ME 254

Large displacement analysis of a cantilever beam

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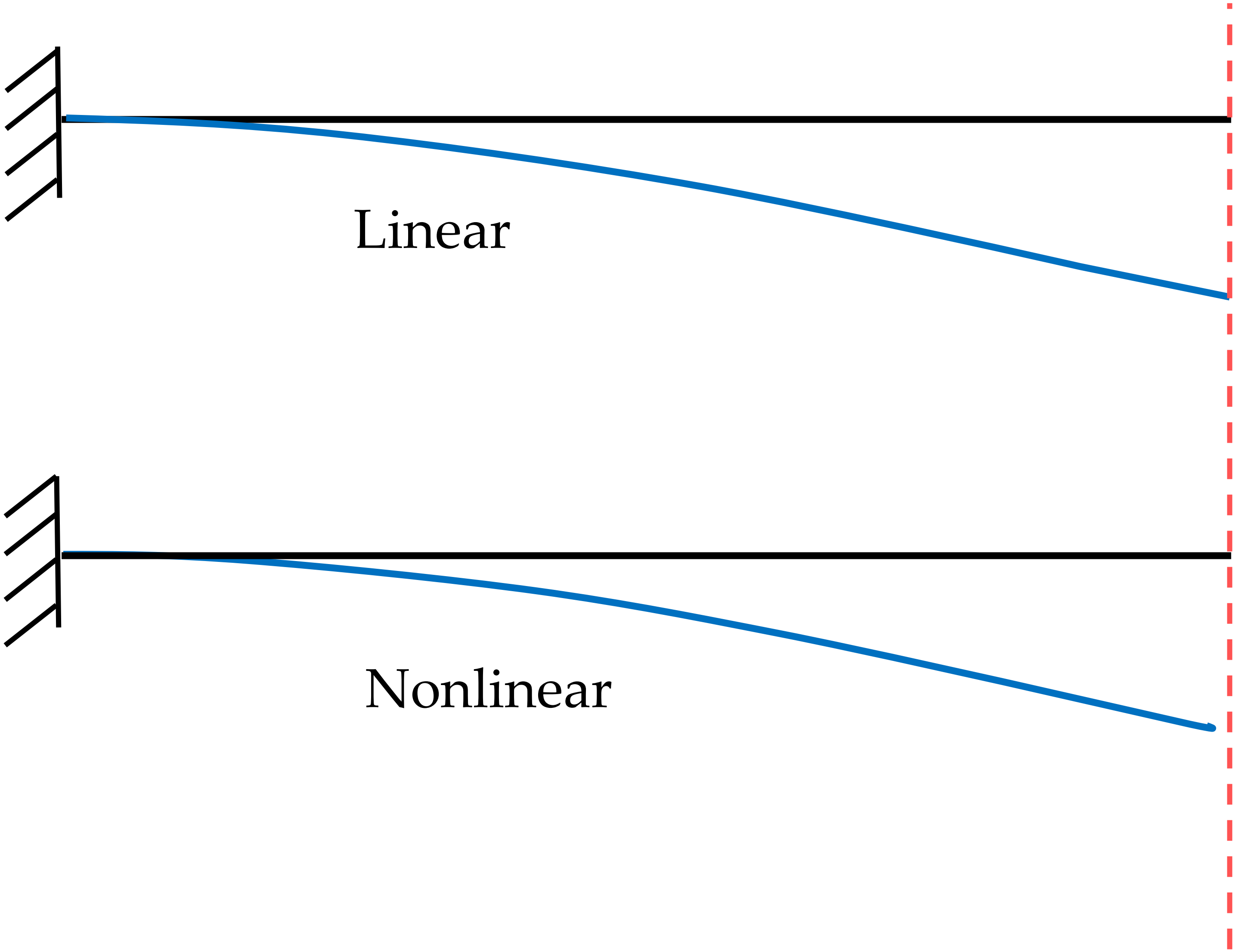
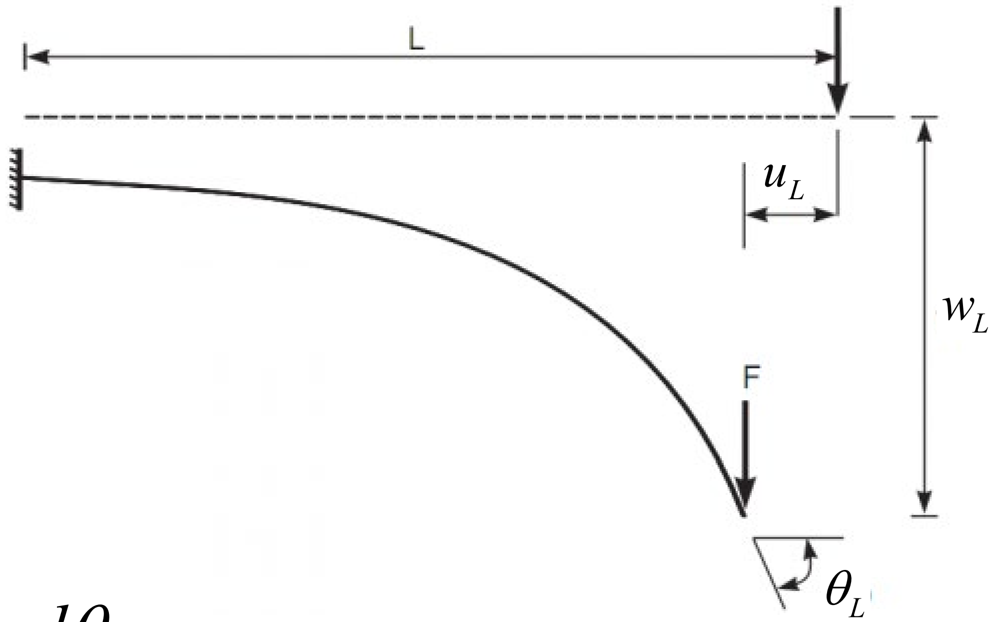
Deformation of a cantilever beam



