ME 254

Elastic similarity calculations using elliptic integrals in Matlab

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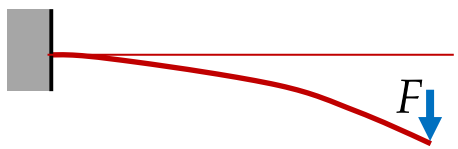
Important equations

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{2\frac{F}{EI}\left(\sin\theta_L - \sin\theta\right)}} = \int_0^L ds = L$$

$$\theta \to \phi$$

$$\sin\theta = 2p^2 \sin^2\phi - 1$$

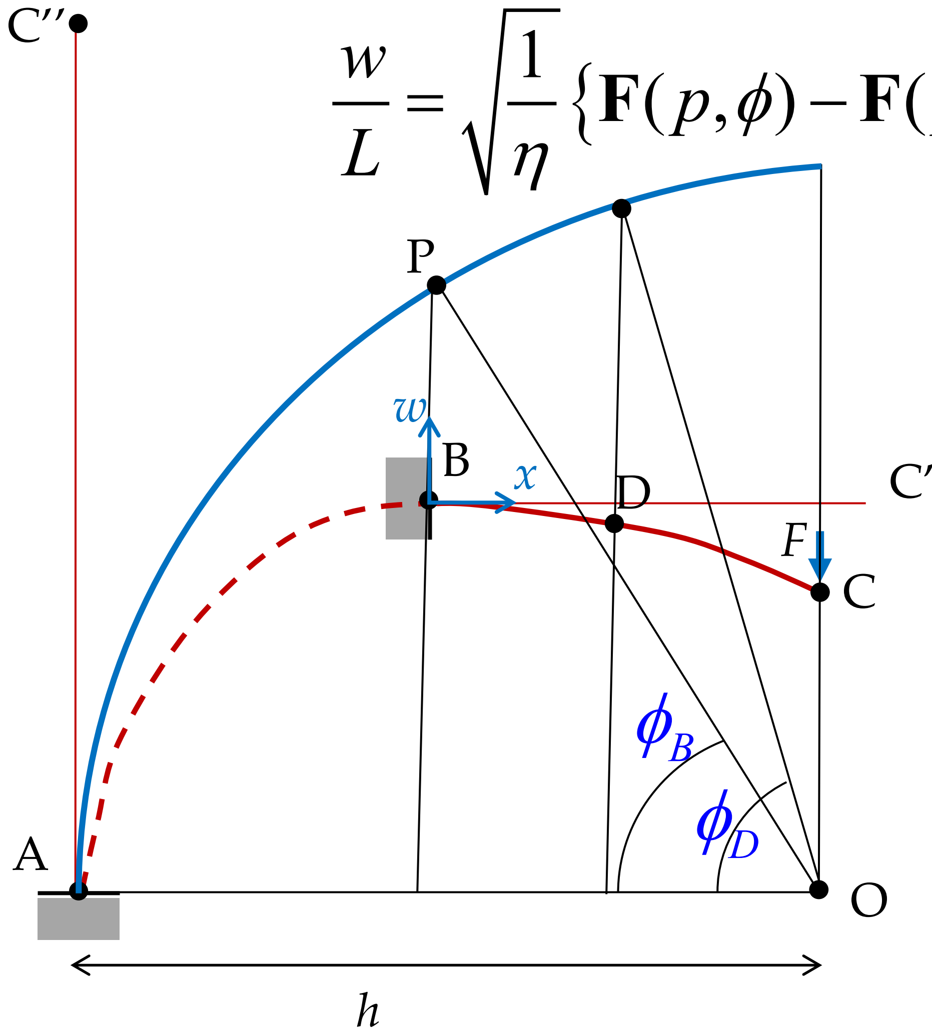
$$p^2 = \frac{1+\sin\theta_L}{2}$$



$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\pi/2} \frac{d\phi}{\sqrt{\frac{F}{EI}}\sqrt{1-p^2\sin^2\phi}} = L$$

Coordinates of the deflected profile at any point beyond B

$$\frac{w}{L} = \sqrt{\frac{1}{\eta}}\left\{\mathbf{F}(p,\phi) - \mathbf{F}(p,\phi_B) - 2\mathbf{E}(p,\phi) + 2\mathbf{E}(p,\phi_B)\right\}$$



$$\frac{x}{L} = \frac{2p}{\sqrt{\eta}}\left(\cos\phi_B - \cos\phi\right)$$

$$h = \frac{2pL}{\sqrt{\eta}} = \frac{2p}{\sqrt{F/EI}}$$

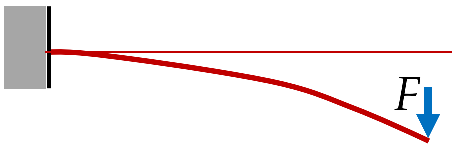
Non-dimensional portrayal

$$\int_{\phi_1}^{\phi_2} \frac{d\phi}{\sqrt{\frac{F}{EI}}\sqrt{1-p^2\sin^2\phi}} = \int_{s_1}^{s_2} ds = s_2 - s_1$$

$$\frac{w_2 - w_1}{L} = \sqrt{\frac{1}{\eta}}\left\{\mathbf{F}(p,\phi_2) - \mathbf{F}(p,\phi_1) - 2\mathbf{E}(p,\phi_2) + 2\mathbf{E}(p,\phi_1)\right\}$$

$$\frac{x_2 - x_1}{L} = \frac{2p}{\sqrt{\frac{F}{EI}}}\left(\cos\phi_2 - \cos\phi_1\right)$$

A numerical example



The force varies from 0 to 12 N.

$$L = 1\ m$$

$$E = 210E9\ Pa$$

$$I = \frac{bd^3}{12}$$

$$b = 5\times10^{-2}\ m$$

$$d = 1\times10^{-3}\ m$$

What we use:

Our codes:

