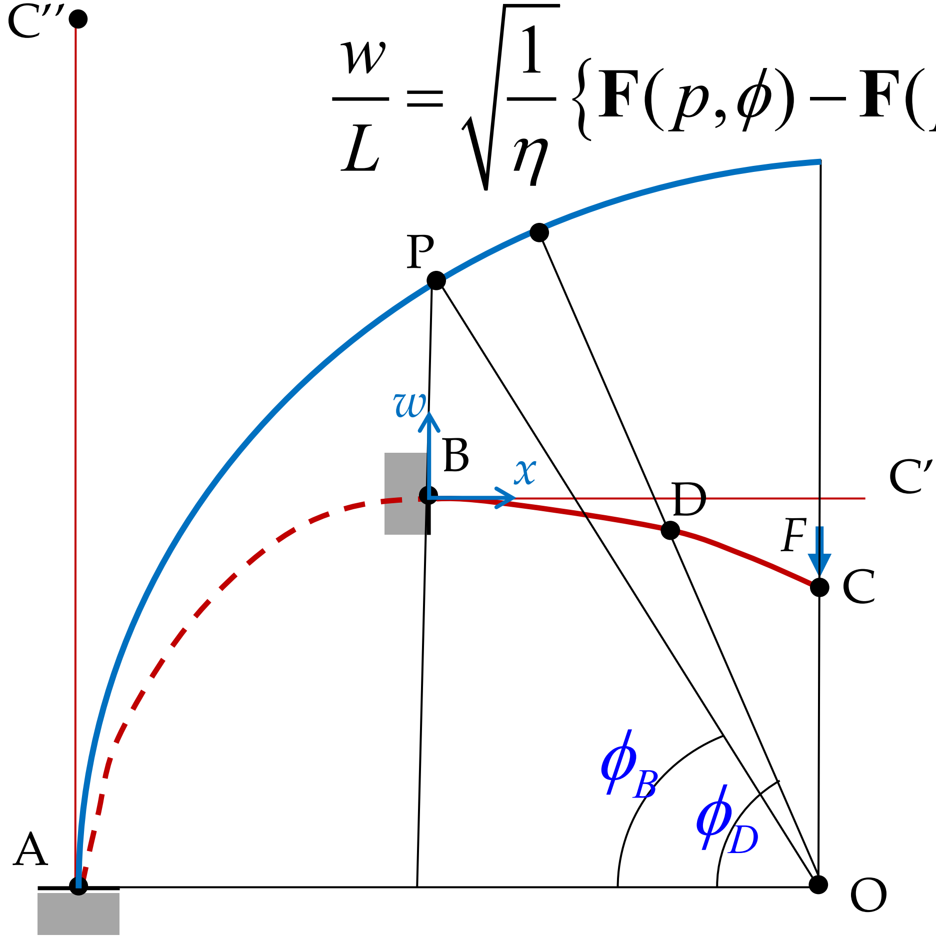
$$\int_0^{\phi} \frac{d\phi}{\sqrt{\frac{F}{EI}}\sqrt{1-p^2\sin^2\phi}} = \int_0^{s_\phi} ds = s_\phi$$

$\boxed{\phi}$

Coordinates of the deflected profile at any point beyond B

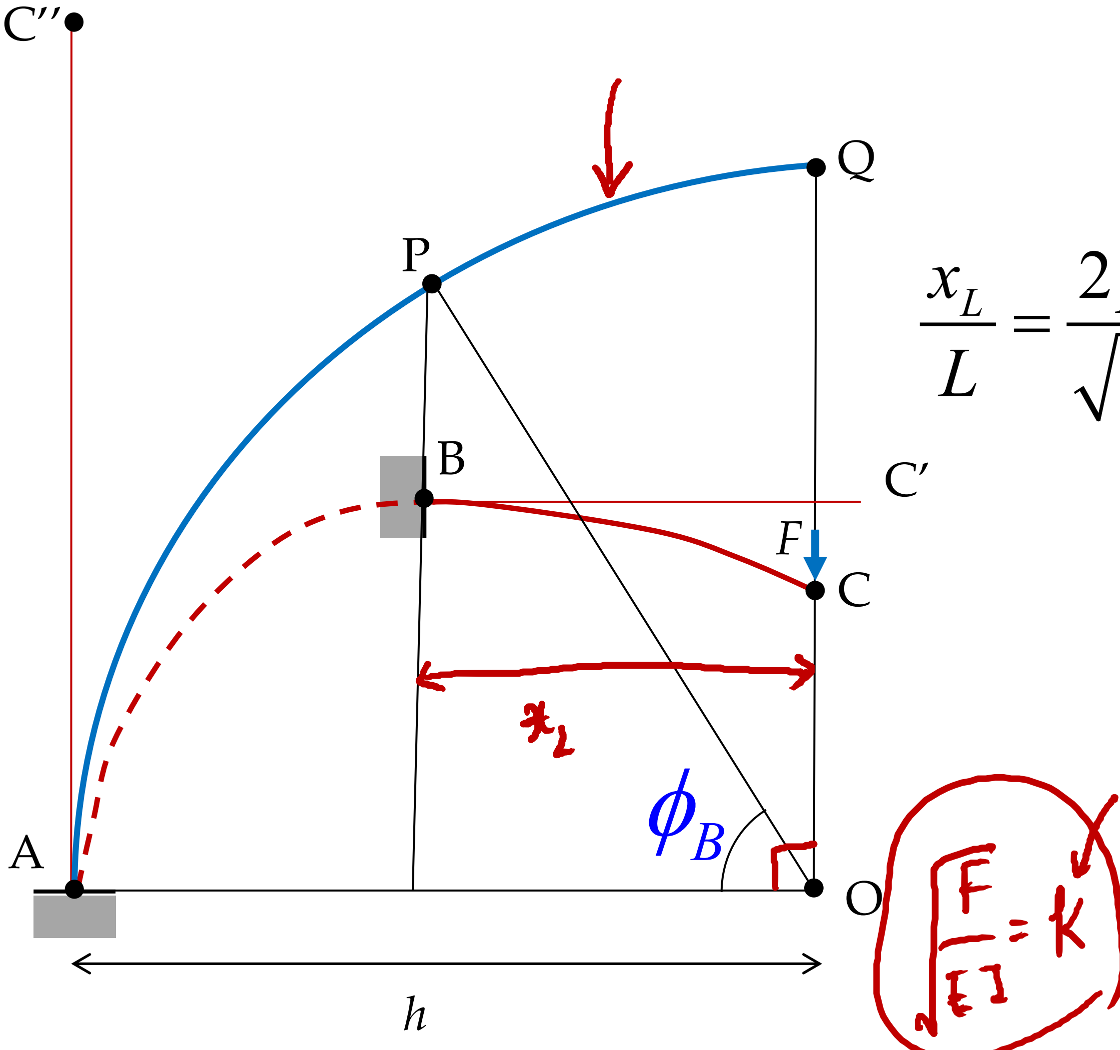
$$\frac{w}{L} = \sqrt{\frac{1}{\eta}} \left\{ \mathbf{F}(p,\phi) - \mathbf{F}(p,\phi_B) - 2\mathbf{E}(p,\phi) + 2\mathbf{E}(p,\phi_B) \right\}$$

$$\frac{x}{L} = \frac{2p}{\sqrt{\eta}} \left(\cos\phi_B - \cos\phi \right)$$



h = ? (radius of the phi-circle)