Important references

- Kirchhoff, G. R., “On the Equilibrium and Movements of an Infinitely Thin Bar,” Crelles Journal Math., 56 (1859).
- Bisshopp, K. E. and Drucker D. C., “Large Deflections of Cantilever Beams,” Quart. Appl. Math., 3 (1945), p. 272.
- Frisch-Fay, R., *Flexible Bars*, Butterworths, London, 1962.

Large displacements analysis of a cantilever beam with a tip-load



$$EI\frac{d\theta}{ds} = F(L - x - u_L)$$ Large displacements

$$EI\frac{d^2 w(x)}{dx^2} = F(L - x)$$ Small displacements

Two differences between linear and nonlinear governing equations

$$EI\frac{d^2w(x)}{dx^2} = F(L-x)$$

$$EI\frac{d\theta}{ds} = F(L-x-u)$$

1. Moment balance in the original (linear) and deformed (nonlinear) configurations.
2. Approximation of beam curvature in the linear case.

Elastica equation

$$EI\frac{d\theta}{ds} = F(L - x - u)$$

Kirchhoff's pendulum analogy

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

Differentiate to get

$$\frac{d^2\theta}{ds^2} = -\frac{F}{EI}\frac{dx}{ds} = -\frac{F}{EI}\cos\theta$$

Elastica equation

$$EI\frac{d\theta}{ds} = F(L - x - u)$$

Differentiate to get

$$\frac{d^2\theta}{ds^2} = -\frac{F}{EI}\frac{dx}{ds} = -\frac{F}{EI}\cos\theta$$

(a little manipulation)

$$\frac{d^2\theta}{ds^2}\frac{d\theta}{ds} = -\frac{F}{EI}\cos\theta\frac{d\theta}{ds}$$

Integrate to get

$$\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 = -\frac{F}{EI}\sin\theta + C$$

(Curvature is zero at the tip)

$$C = \frac{F}{EI}\sin\theta_L$$

Elastica equation (contd.)