nlbeam.m and other files

its data files

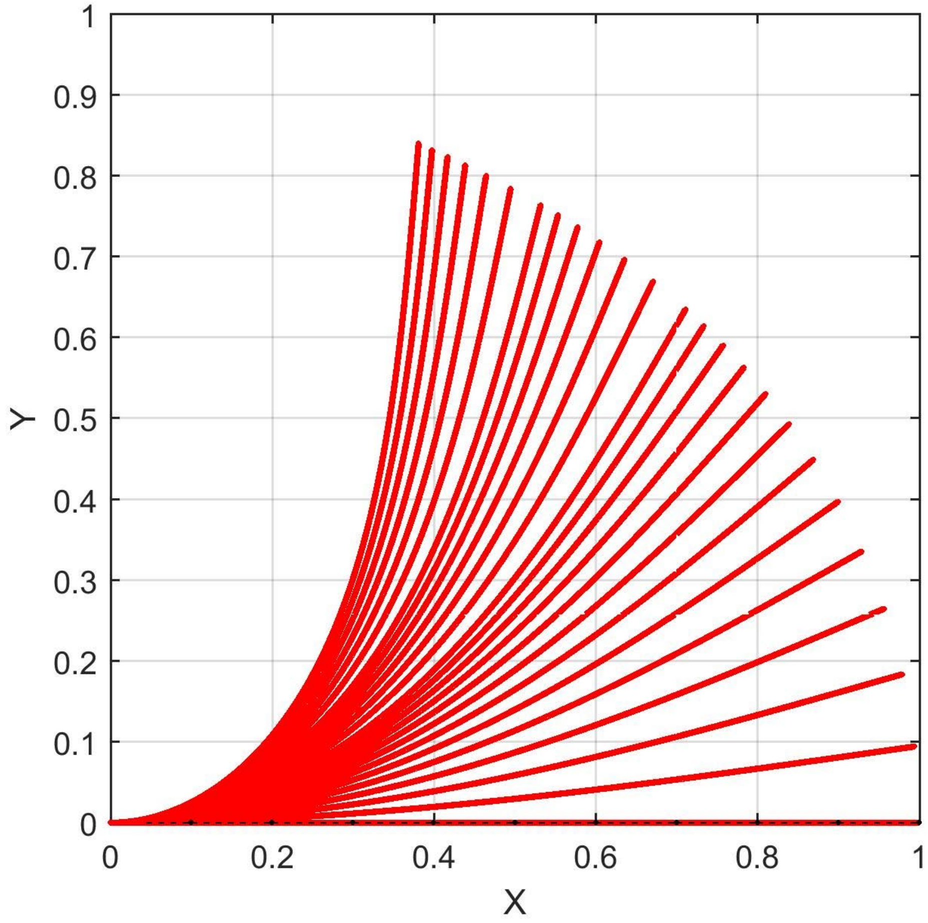
Matlab functions

ellipticF

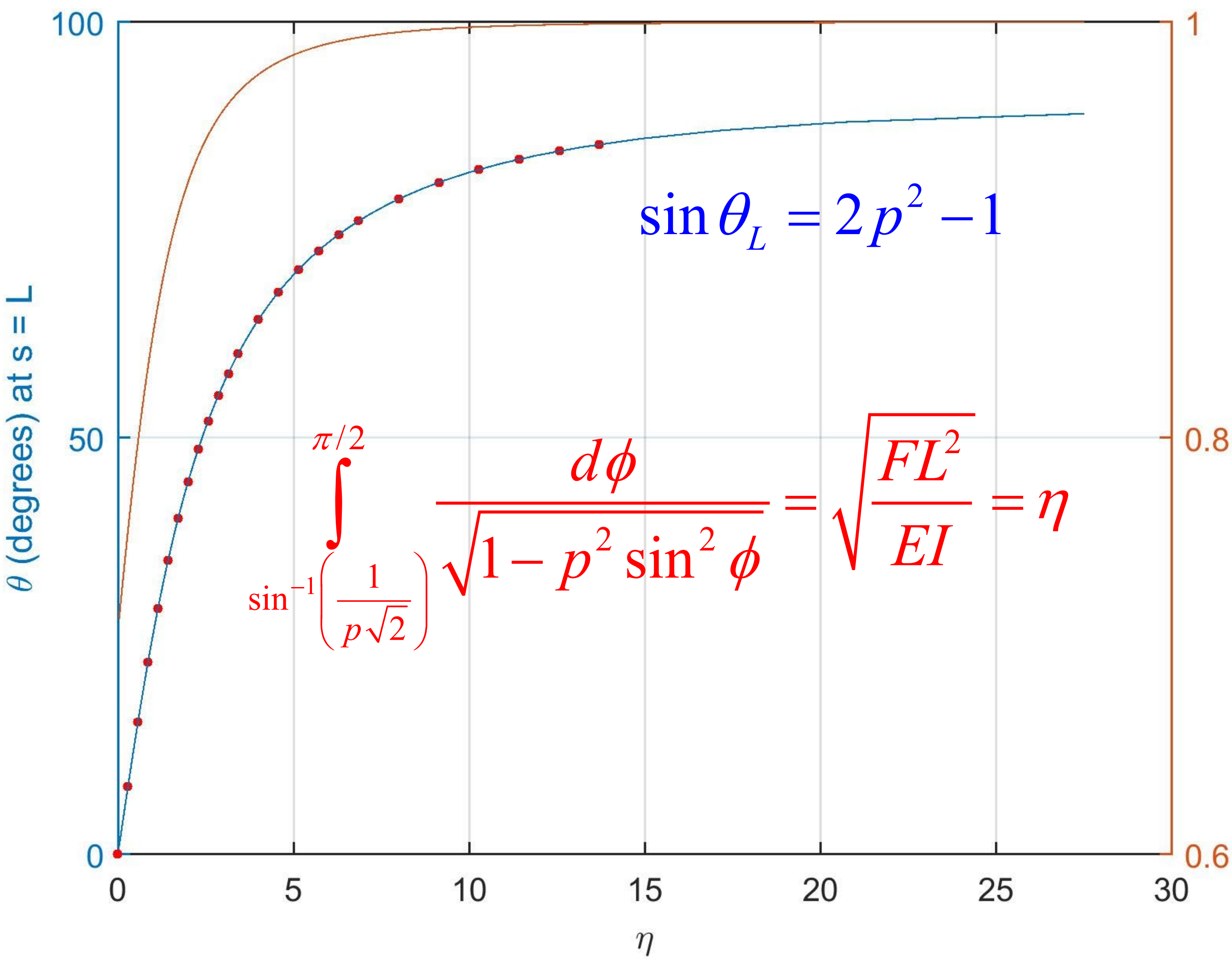
ellipticE

nlfeambeam.m code

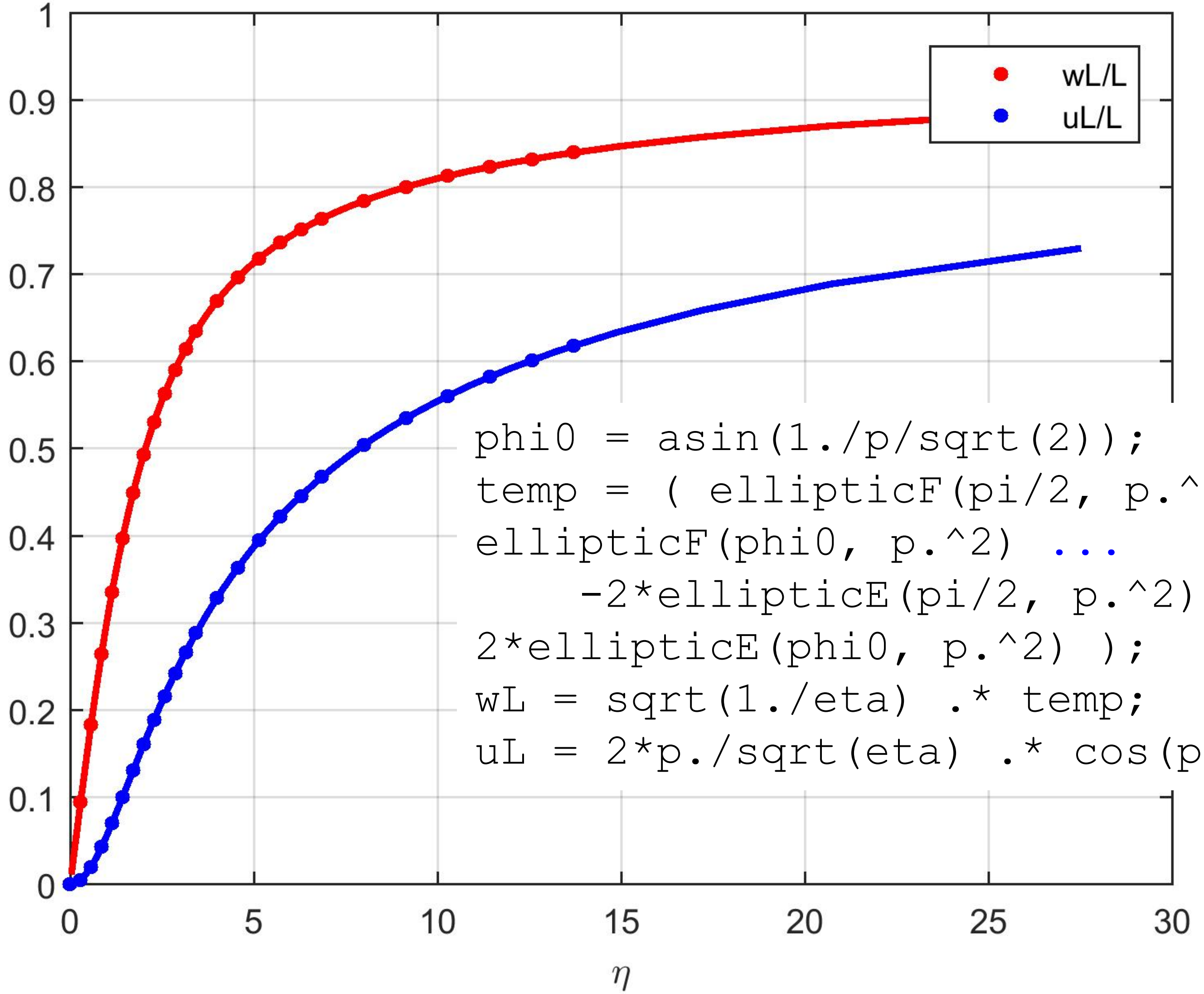
```
f = [0:0.25:3 3.5:0.5:6 7:1:12];
```



