Lecture 6

Maxwell's rule
and
Grübler's formula

G. K. Ananthasuresh

suresh@iisc.ac.in

DoF formula extended to compliant mechanisms

Midha, Murphy, and Howell (1995)
Ananthasuresh and Howell (1996)

3D

$$DoF = 6(n_{seg} - 1) - \sum_{j=1}^{5}(6-j)\,n_{Kj} - \sum_{j=1}^{5}(6-j)\,n_{Cj} - 6n_{fix} + \sum_{j=1}^{6} j\,n_{scj}$$

2D

$$DoF = 3(n_{seg} - 1) - \sum_{j=1}^{2}(3-j)\,n_{Kj} - \sum_{j=1}^{2}(3-j)\,n_{Cj} - 3n_{fix} + \sum_{j=1}^{3} j\,n_{scj}$$

n_{seg} = number of segments (rigid or compliant)

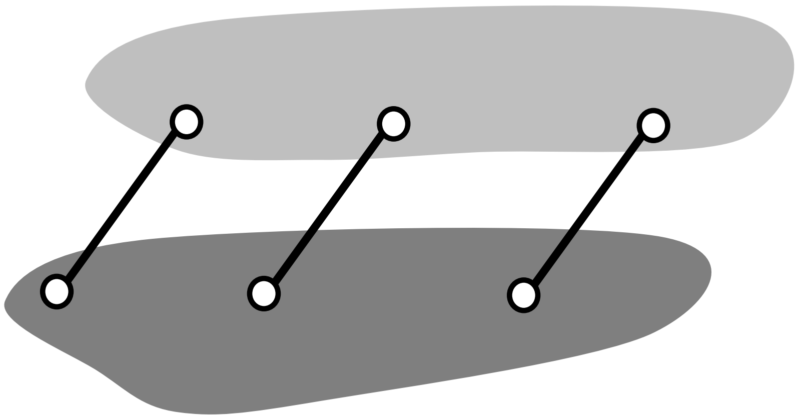
n_{Kj} = number of kinematic pairs allowing j relative dof