Differential equation now:

$$\frac{d\theta}{ds} = \sqrt{\frac{2F}{EI}\left(\sin\theta_L - \sin\theta\right)}$$

Assumption of no stretching.

$$\int_0^L ds = L = \int_0^{\theta_L} \frac{d\theta}{\sqrt{\frac{2F}{EI}\left(\sin\theta_L - \sin\theta\right)}}$$

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{\left(\sin\theta_L - \sin\theta\right)}} = \sqrt{2}\sqrt{\frac{FL^2}{EI}} = \sqrt{2\eta}$$

$$\eta = \frac{FL^2}{EI}$$

Change of variable

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{(\sin\theta_L - \sin\theta)}} = \sqrt{2\eta}$$

$$\sin\theta = 2p^2 \sin^2\phi - 1$$

$$p^2 = \frac{1+\sin\theta_L}{2}$$

$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\pi/2} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} = \sqrt{\eta}$$

Let us see how…

Change of variable

$\theta \rightarrow \phi$

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{\left(\sin\theta_L - \sin\theta\right)}} = \sqrt{2\eta}$$

$$\sin\theta = 2p^2 \sin^2\phi - 1$$

$$p^2 = \frac{1+\sin\theta_L}{2}$$

$$\sin\theta = 2p^2 \sin^2\phi - 1$$

$$2p^2 - 1 = \sin\theta_L$$

Limits:

Change of variable

$$\theta \to \phi$$

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{\left(\sin\theta_L - \sin\theta\right)}} = \sqrt{2\eta}$$

$$\sin\theta = 2p^2 \sin^2\phi - 1$$

$$p^2 = \frac{1+\sin\theta_L}{2}$$

$$\sin\theta = 2p^2 \sin^2\phi - 1$$

$$2p^2 - 1 = \sin\theta_L$$

Change of variable $\theta \to \phi$

$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{\left(\sin\theta_L - \sin\theta\right)}} = \sqrt{2\eta}$$

$$\sin\theta = 2p^2\sin^2\phi - 1$$

$$p^2 = \frac{1+\sin\theta_L}{2}$$

$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\pi/2} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} = \sqrt{\eta}$$

Elliptic integrals

I kind (incomplete)

$$\int_0^\theta \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} = \mathbf{F}(p,\theta)$$

I kind (complete)

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-p^2\sin^2\phi}} = \mathbf{F}(p,\pi/2)$$

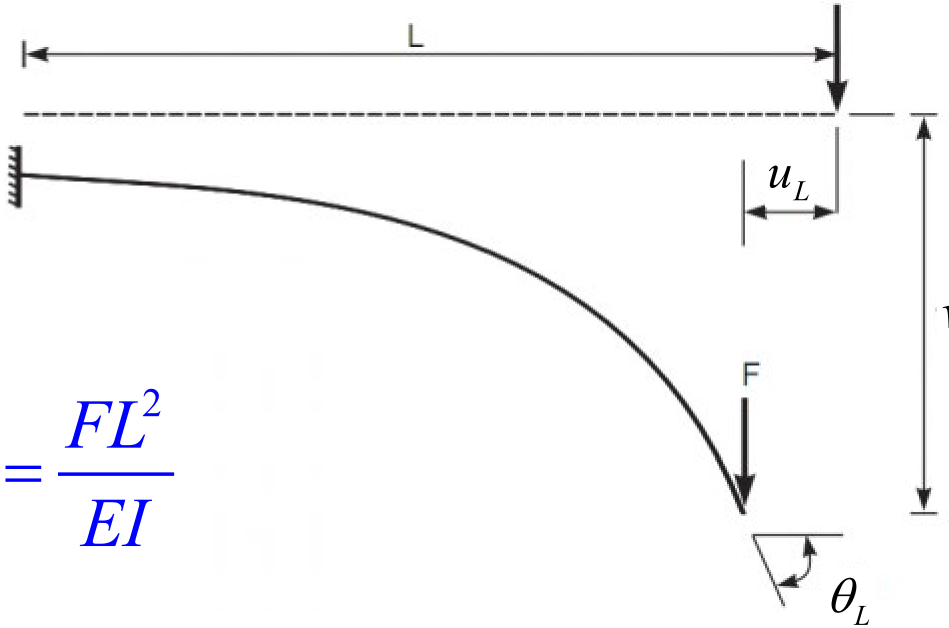
II kind (incomplete)

$$\int_0^\theta \sqrt{1-p^2\sin^2\phi}\; d\phi = \mathbf{E}(p,\theta)$$

II kind (complete)

$$\int_0^{\pi/2} \sqrt{1-p^2\sin^2\phi}\; d\phi = \mathbf{E}(p,\pi/2)$$

Finally, the solution!