FEA and Elliptic integral solutions



Locus of the loaded tip



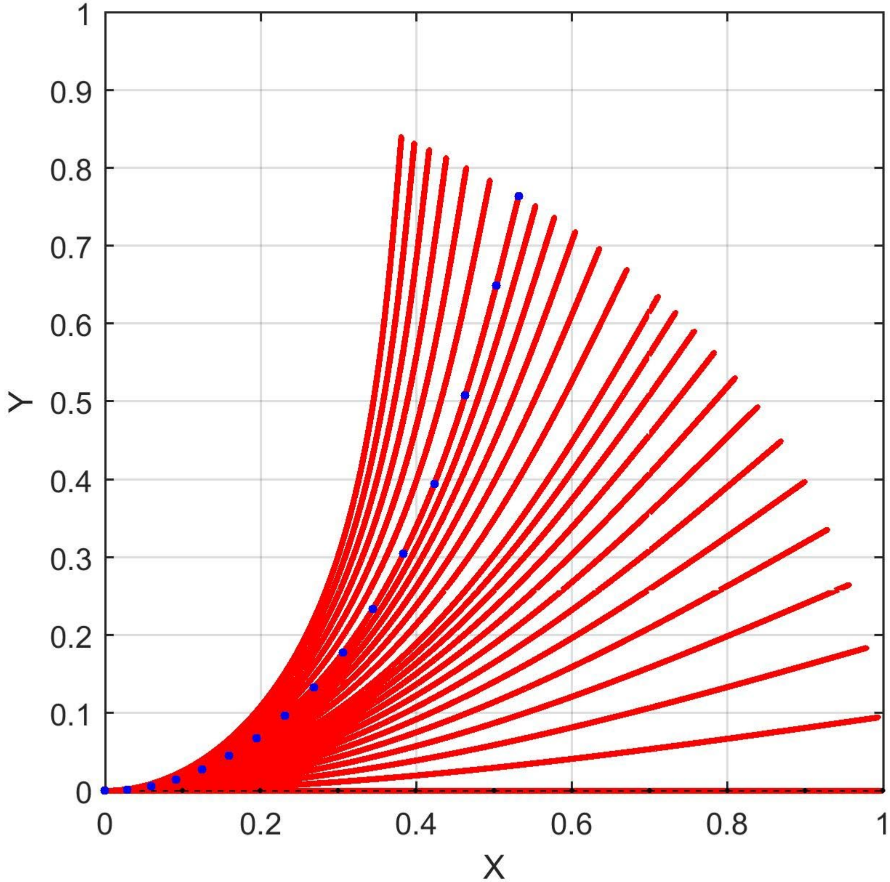

```
phi0 = asin(1./p/sqrt(2));
temp = ( ellipticF(pi/2, p.^2) -
ellipticF(phi0, p.^2) ...
    -2*ellipticE(pi/2, p.^2) +
2*ellipticE(phi0, p.^2) );
wL = sqrt(1./eta) .* temp;
uL = 2*p./sqrt(eta) .* cos(phi0);
```

Matlab code using elliptic integrals

```
for thetaL = pi/180:pi/180:89*pi/180,
    i = i + 1;
    p(i) = sqrt((1+sin(thetaL))/2);
    phi0 = asin(1/p(i)/sqrt(2));
    eta(i) = ( ellipticF(pi/2,p(i)^2)
- ellipticF(phi0, p(i)^2) )^2;
    thetaLdeg(i) = thetaL*180/pi;
end
```

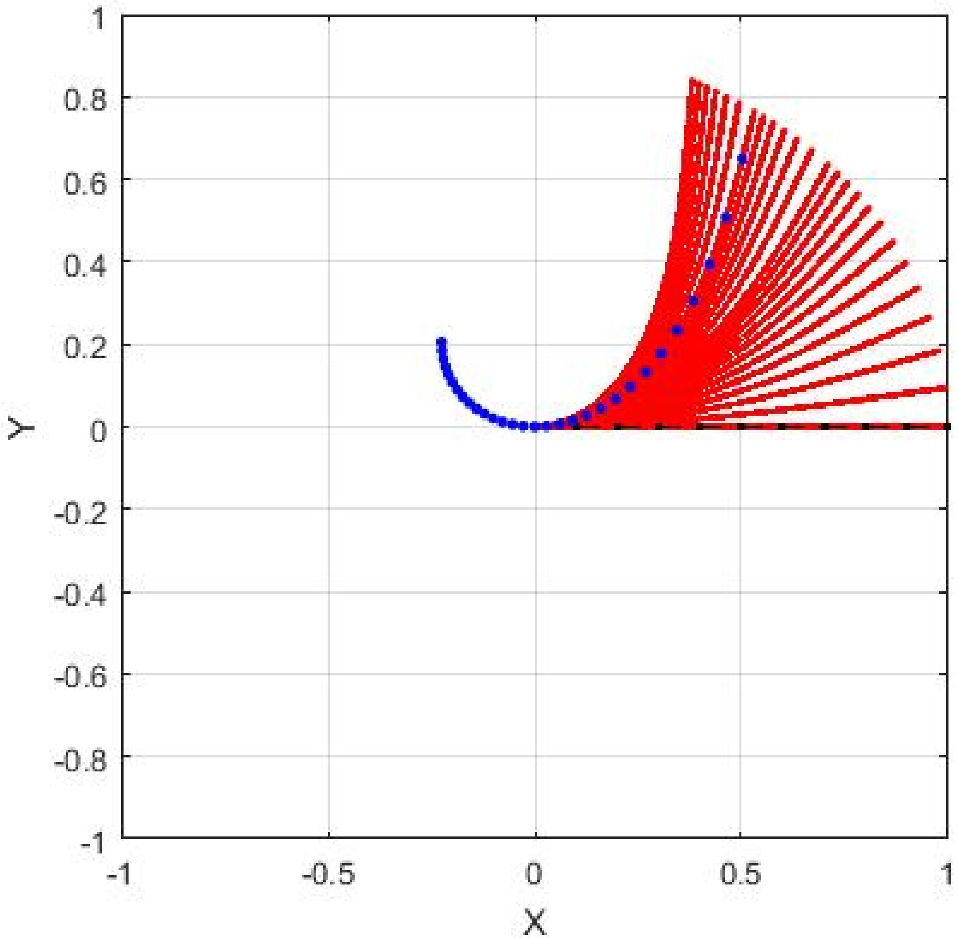
FEA and elliptic solutions

$$\frac{w}{L} = \sqrt{\frac{1}{\eta}} \left\{ \mathbf{F}(p,\phi) - \mathbf{F}(p,\phi_B) - 2\mathbf{E}(p,\phi) + 2\mathbf{E}(p,\phi_B) \right\}$$