n_{Cj} = number of elastic pairs allowing j relative dof

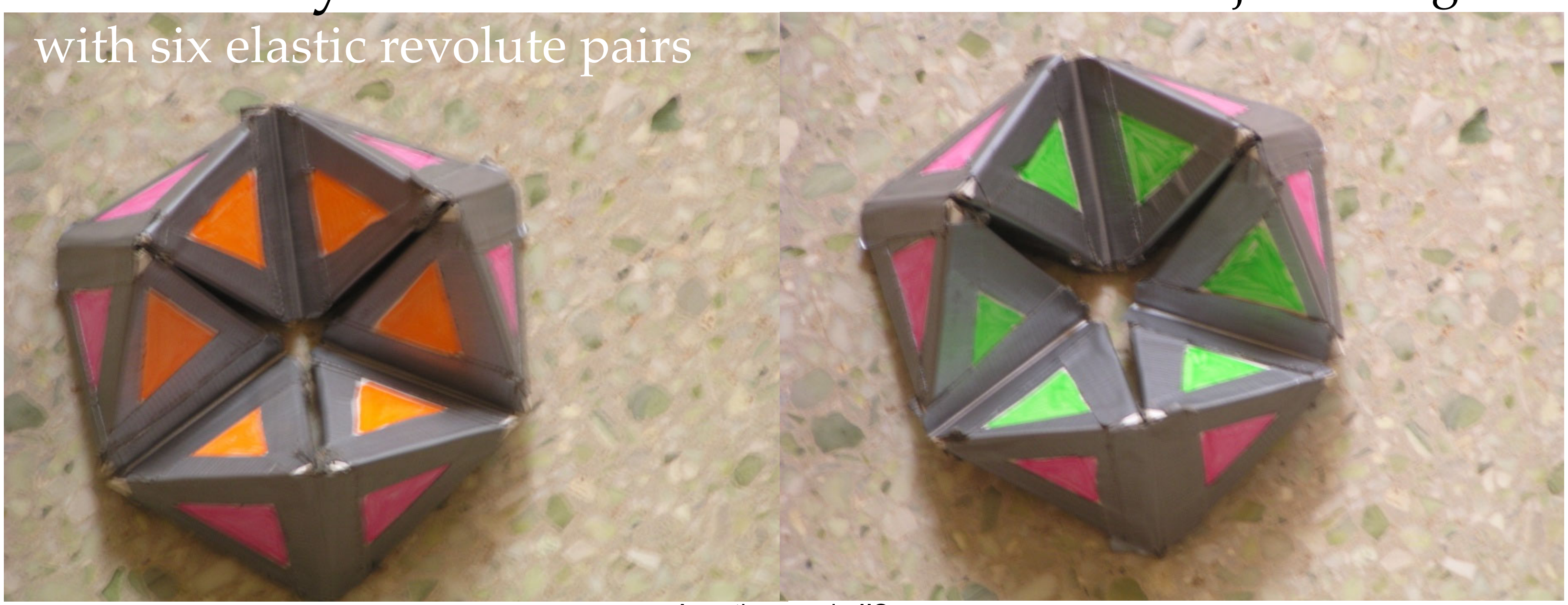
n_{fix} = number of fixed connections

n_{scj} = number of segments with segment compliance of j

An example that does not work



Kaleidocycle mechanism: six-tetrahedrons joined together with six elastic revolute pairs

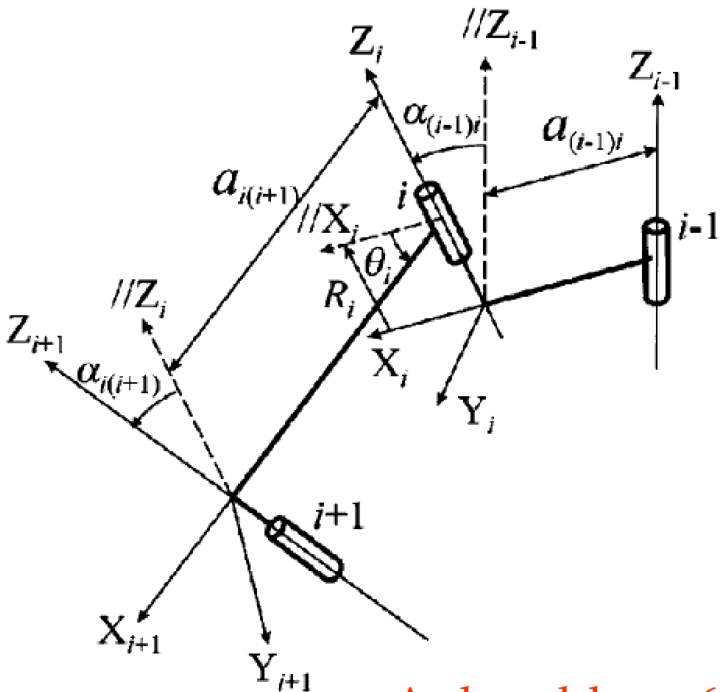


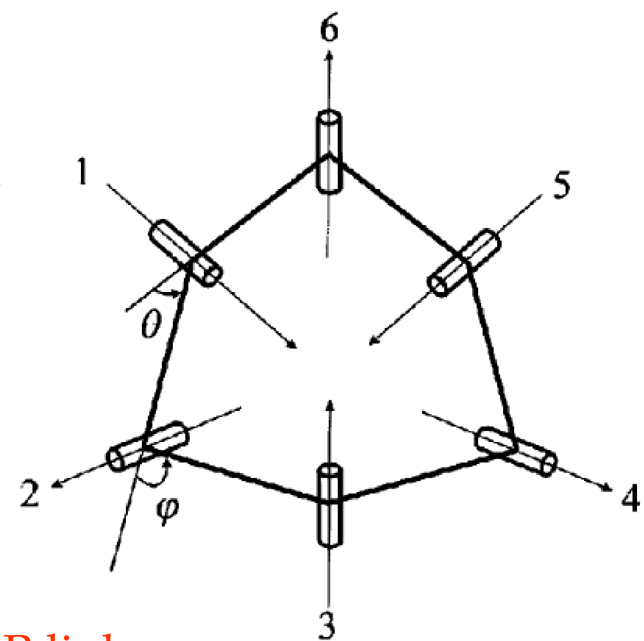
Bricard linkage: the formula fails!