$$2p^2 - 1 = \sin\theta_L$$

$$\int_{\sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)}^{\pi/2} \frac{d\phi}{\sqrt{1 - p^2\sin^2\phi}} = \sqrt{\eta}$$

$$\eta = \frac{FL^2}{EI}$$

$$\phi_0 = \sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)$$

$$\frac{w_L}{L} = \sqrt{\eta}\left\{\mathbf{F}(p,\pi/2) - \mathbf{F}(p,\phi_0) - 2\mathbf{E}(p,\pi/2) + 2\mathbf{E}(p,\phi_0)\right\}$$

$$\frac{u_L}{L} = 1 - \sqrt{\frac{2EI}{FL^2}\left(2p^2 - 1\right)} = 1 - \sqrt{\frac{2}{\eta}\left(2p^2 - 1\right)}$$

Let us see how…

Transverse displacement

$$dw = ds\ \sin\theta$$

$$dw = \frac{\sqrt{EI}\sin\theta}{\sqrt{2F\left(\sin\theta_L - \sin\theta\right)}}$$

since $\dfrac{d\theta}{ds} = \sqrt{\dfrac{2F}{EI}\left(\sin\theta_L - \sin\theta\right)}$

$$\Rightarrow w_L = \int_0^{\theta_L} \frac{\sqrt{EI}\sin\theta}{\sqrt{2F\left(\sin\theta_L - \sin\theta\right)}} d\theta$$

$$\Rightarrow w_L = \sqrt{\frac{EI}{F}} \int_{\phi_0}^{\pi/2} \frac{\left(2p^2\sin^2\phi - 1\right)}{\sqrt{1-p^2\sin^2\phi}} d\theta$$

$$\phi_0 = \sin^{-1}\left(\frac{1}{p\sqrt{2}}\right)$$

Transverse displacement (contd.)

$$w_L = \sqrt{\frac{EI}{F}} \int_{\phi_0}^{\pi/2} \frac{\left(2p^2\sin^2\phi - 1 - 1 + 1\right)}{\sqrt{1 - p^2\sin^2\phi}} d\theta$$

$$\Rightarrow w_L = \sqrt{\frac{EI}{F}} \int_{\phi_0}^{\pi/2} \frac{2\left(p^2\sin^2\phi - 1)\right)}{\sqrt{1 - p^2\sin^2\phi}} d\theta + \sqrt{\frac{EI}{F}} \int_{\phi_0}^{\pi/2} \frac{1}{\sqrt{1 - p^2\sin^2\phi}} d\theta$$