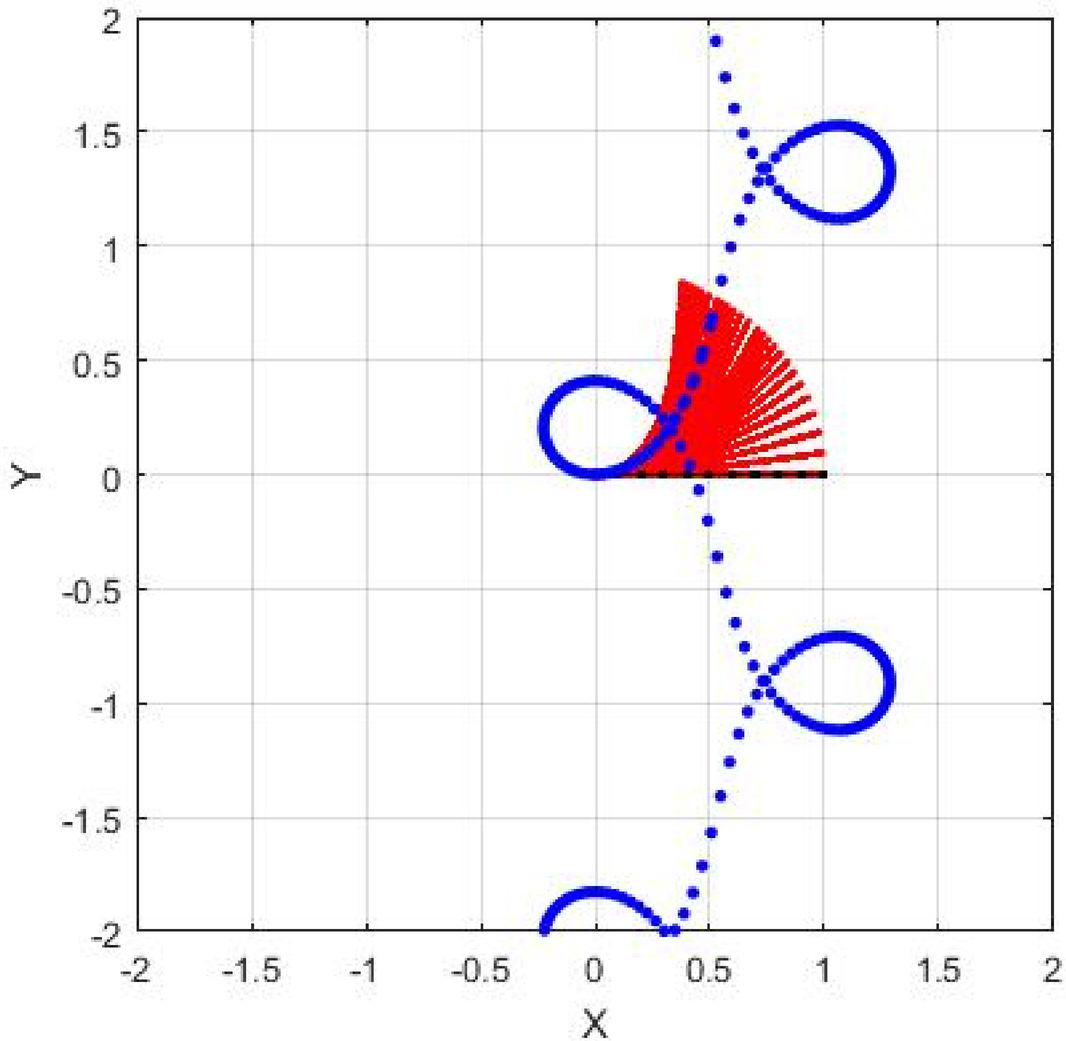


$$\frac{x}{L} = \frac{2p}{\sqrt{\eta}} \left(\cos\phi_B - \cos\phi \right)$$

Extrapolation to a column using elastic similarity

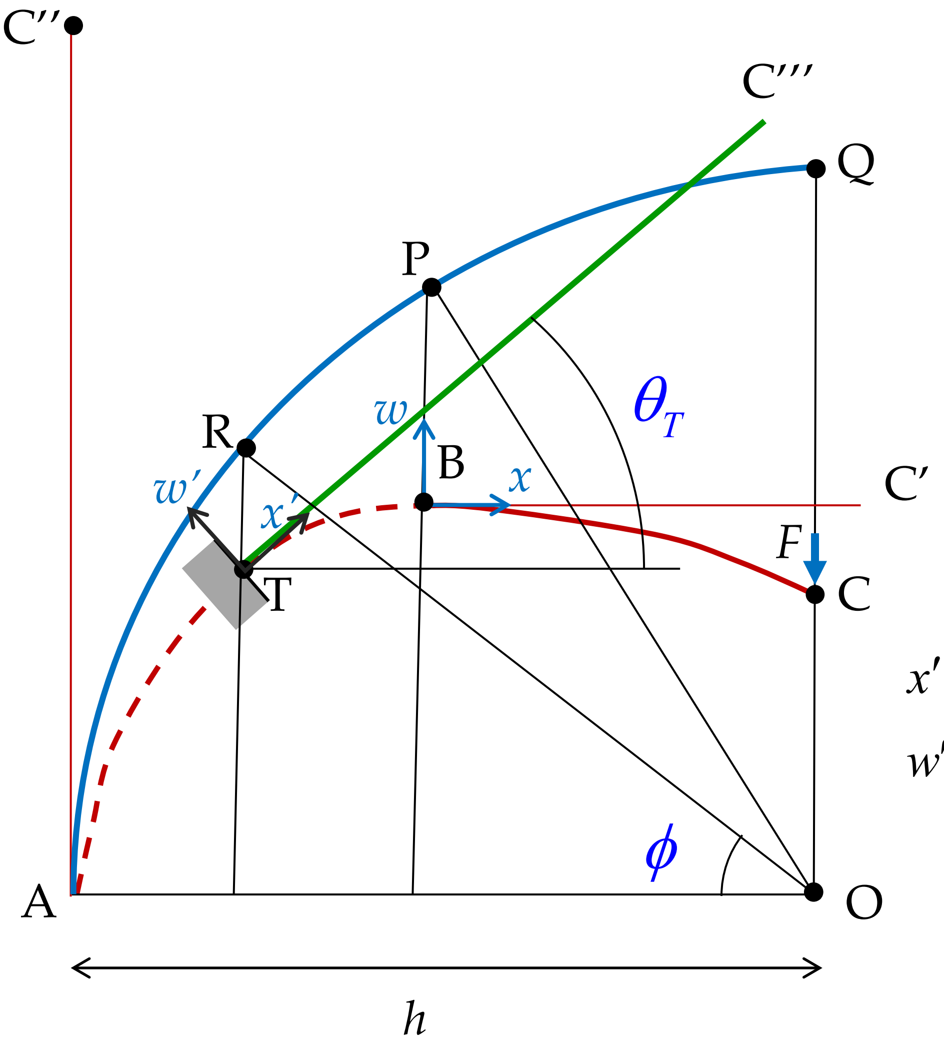


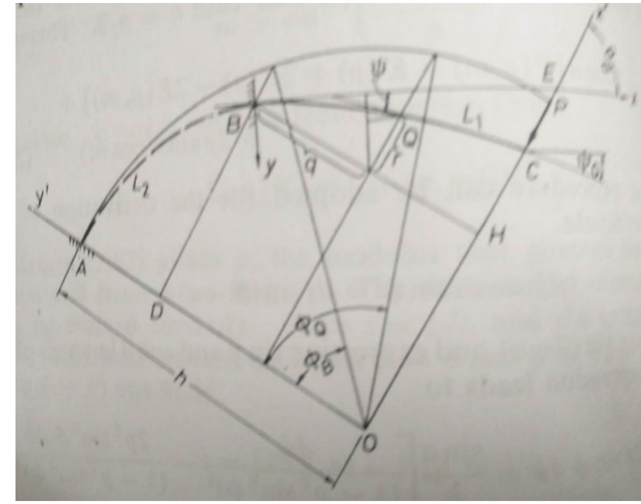
Nodal elastica



Inclined load

Frisch-Fay, R., *Flexible Bars*, 1962.

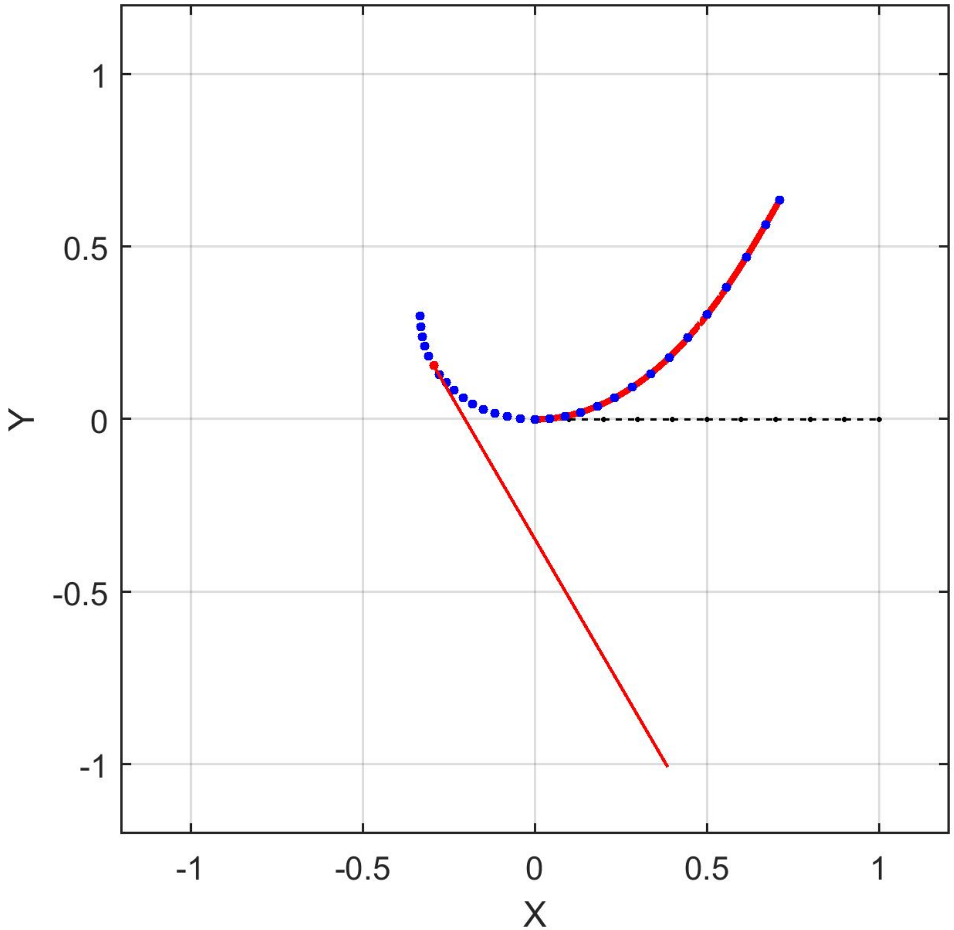




$$x' = \quad x\cos\theta_T + w\sin\theta_T - \left(x_T\cos\theta_T + w_T\sin\theta_T\right)$$