$\eta = \sqrt{\frac{FL^2}{EI}}$



$$\frac{x}{L} = \frac{2p}{\sqrt{\eta}}\left(\cos\phi_B - \cos\phi\right)$$

$$\frac{x_L}{L} = \frac{2p}{\sqrt{\eta}}\cos\phi_B \Rightarrow \cos\phi_B = \frac{\sqrt{\eta}\,x_L}{2pL}$$

$$\cos\phi_B = \frac{x_L}{h}$$

$$\cos\phi_B = \frac{x_L}{h} = \frac{\sqrt{\eta}\,x_L}{2pL}$$

$$\Rightarrow h = \frac{2pL}{\sqrt{\eta}} = \frac{2p}{\sqrt{F/EI}}$$

Non-dimensional portrayal

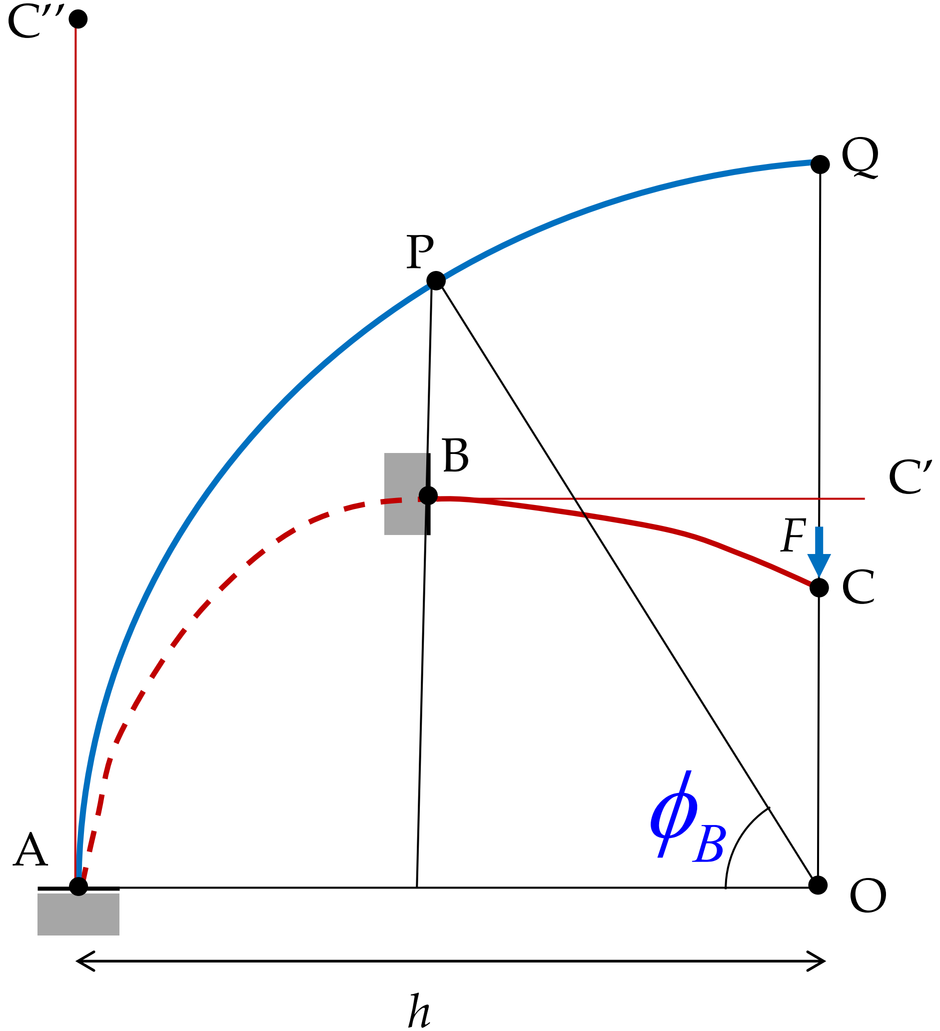
$$\int_{\sin^{-1}\left(1/p\sqrt{2}\right)}^{\pi/2} \frac{d\phi}{\sqrt{\frac{F}{EI}}\sqrt{1-p^2\sin^2\phi}} = L$$

$$\phi_0 = \phi_B$$

$$\frac{w_L}{L} = \sqrt{\frac{1}{\eta}}\left\{\mathbf{F}(p,\pi/2) - \mathbf{F}(p,\phi_0) - 2\mathbf{E}(p,\pi/2) + 2\mathbf{E}(p,\phi_0)\right\}$$

$$\frac{x_L}{L} = \sqrt{\frac{2EI}{FL^2}\left(2p^2-1\right)} = \frac{2p}{\sqrt{\eta}}\cos\phi_B$$