$$DoF = 6(n-1) - 5f_1 - 4f_2 - 3f_3 - 2f_4 - f_5$$

$$= 6(6-1) - (5\times 6) - (4\times 0) - (3\times 0) - (2\times 0) - (1\times 0)$$

$$= 30 - 30 = 0$$





A closed-loop 6R linkage

Maxwell's rule
—structures perspective

2D

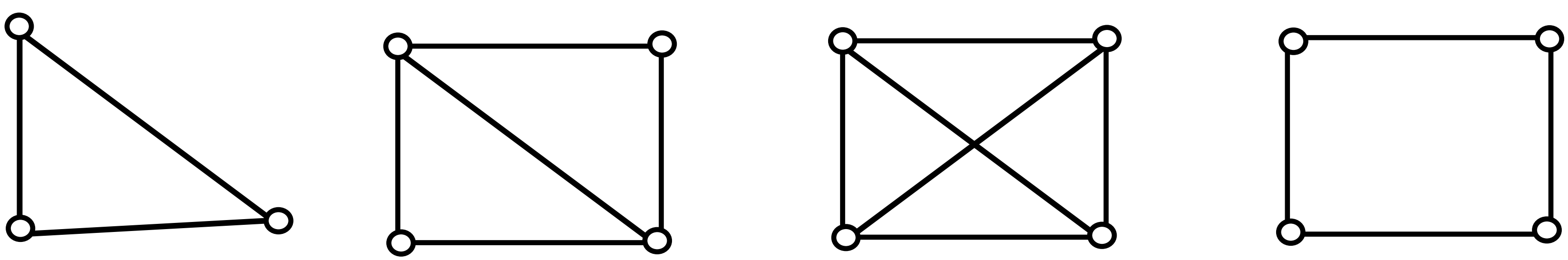
$$2v - 3 - b = 0$$

$v =$ number of vertices

$b =$ number of bars

3D

$$3v - 6 - b = 0$$



Maxwell's rule —structures perspective

2D

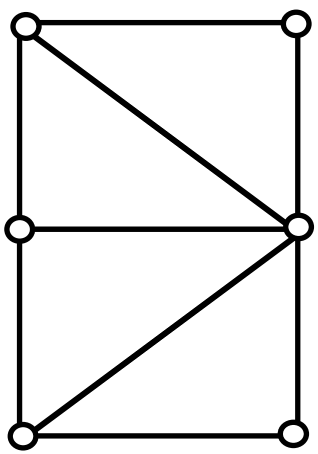
$$2v-3-b=0$$

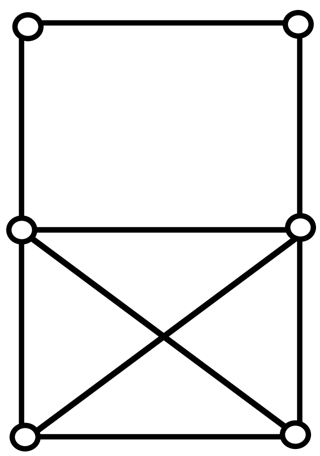
3D

$$3v-6-b=0$$

v = number of vertices

b = number of bars





Maxwell's rule —structures perspective

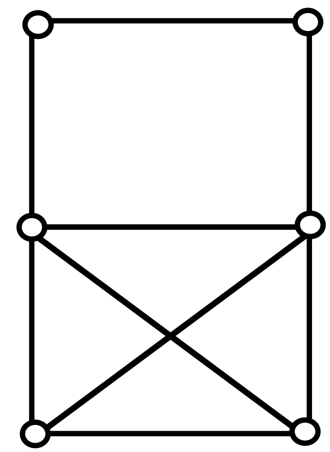
2D

$$2v - 3 - b = DoF - SoSS$$

Modified by Calladine

3D

$$3v - 6 - b = DoF - SoSS$$



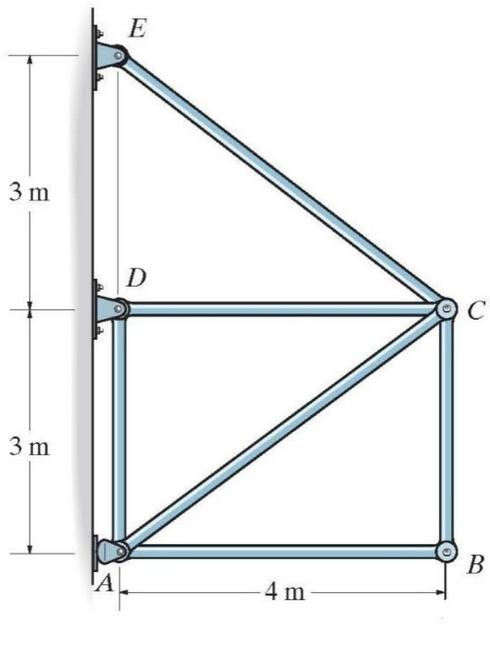
v = number of vertices

b = number of bars

DoF = number of degrees of freedom

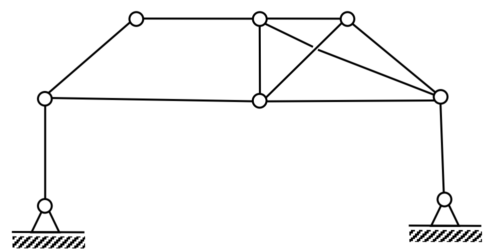
$SoSS$ = number of states of self-stress