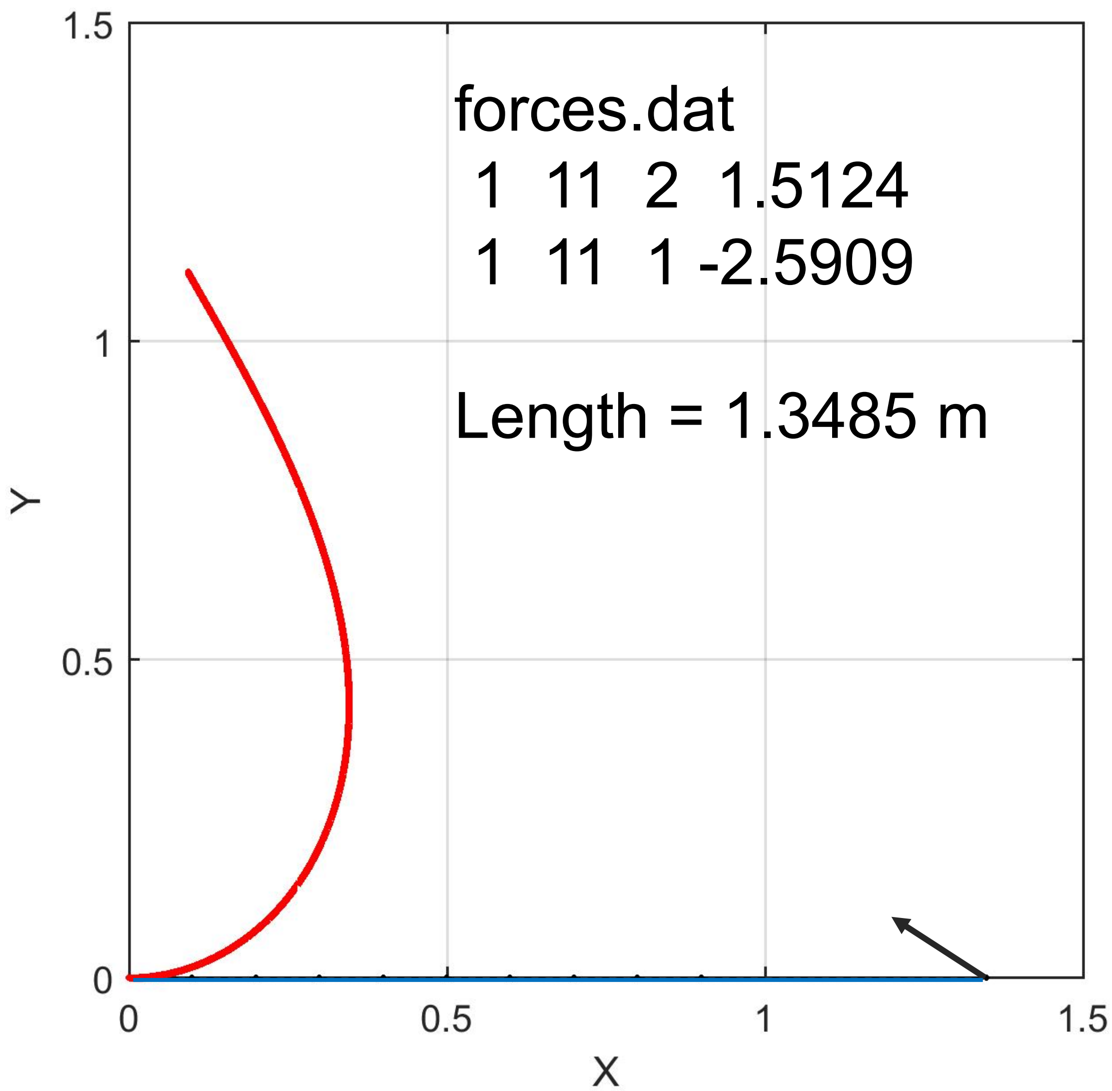
$$w' = -x\sin\theta_T + w\cos\theta_T - \left(-x_T\sin\theta_T + w_T\cos\theta_T\right)$$

Solution for the inclined load




```
>> theta*180/pi

ans =

  -59.7263

>> Lphi

Lphi =

    1.3485

>> f*cos(theta)

ans =

    1.5124

>> f*sin(theta)

ans =

   -2.5909
```

FEA with the inclined load



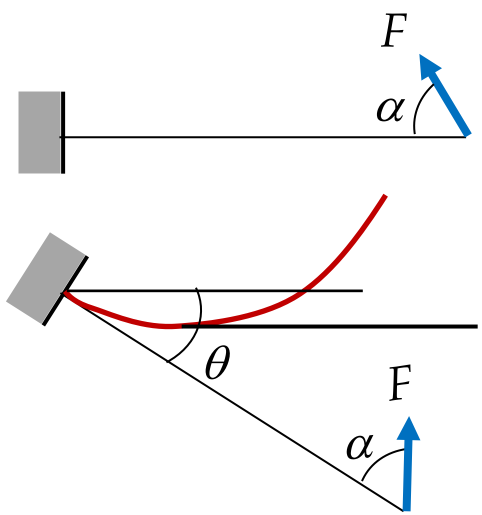
xp =

0.0932

wp =

1.1092

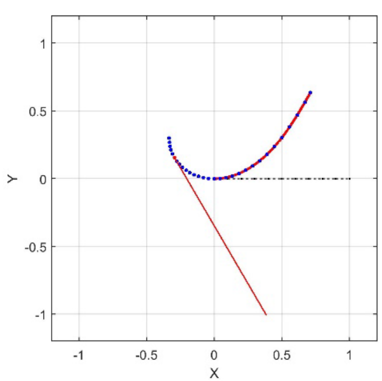
How do we proceed with an inclined load for a beam of length L'?



$$\theta = \frac{\pi}{2} - \alpha$$

$$\phi = \sin^{-1}\left(\sqrt{\frac{1+\sin\theta}{2p^2}}\right)$$

Find p such that $$\int_{\phi}^{\pi/2} \frac{d\phi}{\sqrt{\frac{F}{EI}}\sqrt{1-p^2\sin^2\phi}} = L'$$