Arc-length interpretation

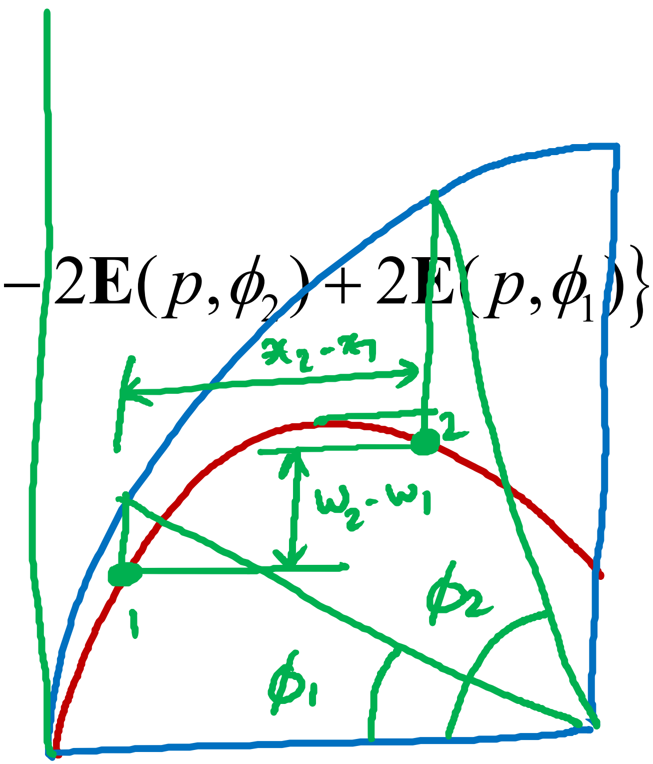


Non-dimensional portrayal

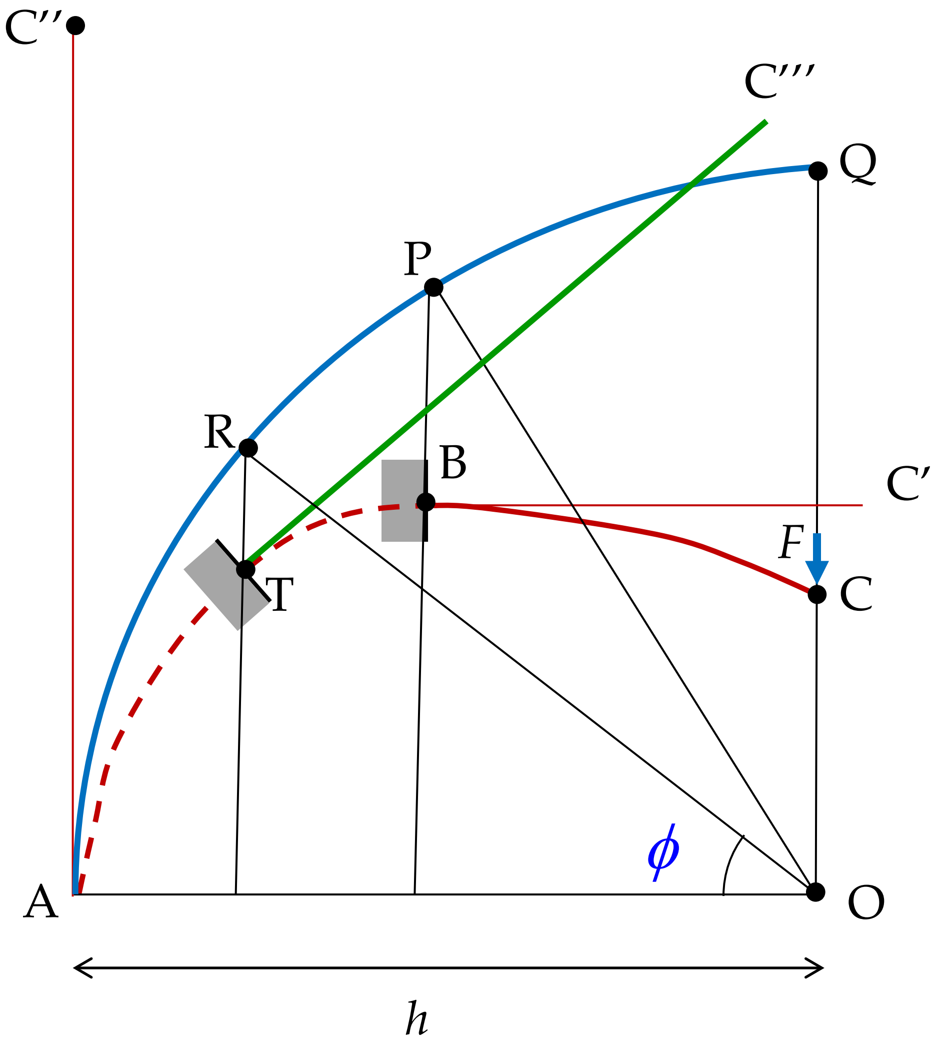
$$\int_{\phi_1}^{\phi_2} \frac{d\phi}{\sqrt{\frac{F}{EI}}\sqrt{1-p^2\sin^2\phi}} = \int_{s_1}^{s_2} ds = s_2 - s_1$$

$$\frac{w_2 - w_1}{L} = \sqrt{\frac{1}{\eta}}\left\{\mathbf{F}(p,\phi_2) - \mathbf{F}(p,\phi_1) - 2\mathbf{E}(p,\phi_2) + 2\mathbf{E}(p,\phi_1)\right\}$$

$$\frac{x_2 - x_1}{L} = \frac{2p}{\sqrt{\frac{F}{EI}}}\left(\cos\phi_2 - \cos\phi_1\right)$$