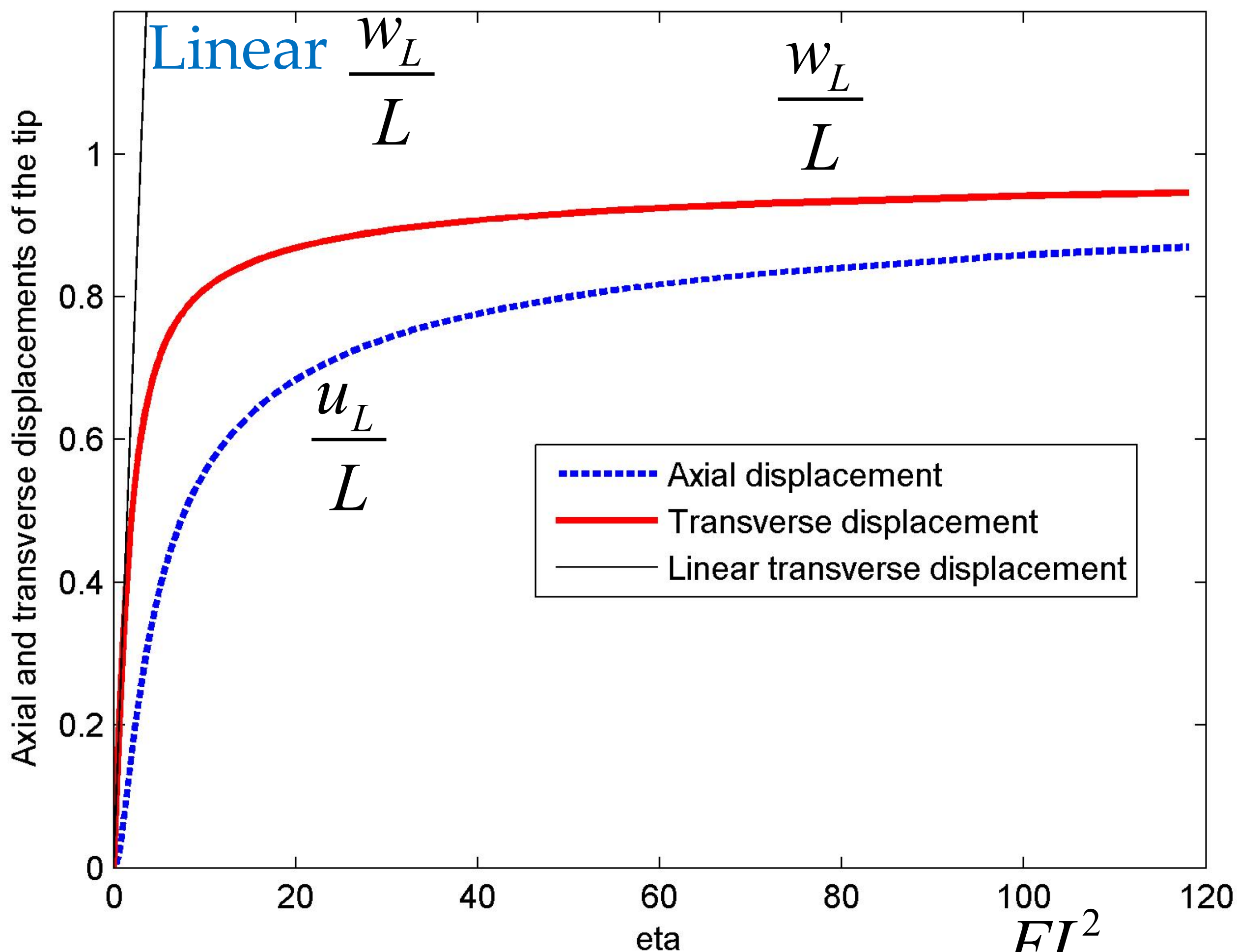
$$\Rightarrow \frac{w_L}{L} = \sqrt{\eta}\left\{\mathbf{F}(p,\pi/2) - \mathbf{F}(p,\phi_0) - 2\mathbf{E}(p,\pi/2) + 2\mathbf{E}(p,\phi_0)\right\}$$

Axial displacement

$$F(L-u_L) = EI\left(\frac{d\theta}{ds}\right)\Bigg|_{x=0} = EI\sqrt{\frac{2F\sin\theta_L}{EI}}$$