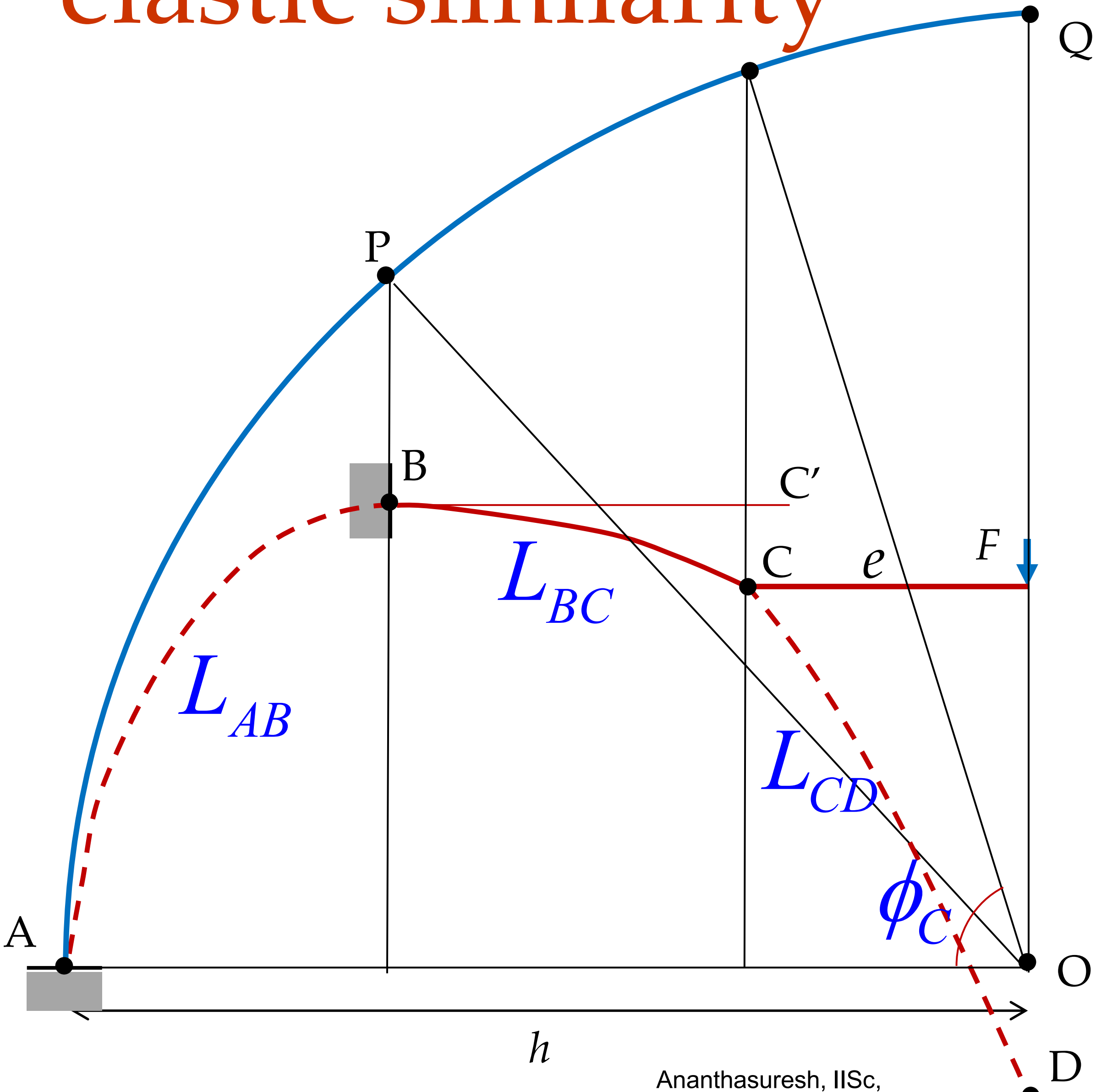


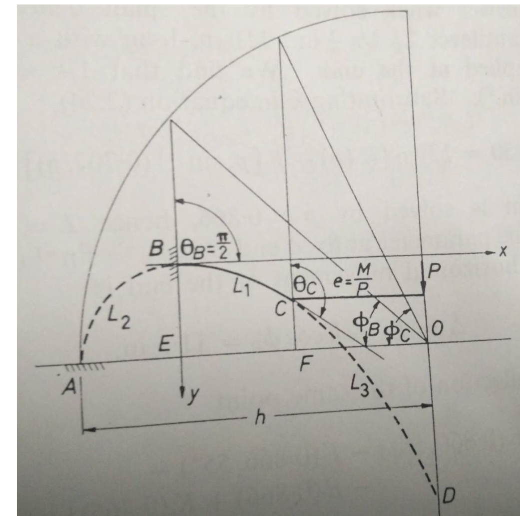
Then, $$L = \int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\pi/2} \frac{d\phi}{\sqrt{\frac{F}{EI}}\sqrt{1-p^2\sin^2\phi}}$$

Let us consider a moment load now.

Moment load added using elastic similarity

Frisch-Fay, R., *Flexible Bars*, 1962.



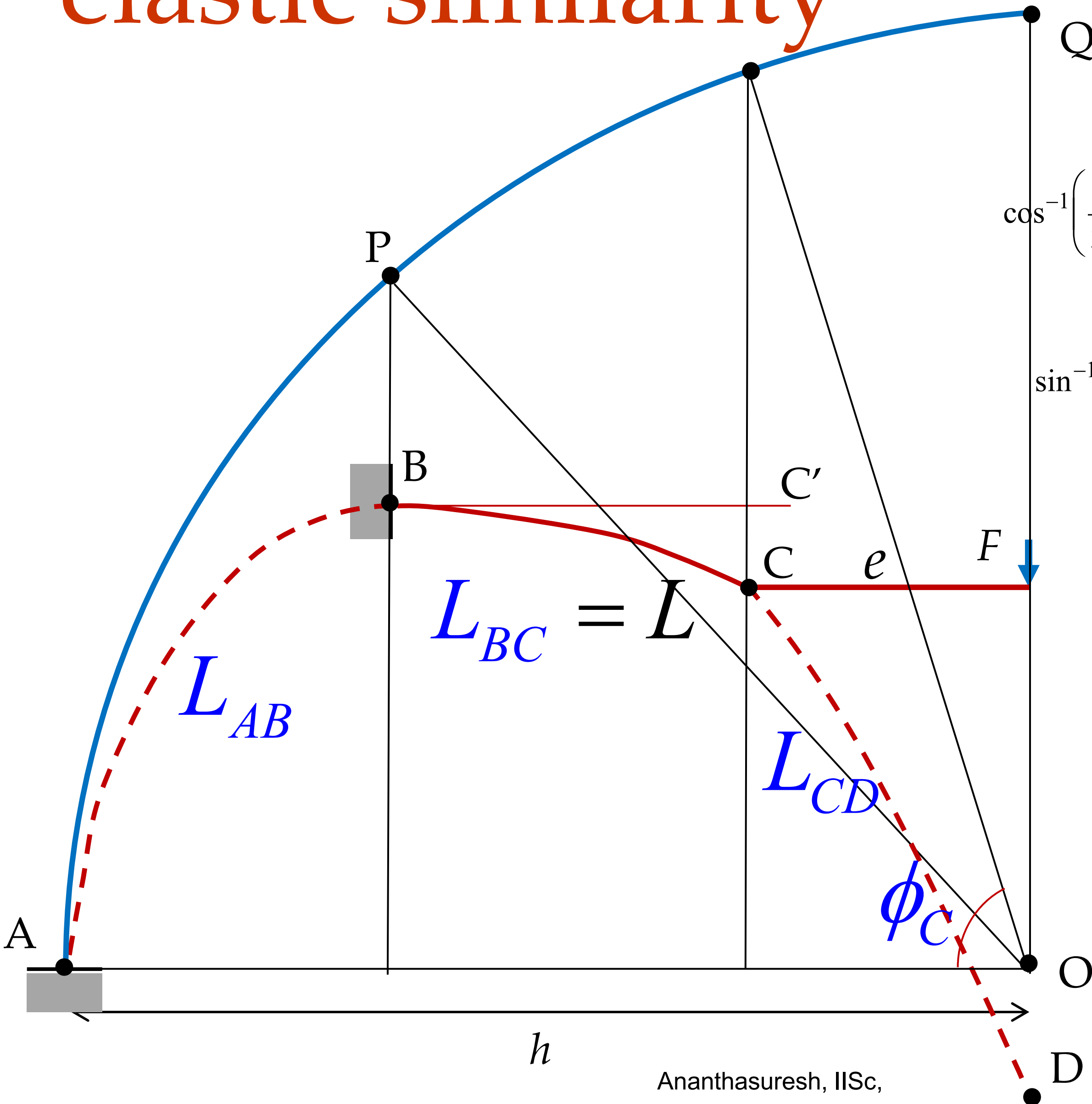


$$e = \frac{M}{F}$$

$$\frac{e}{h} = \cos\phi_C = \frac{e}{2p}\sqrt{\frac{F}{EI}}$$

$$\phi_C = \cos^{-1}\left(\frac{M}{2p}\sqrt{\frac{1}{FEI}}\right)$$

Moment load added using elastic similarity



Find p such that

$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\cos^{-1}\left(\frac{M}{2p}\sqrt{\frac{1}{FEI}}\right)} \frac{d\phi}{\sqrt{\frac{F}{EI}}\sqrt{1-p^2\sin^2\phi}} = L$$