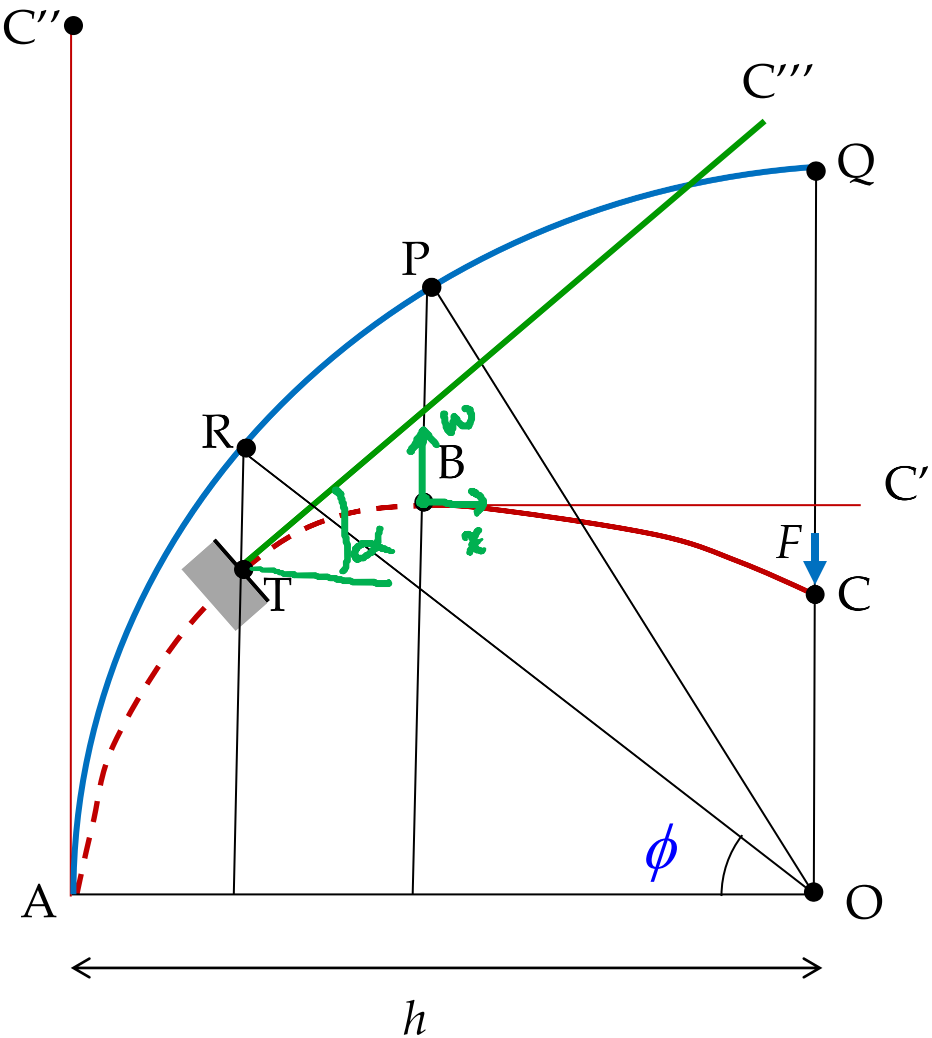


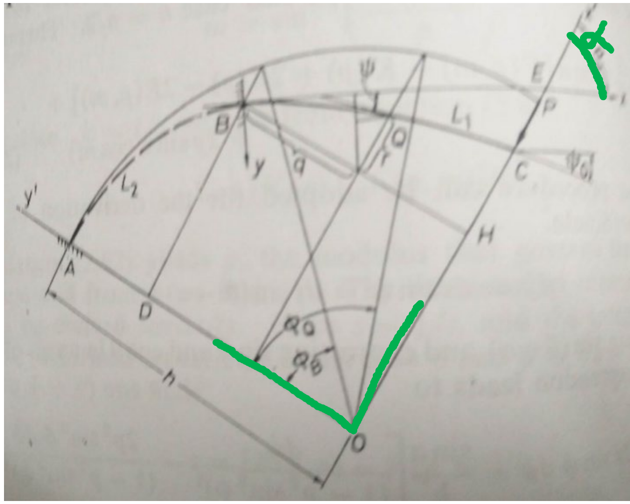
Inclined load



Inclined load

Frisch-Fay, R., *Flexible Bars*, 1962.





Inclined load