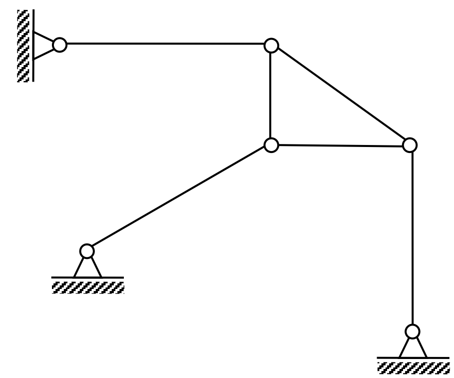
Maxwell's rule applied to trusses



$$2v-3-b=DoF-SoSS$$

$$16-3-12=1=2-1$$





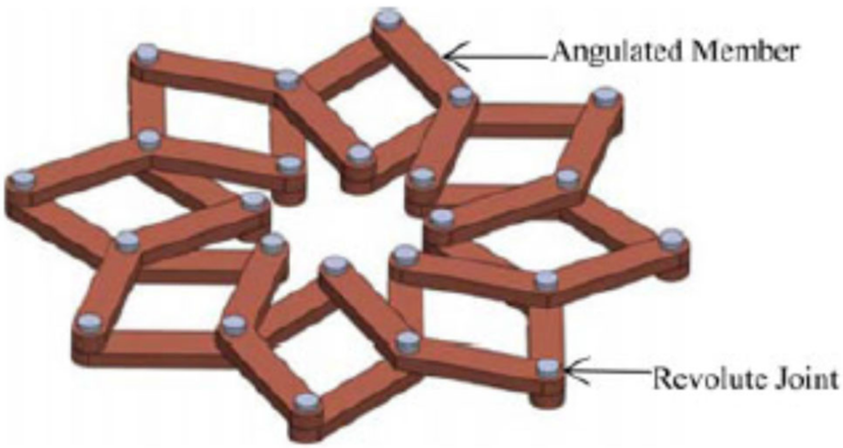
$$2v-3-b=DoF-SoSS$$

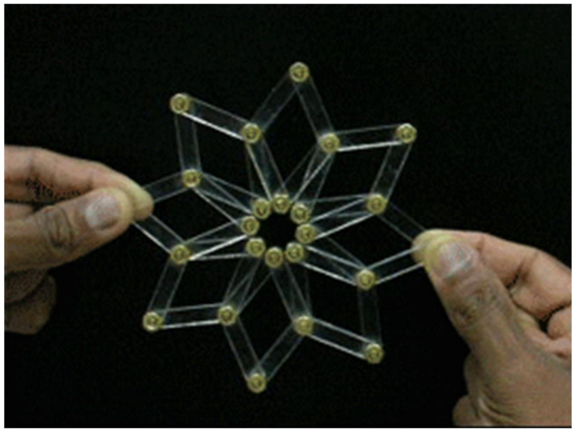
$$10-3-7=0$$

$$2v-3-b=DoF-SoSS$$

$$12-3-9=0$$

Maxwell's rule can be applied to trusses and linkages alike…





$$2v - 3 - b = DoF - SoSS$$

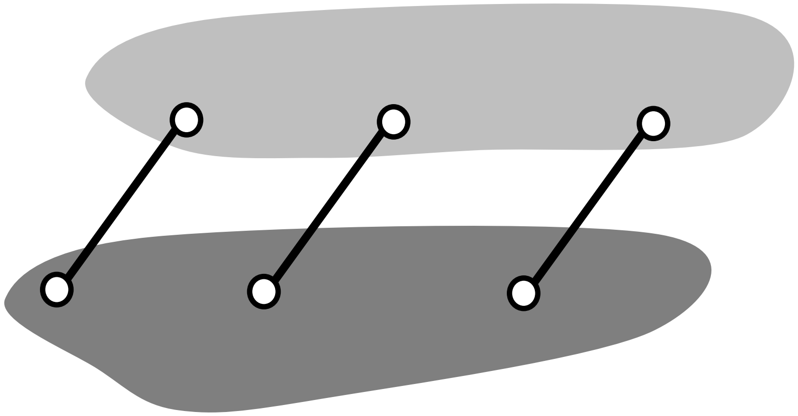
$$48 - 3 - 48 = -3$$

$$DoF = 3(n-1) - 2n_{K1}$$

$$DoF = 3(16-1) - 2 \times 24 = -3$$

Fowler and **Guest** have developed nice theory for counting **symmetry groups** and resolve the conflicts arising due to special geometric conditions.

An example that does not work



Let us understand the Maxwell's rule

2D

$$2v - 3 - b = 0$$

Equilibrium matrix

Equilibrium matrix

Bar forces

Forces at vertices

$$\mathbf{H}_{2v\times b}\mathbf{p}_{b\times 1} = \mathbf{f}_{2v\times 1}$$

Compatibility matrix

Compliance matrix | Disp. at vertices | Bar elongations

$$\mathbf{C}_{b\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{e}_{b\times 1}$$

Compatibility and Equilibrium matrices

Compliance matrix | Disp. at vertices | Bar elongations