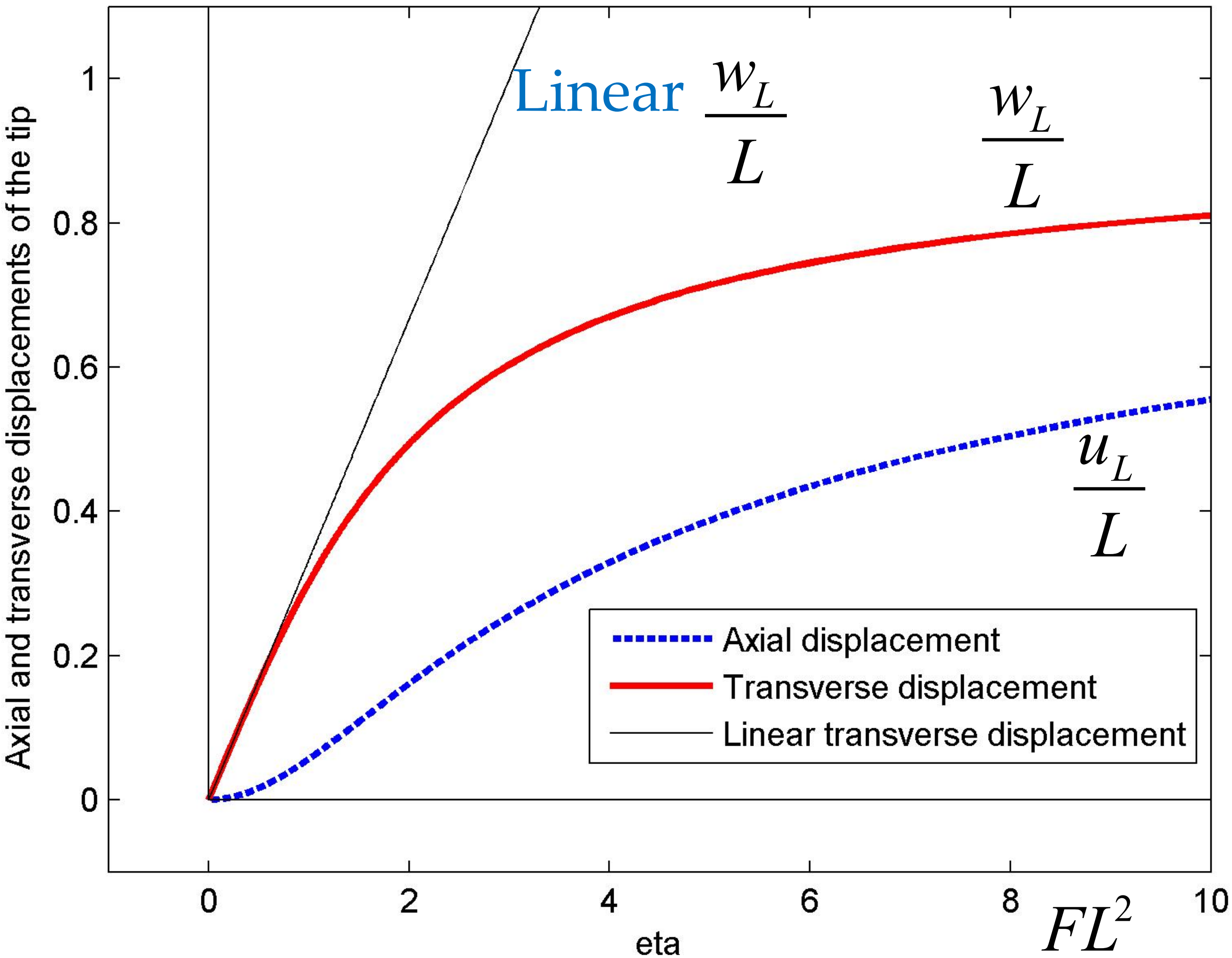
$$\Rightarrow \frac{u_L}{L} = 1-\sqrt{\frac{2EI}{FL^2}\left(2p^2-1\right)}$$