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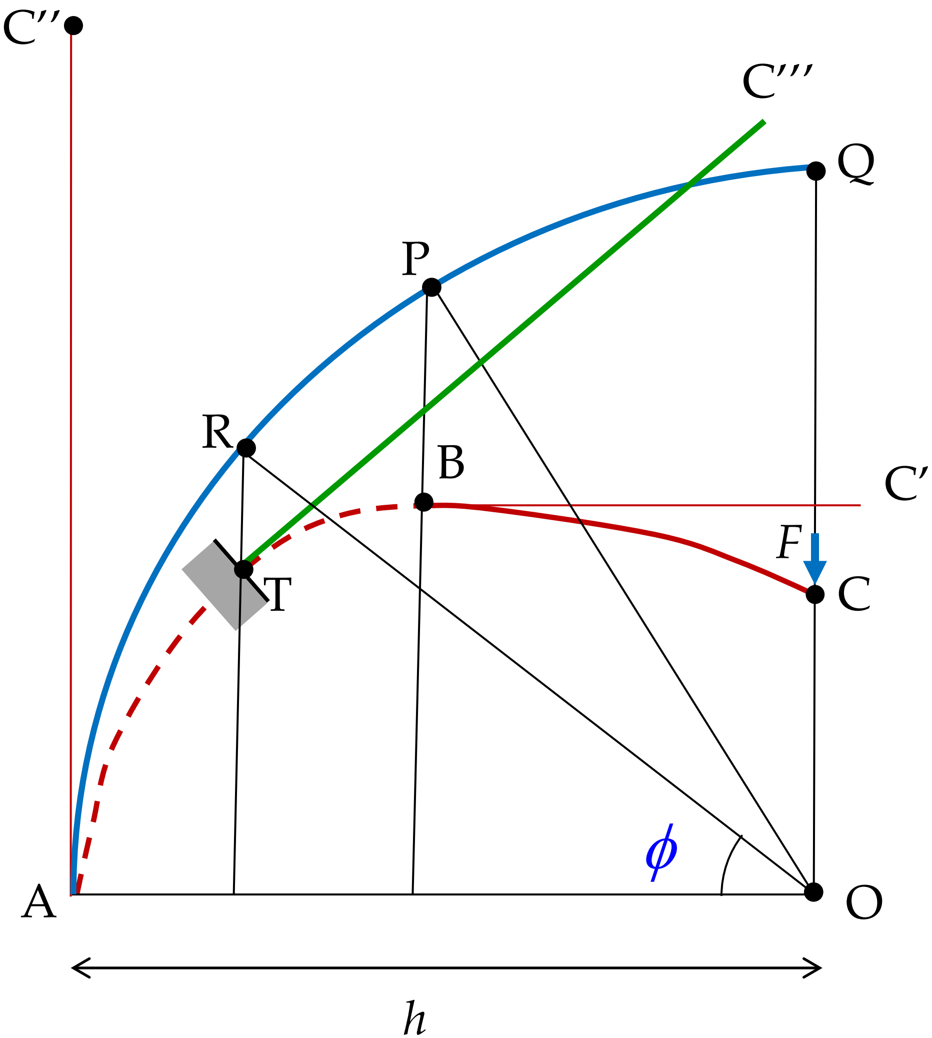
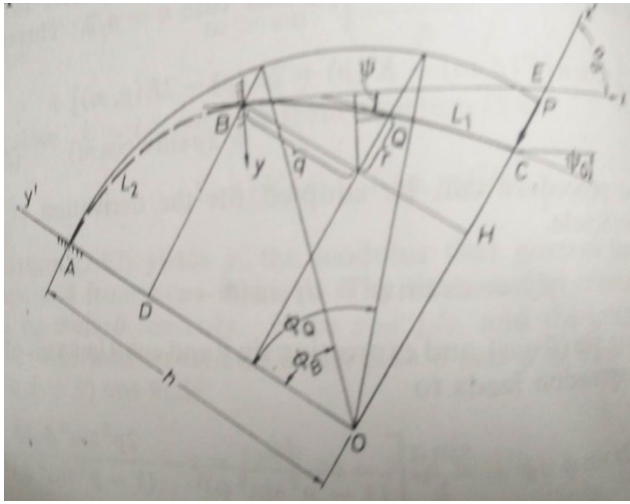
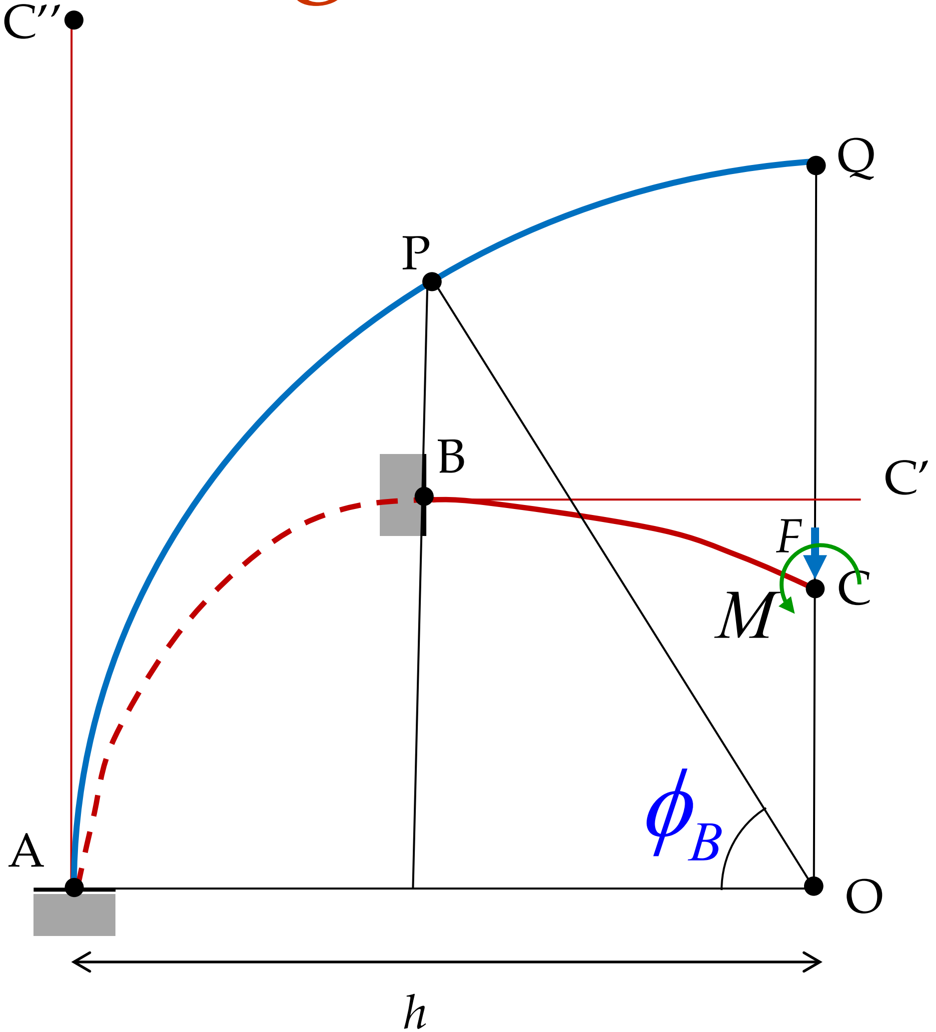