$$\mathbf{C}_{b\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{e}_{b\times 1}$$

Equilibrium matrix | Bar forces | Forces at vertices

$$\mathbf{H}_{2v\times b}\mathbf{p}_{b\times 1} = \mathbf{f}_{2v\times 1}$$

$$\mathbf{p}^T \delta\mathbf{e} = \mathbf{f}^T\ \delta\mathbf{u}$$

$$\Rightarrow \mathbf{p}^T \mathbf{C}\,\delta\mathbf{u} = \mathbf{p}^T \mathbf{H}^T\ \delta\mathbf{u}$$

$$\Rightarrow \mathbf{C} = \mathbf{H}^T$$

Compatibility and Equilibrium matrices

Compliance matrix — Disp. at vertices — Bar elongations

$$\mathbf{C}_{b\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{e}_{b\times 1}$$

Equilibrium matrix — Bar forces — Forces at vertices

$$\mathbf{H}_{2v\times b}\mathbf{p}_{b\times 1} = \mathbf{f}_{2v\times 1}$$

$$\mathbf{p}^T\delta\mathbf{e} = \mathbf{f}^T\ \delta\mathbf{u}$$

$$\Rightarrow \mathbf{p}^T\mathbf{C}\delta\mathbf{u} = \mathbf{p}^T\mathbf{H}^T\ \delta\mathbf{u}$$

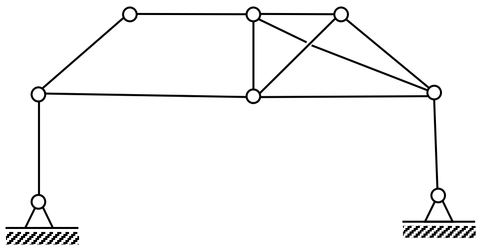
$$\Rightarrow \mathbf{C} = \mathbf{H}^T$$

Rank-deficiency of **C** indicates DoF
Rank-deficiency of **H** indicates SoSS.

Null-space of **C** indicates instantaneous rigid-body modes.

Null-space of **H** indicates self-stress modes.