An example with transverse and moment loads

forces.dat

```
 1  11  2 3
 1  11  3 0.4
```

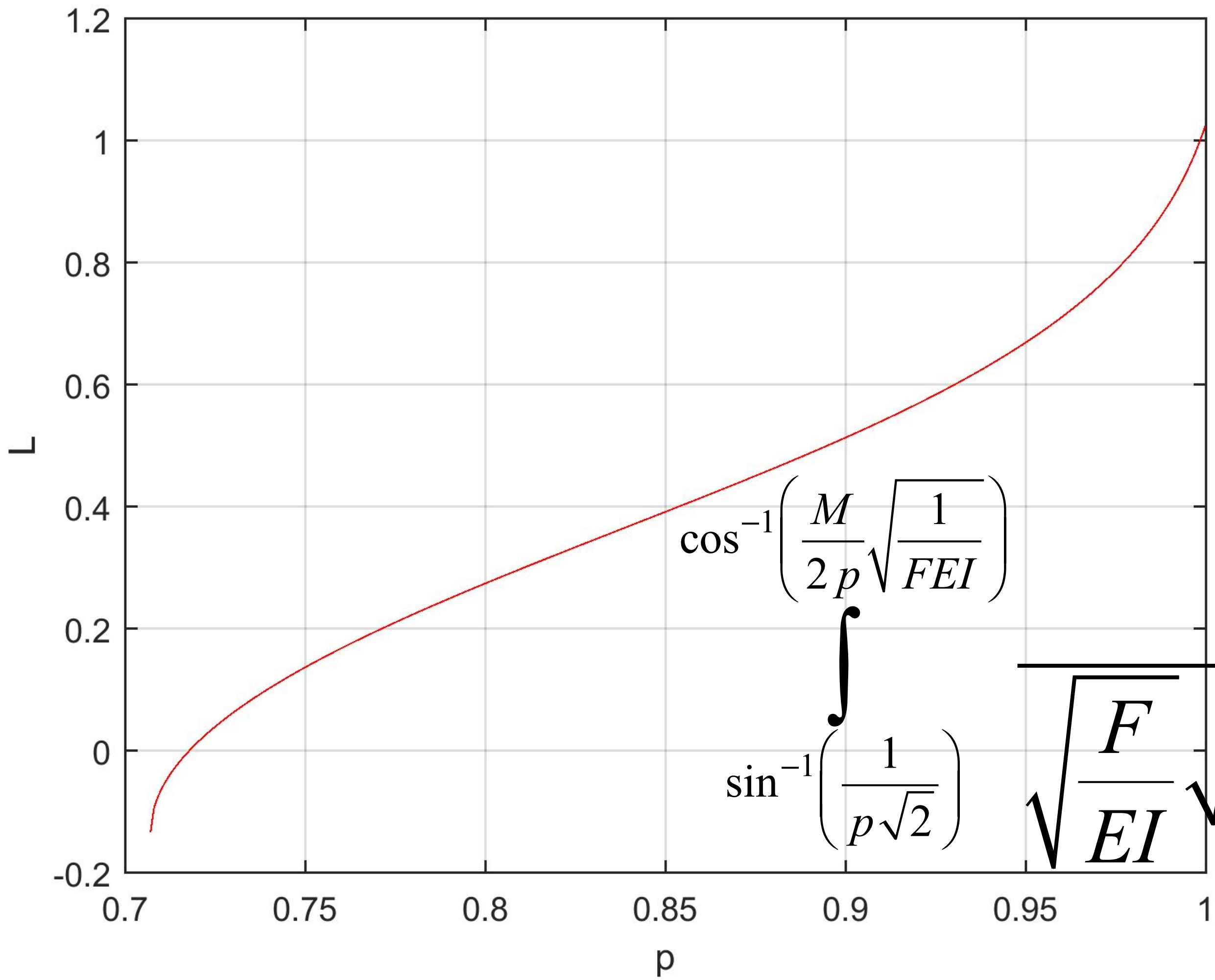
$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\cos^{-1}\left(\frac{M}{2p}\sqrt{\frac{1}{FEI}}\right)} \frac{d\phi}{\sqrt{\frac{F}{EI}}\sqrt{1-p^2\sin^2\phi}} = L$$

L = 1 m

E = 210 GPa

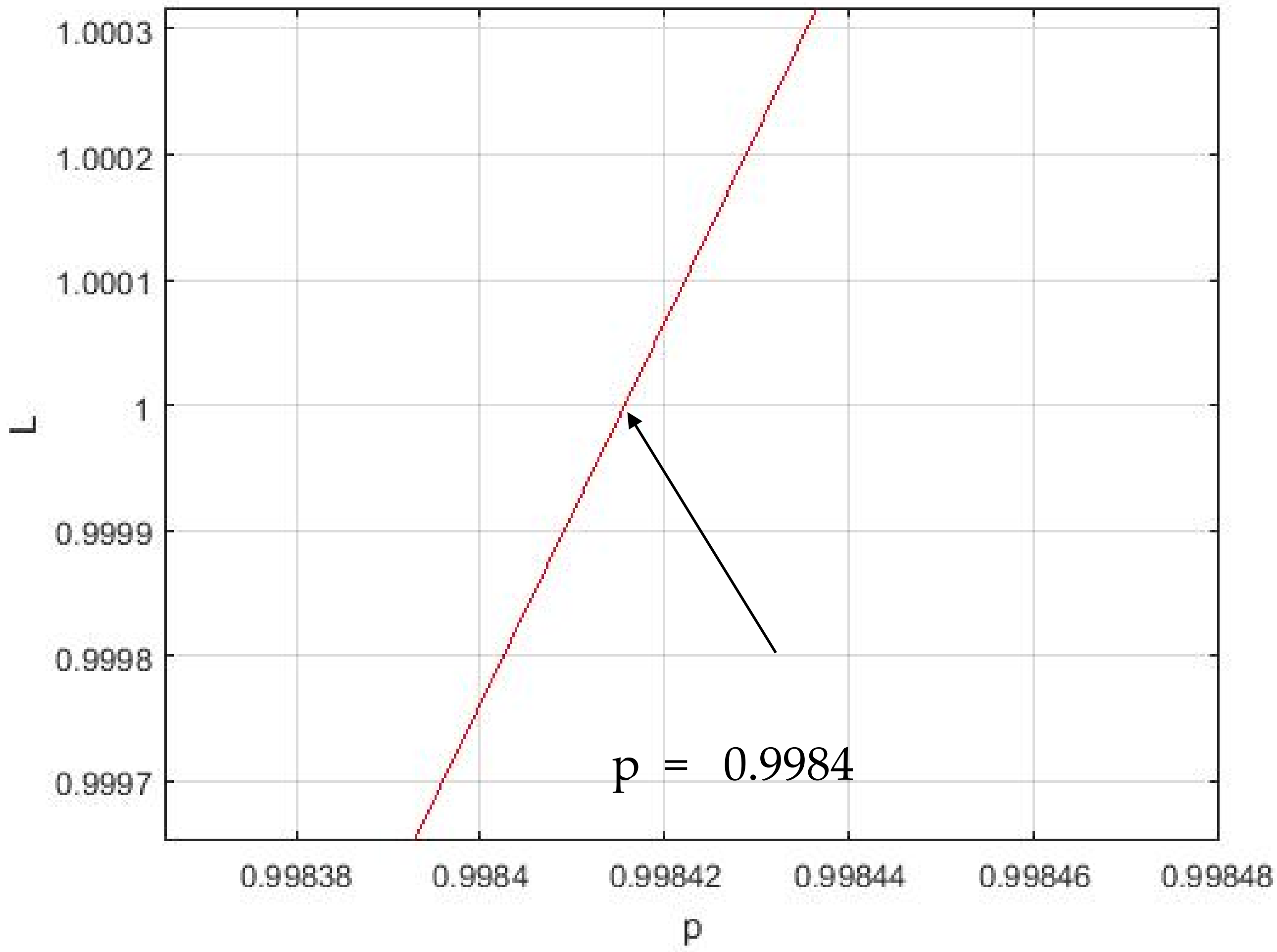
I = $(5 \times 10^{-2})(1 \times 10^{-3})^3/12$ m^4

Finding p with moment load equation



$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\cos^{-1}\left(\frac{M}{2p}\sqrt{\frac{1}{FEI}}\right)} \frac{d\phi}{\sqrt{\frac{F}{EI}}\sqrt{1-p^2\sin^2\phi}} = L$$

Finding p with moment load equation (close-up view)