Transverse and axial displacements of the loaded tip



$$\eta = \frac{FL^2}{EI}$$

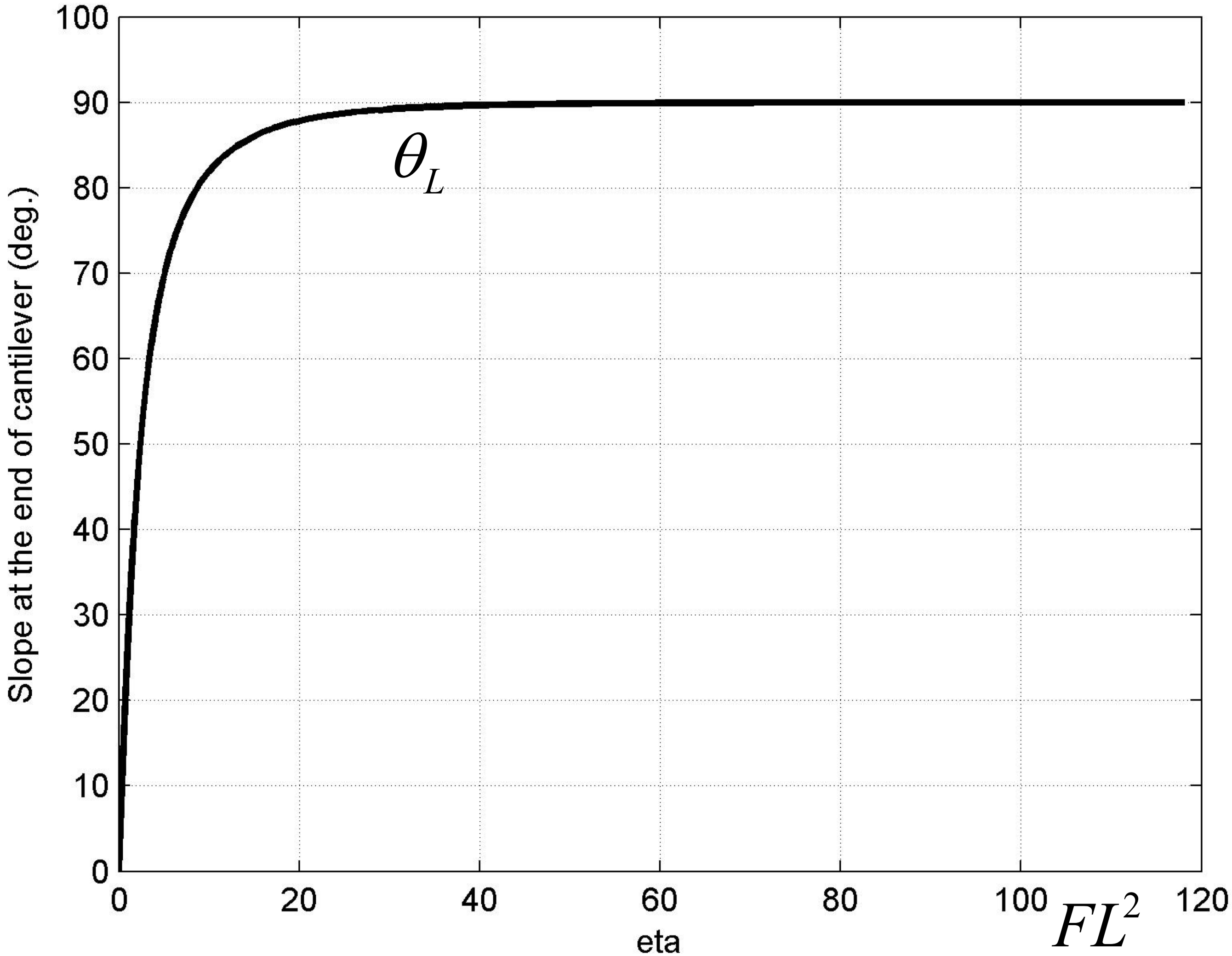
A close-up view



$$\eta = \frac{FL^2}{EI}$$

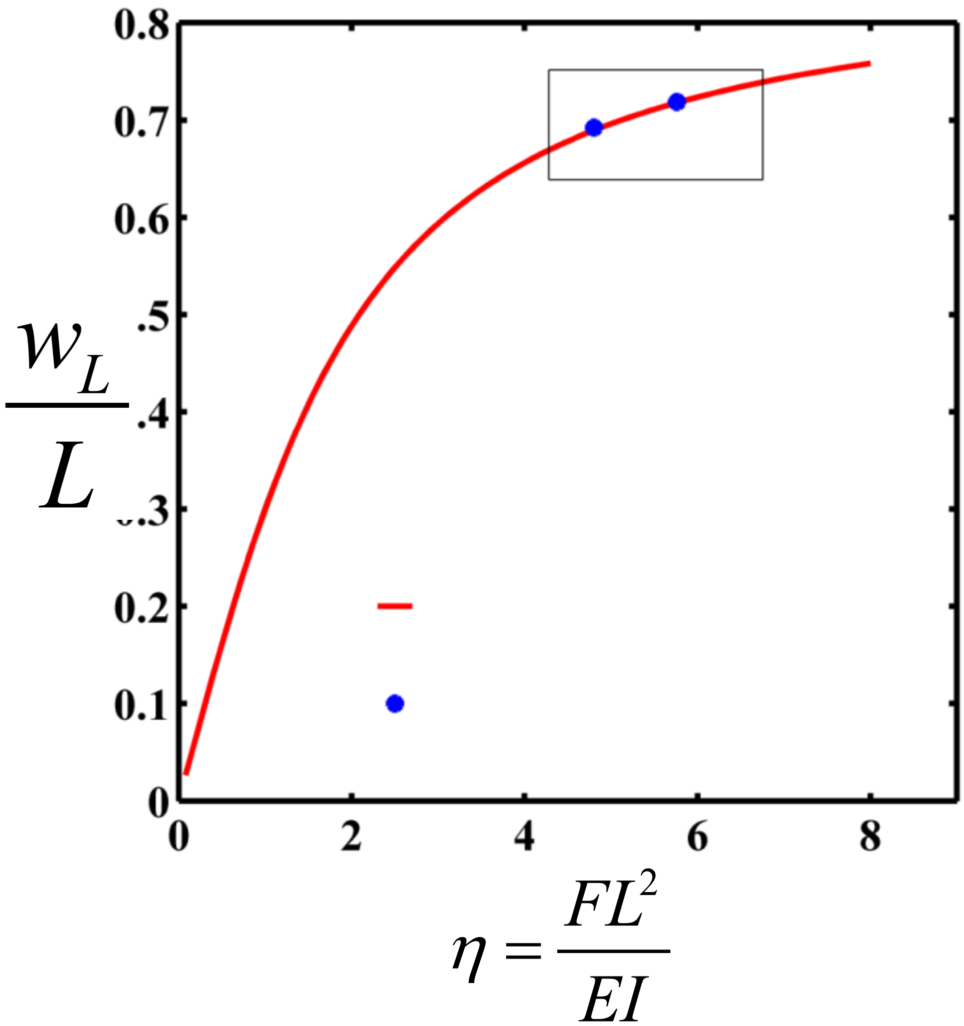
Index of bending

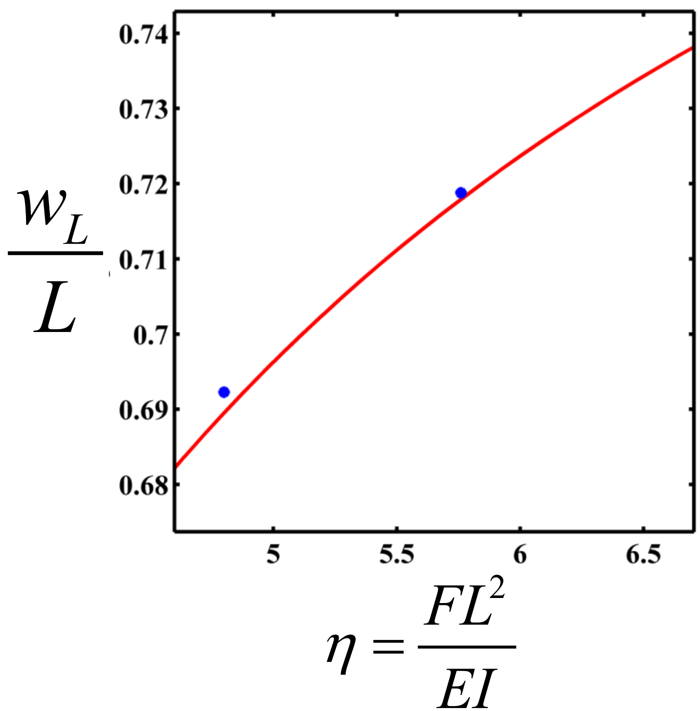
$$\int_0^{\theta_L} \frac{d\theta}{\sqrt{(\sin\theta_L - \sin\theta)}} = \sqrt{2\eta}$$



$$\eta = \frac{FL^2}{EI}$$

Comparing with geometrically nonlinear finite element analysis





Main steps

$$EI\frac{d\theta}{ds} = F(L - x - u)$$

- Differentiation
- And then integration to reduce to an equation involving slope at the tip
- No-stretching assumption
- Change of variable to reduce to an elliptic integral of the I kind
- Then, computing the transverse and axial displacements of the tip

Main points

- Non-dimensional quantity $\eta = \dfrac{FL^2}{EI}$
  - Index of bending
- Analytical solution in terms of elliptic integral for the locus of the loaded tip

Further reading

- Frisch-Fay, R., *Flexible Bars*, Butterworths, London, 1962.