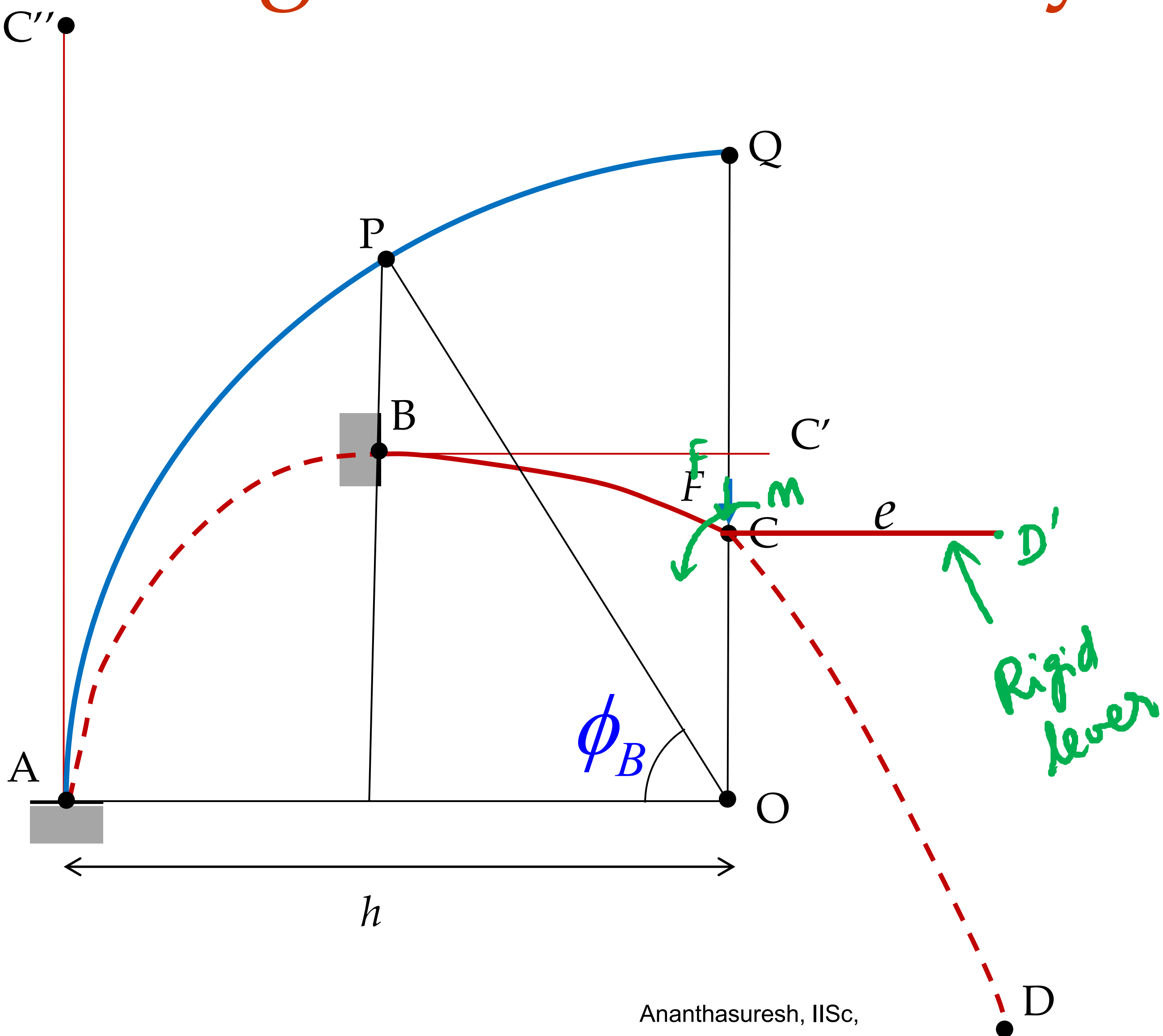
$$x = r\cos\alpha + q\sin\alpha + x_0$$

$$w = q\cos\alpha - r\sin\alpha + w_0$$

Dealing with the moment load using elastic similarity



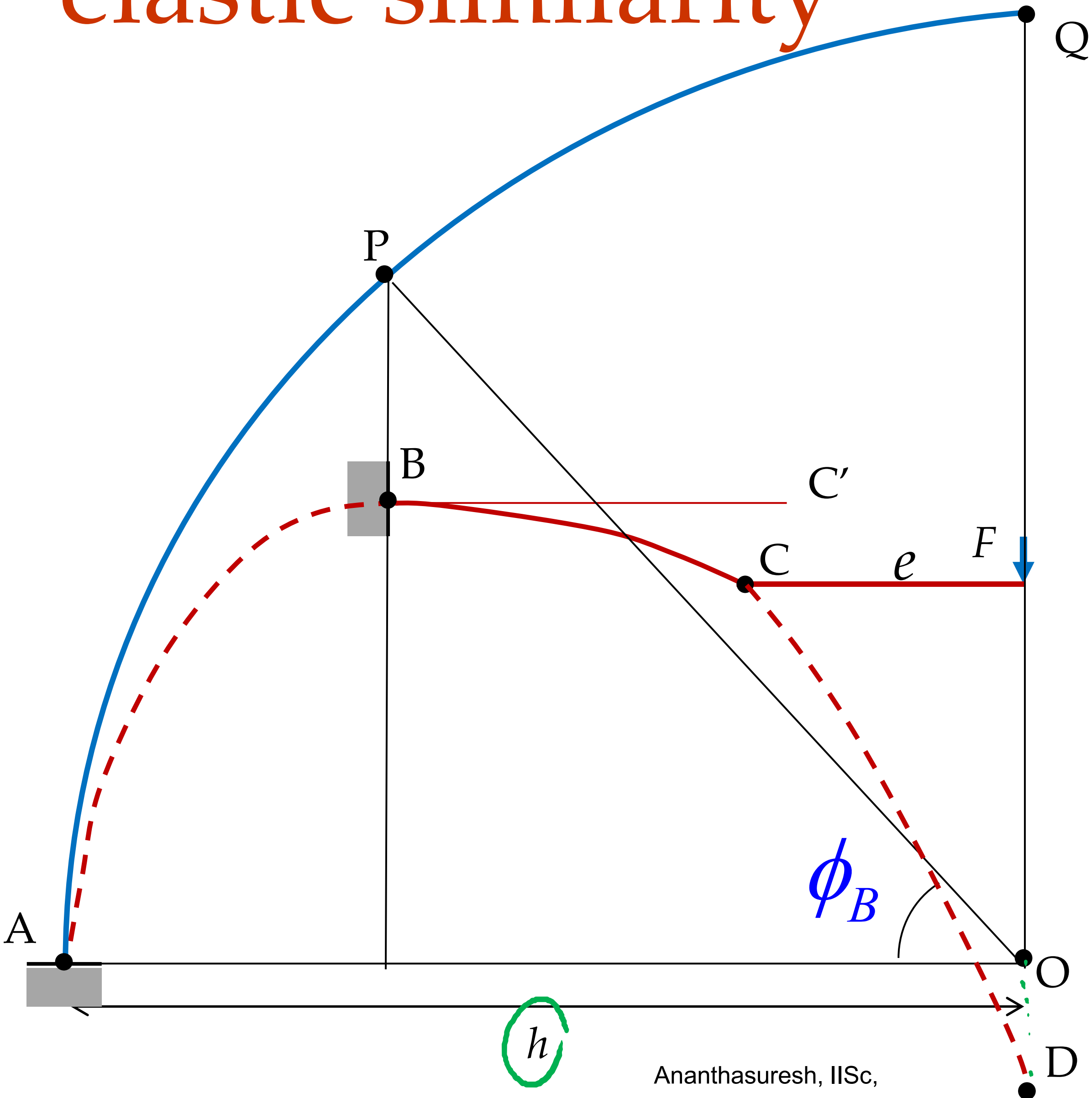
Dealing with the moment load using elastic similarity

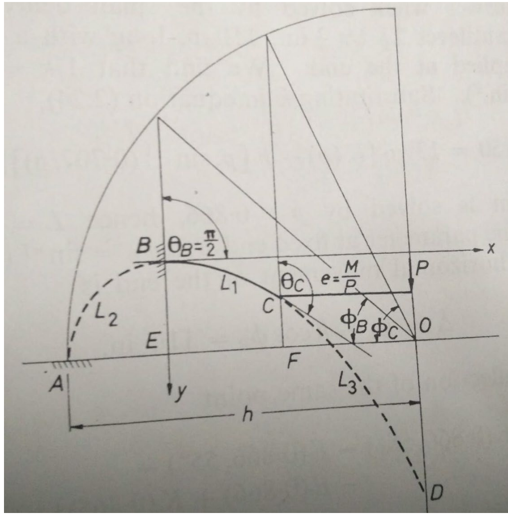


$$e = \frac{M}{F}$$

Moment load added using elastic similarity

Frisch-Fay, R., *Flexible Bars*, 1962.