DoF and SoSS



$$2v - 3 - b = DoF - SoSS$$

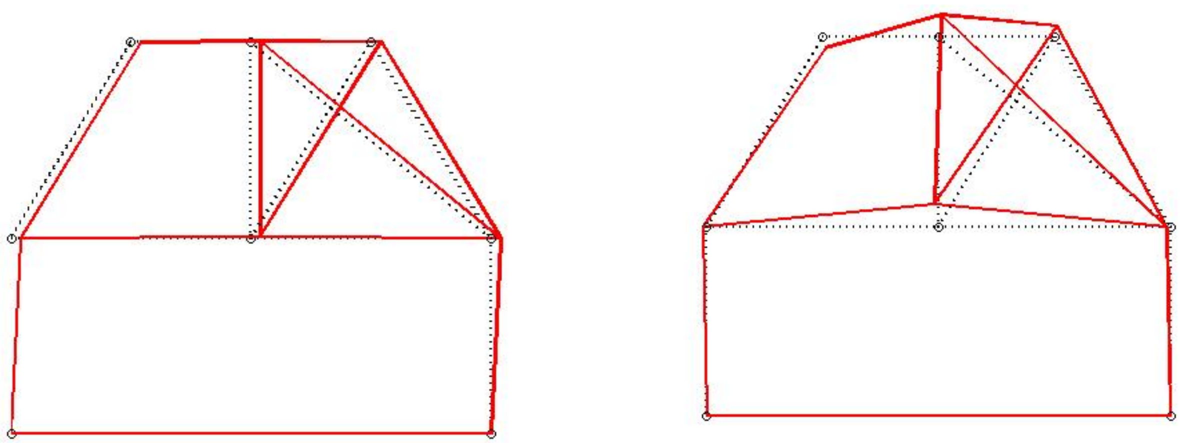
$$16 - 3 - 12 = 1 = 2 - 1$$

$$\mathbf{C}_{12\times16}\mathbf{u}_{16\times1} = \mathbf{e}_{12\times1}$$

Rank deficiency = 2 (not counting rigid-body modes)
→ 2 DoF

$$\mathbf{H}_{16\times12}\mathbf{p}_{12\times1} = \mathbf{f}_{16\times1}$$

Rank deficiency = 1
➔ 1 SoSS



Null-space “modes” of **C**.

Can also use stiffness matrix.
(finite element framework)

Compliance matrix | Disp. at vertices | Bar elongations

$$\mathbf{C}_{b\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{e}_{b\times 1}$$

Equilibrium matrix | Bar forces | Forces at vertices

$$\mathbf{H}_{2v\times b}\mathbf{p}_{b\times 1} = \mathbf{f}_{2v\times 1}$$

$$\mathbf{p}_{b\times 1} = \mathbf{D}_{b\times b}\mathbf{e}_{b\times 1}$$

$$\mathbf{K}_{2v\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{f}_{2v\times 1}$$

$$\Rightarrow \mathbf{p}_{b\times 1} = \mathbf{D}_{b\times b}\mathbf{C}_{b\times 2v}\mathbf{u}_{2v\times 1}$$

$$\Rightarrow \mathbf{H}_{2v\times b}\mathbf{D}_{b\times b}\mathbf{C}_{b\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{H}_{2v\times b}\mathbf{p}_{b\times 1}$$

$$\Rightarrow \mathbf{H}_{2v\times b}\mathbf{D}_{b\times b}\mathbf{C}_{b\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{f}_{2v\times 1}$$

Rank deficiency of the stiffness matrix

$$\mathbf{K}_{2v\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{f}_{2v\times 1}$$

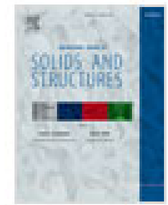
Main Points

- Maxwell's rule and Grübler's formula are equivalent for trusses/linkages.
- Compatibility and equilibrium matrices give correct but only instantaneous DoF and SoSS.
- Stiffness matrix can also be used for finding instantaneous (infinitesimal) DoF.

Further reading

International Journal of Solids and Structures

Volume 14, Issue 2, 1978, Pages 161-172



Buckminster Fuller's "Tensegrity" structures and Clerk Maxwell's rules for the construction of stiff frames

C.R. Calladine

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