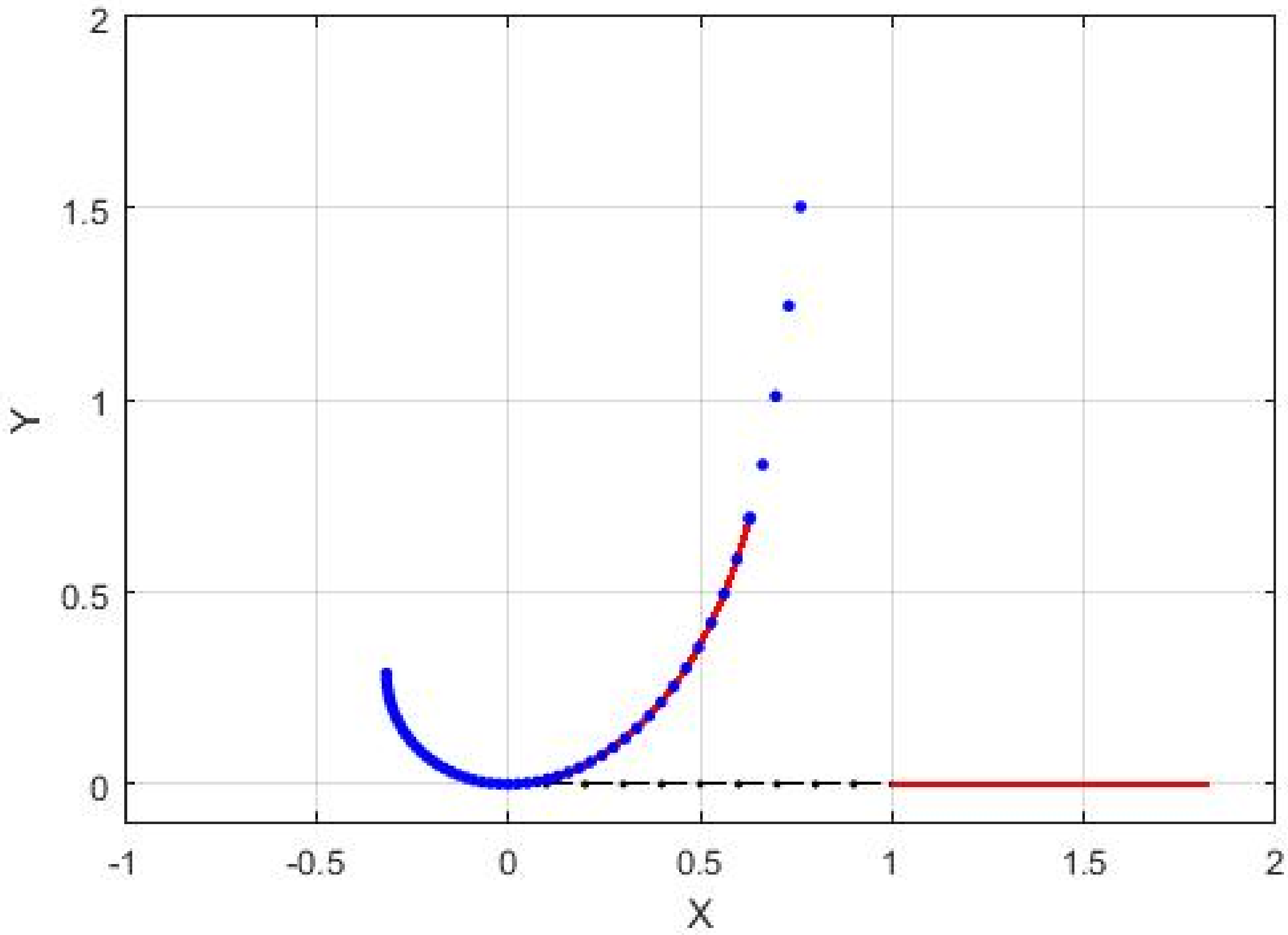
Solution

e = M/F = 0.4/3 = 0.1333 m

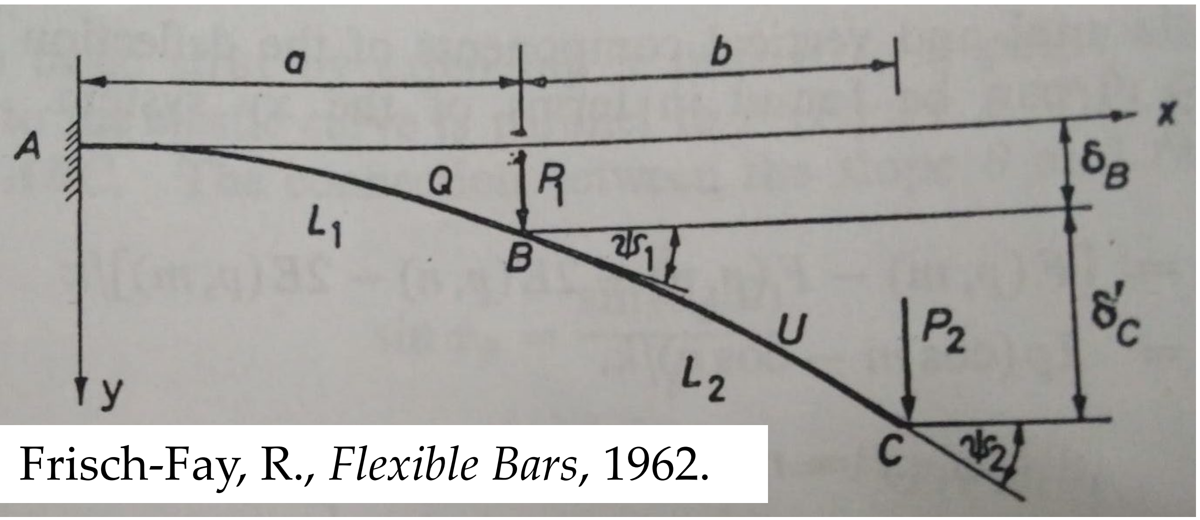
p = 0.9984

LM = sqrt(EI/Fa) * (ellipticF(pi/2,p^2) - ellipticF(phi1,p^2))

LM = 1.8257 m



Two transverse loads

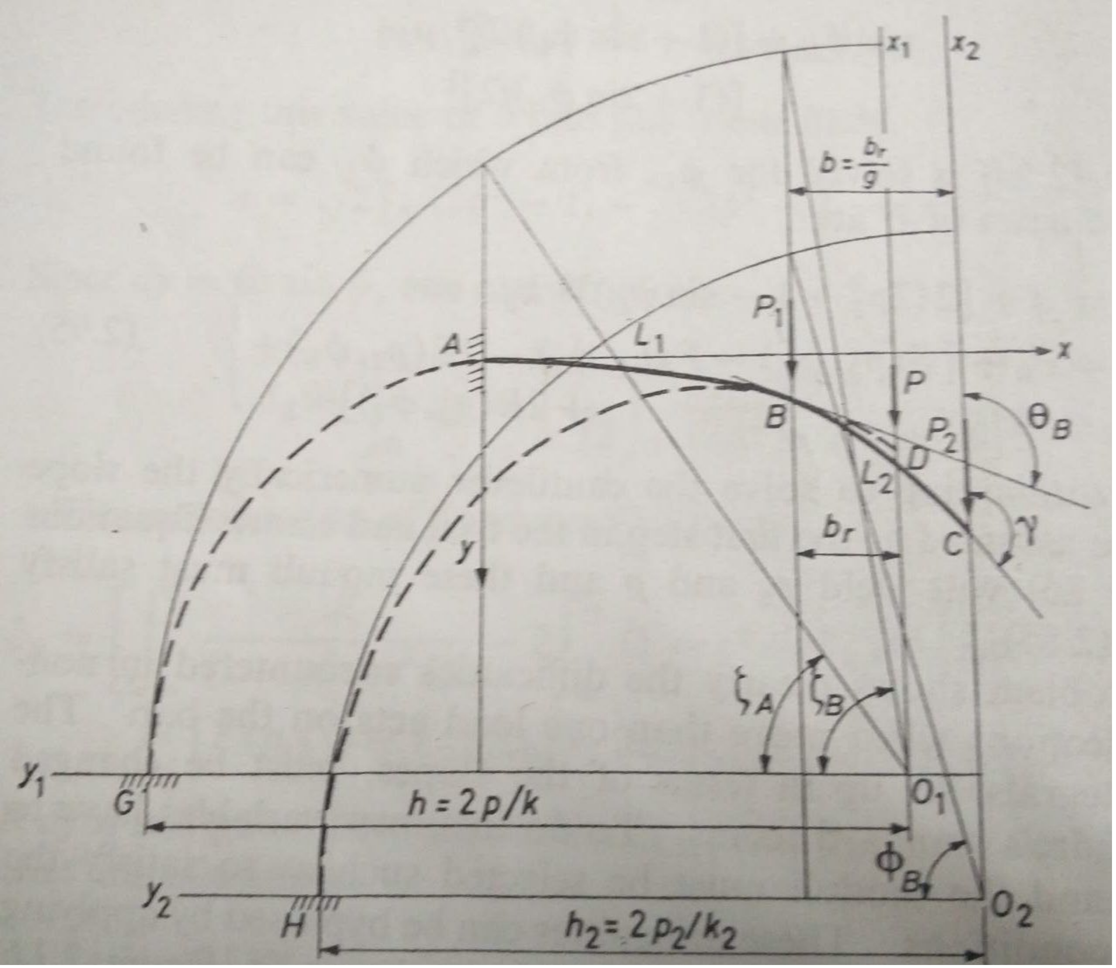


Frisch-Fay, R., *Flexible Bars*, 1962.

Two transverse loads using elastic similarity

Frisch-Fay, R., *Flexible Bars*, 1962.

$$b = \frac{b_r}{g} \qquad g = \frac{P_2}{P}$$



What if there are n loads?

- There will be as many p-unknowns as the number of loads.
- There will be as many ϕ–unknowns as the number of loads less one.
- So, $(2n\text{-}1)$ unknowns and as many equations in terms of elliptic integrals.

Further reading

- Frisch-Fay, R., *Flexible Bars*, Butterworths, London, 1962.