Moment load added using elastic similarity

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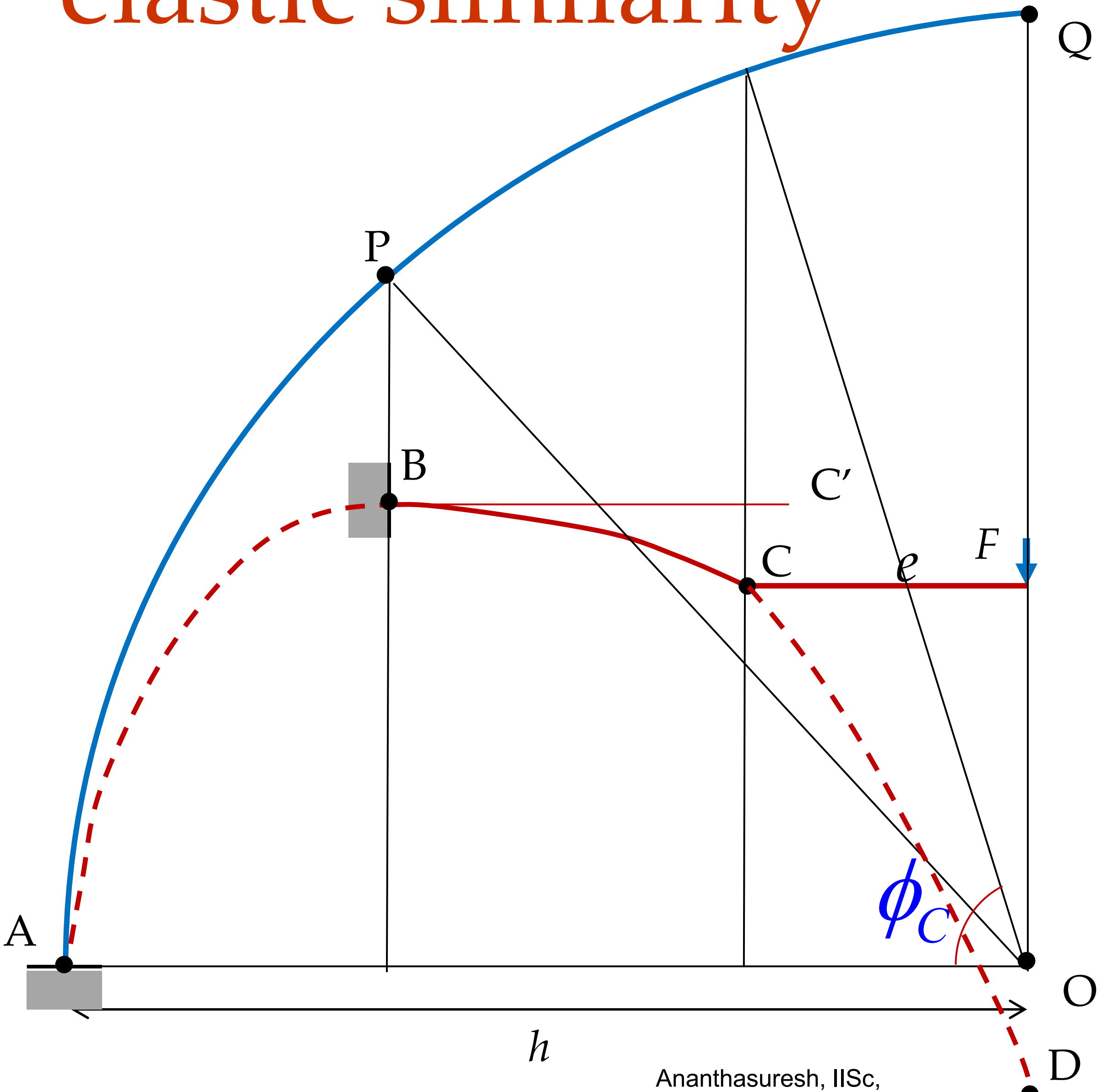
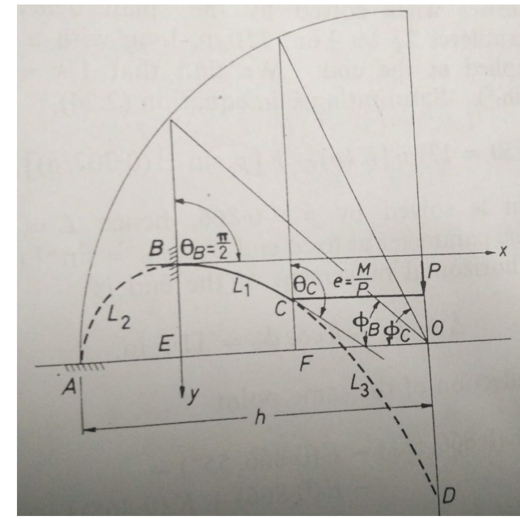
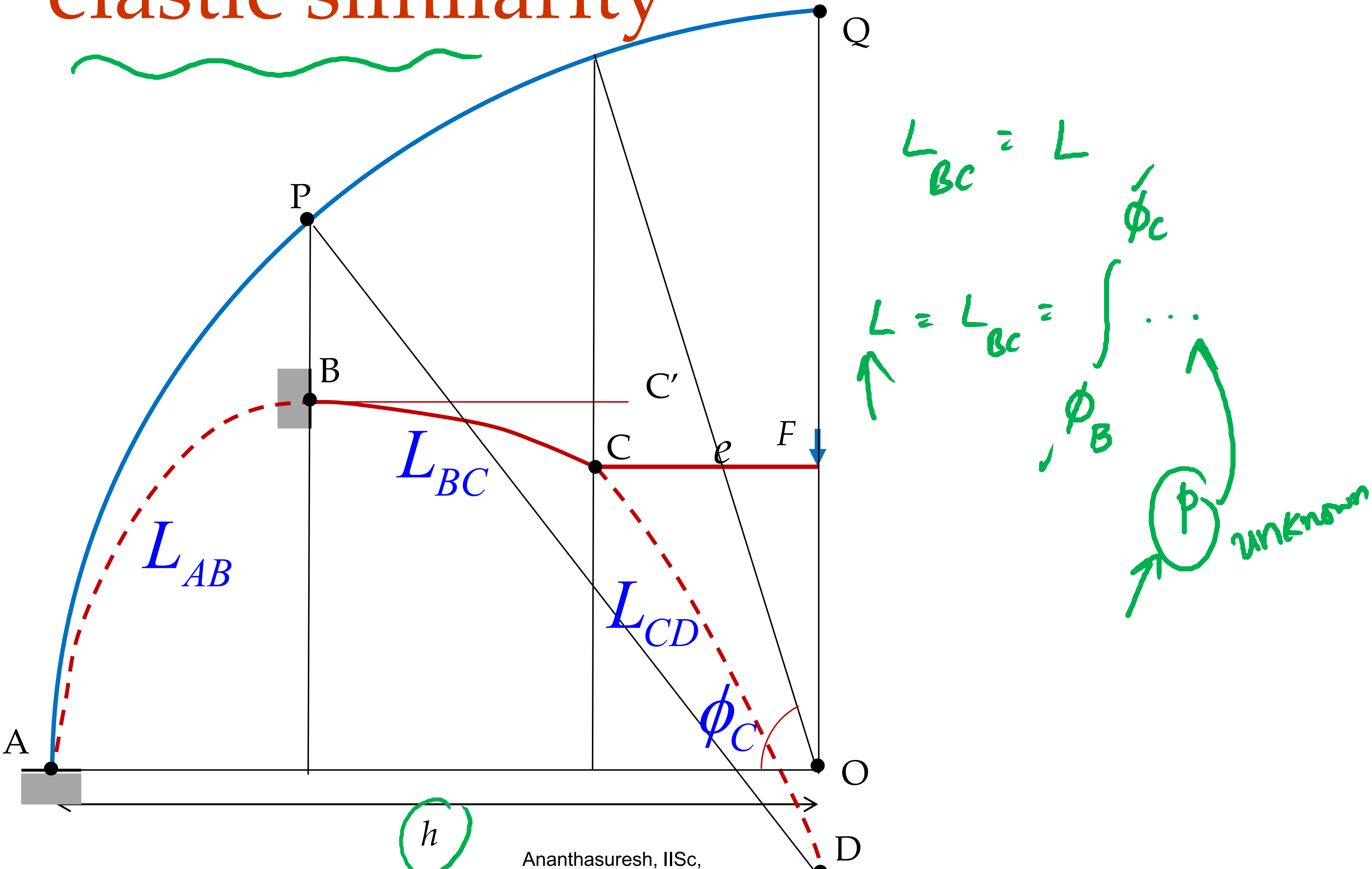


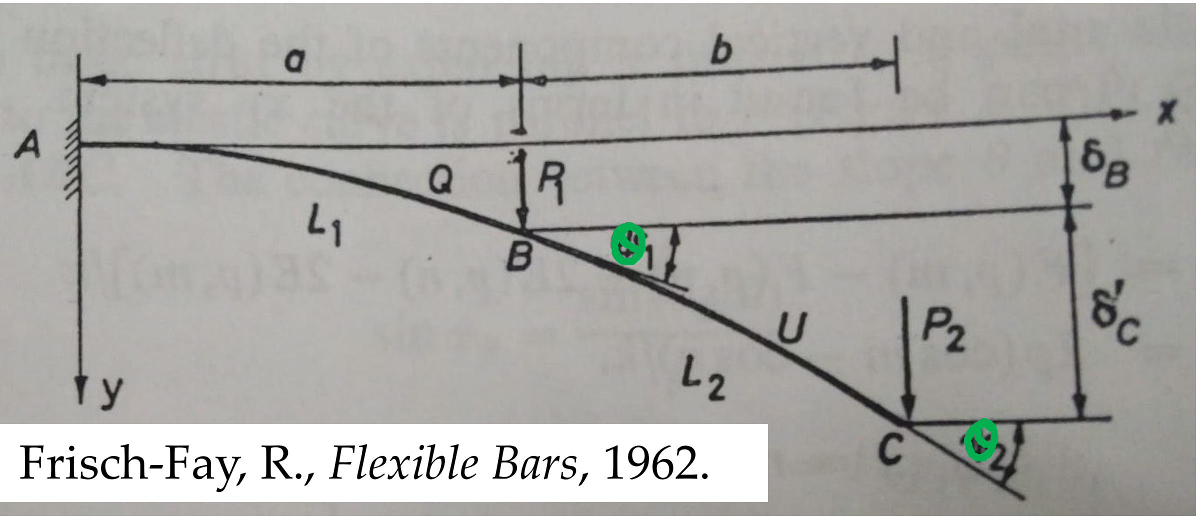


$$\frac{e}{h} = \cos\phi_C = \frac{e}{2p}\sqrt{\frac{F}{EI}}$$

Moment load added using elastic similarity



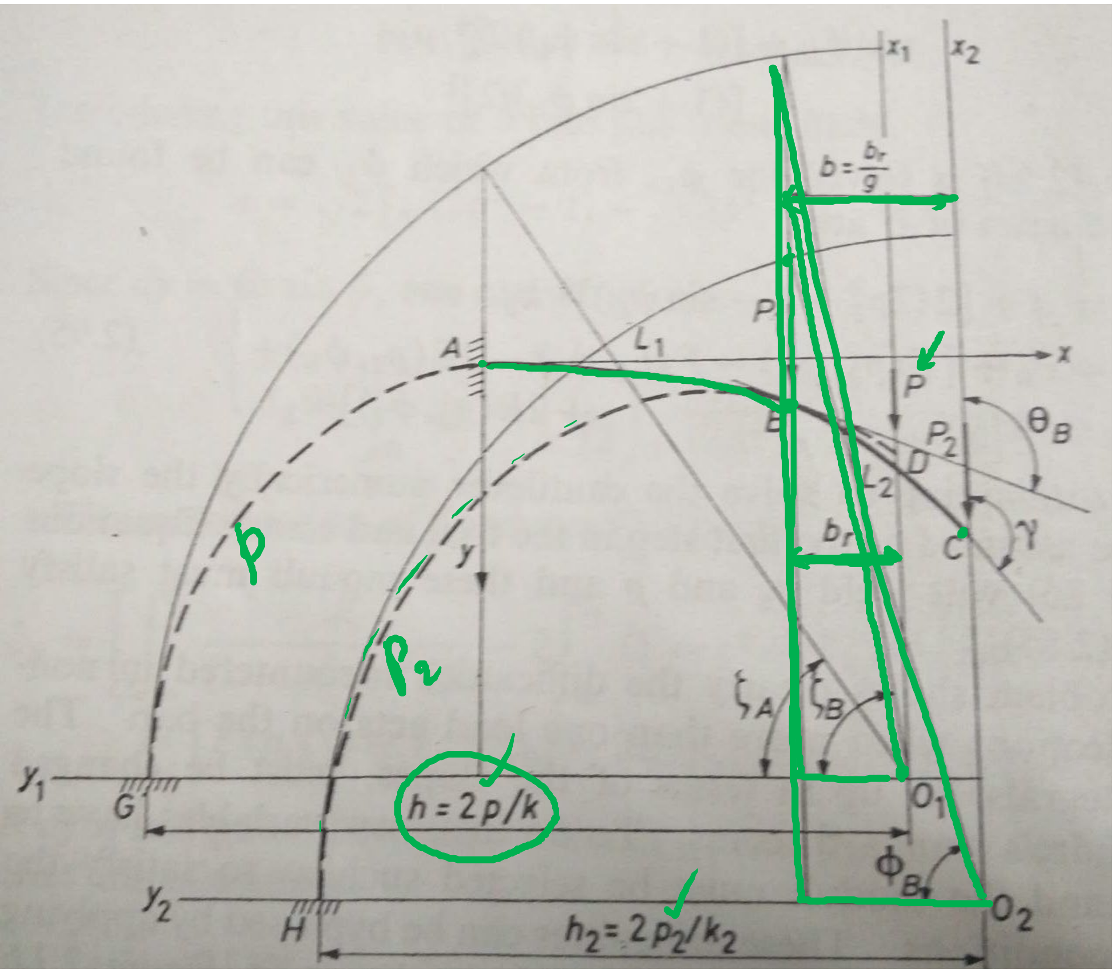
Two transverse loads



Frisch-Fay, R., *Flexible Bars*, 1962.

Two transverse loads using elastic similarity

Frisch-Fay, R., *Flexible Bars*, 1962.



$$b = \frac{b_r}{g}$$

$$g = \frac{P_2}{P}$$

$P = P_1 + P_2$

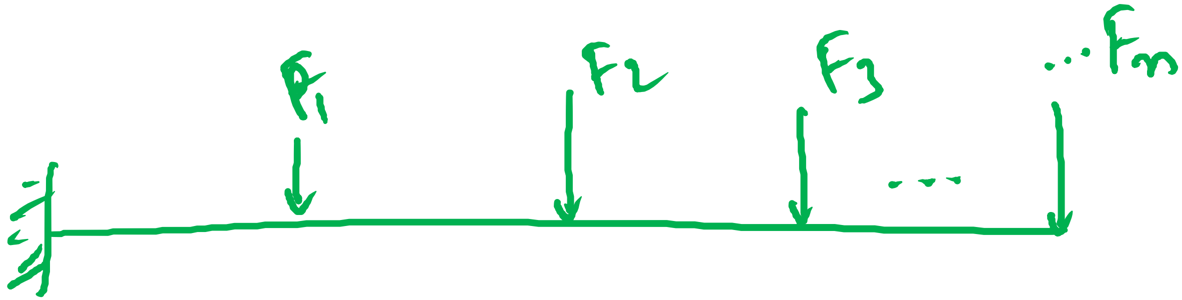
$\phi_B \neq \frac{\pi}{2}$

unknown

P unknown

P_2 unknown

What if there are n loads?



$(2n-1)$ unknowns

$n=0 \Rightarrow$ 1 unknown

$n=2 \Rightarrow$ 3 unknowns P, P_2, ϕ_B

θ_L

p

Further reading

- Frisch-Fay, R., *Flexible Bars*, Butterworths, London, 1962.