ME 254

Analysis of Compliant Mechanisms using Pseudo Rigid-body Modeling

Illustration with an Example

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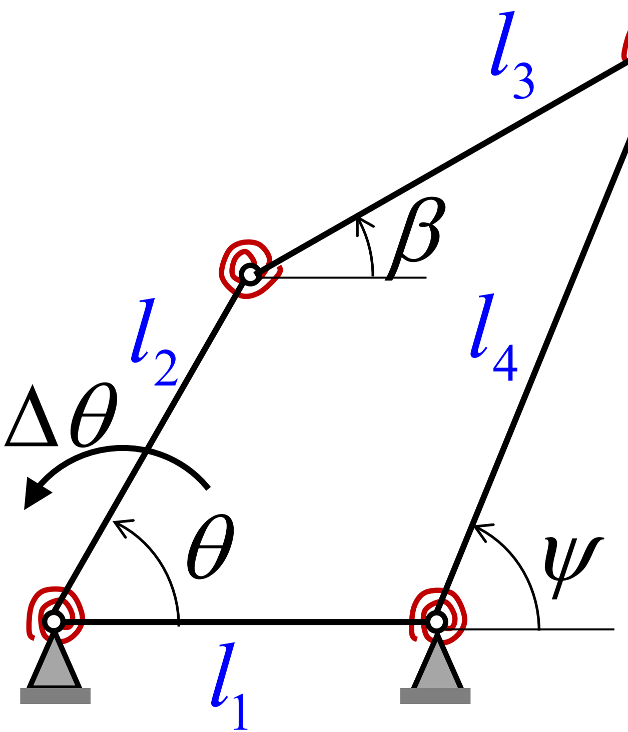
Analysis of a PRB model of a compliant mechanism = kinematic analysis + elastostatic analysis

Kinematic update equations

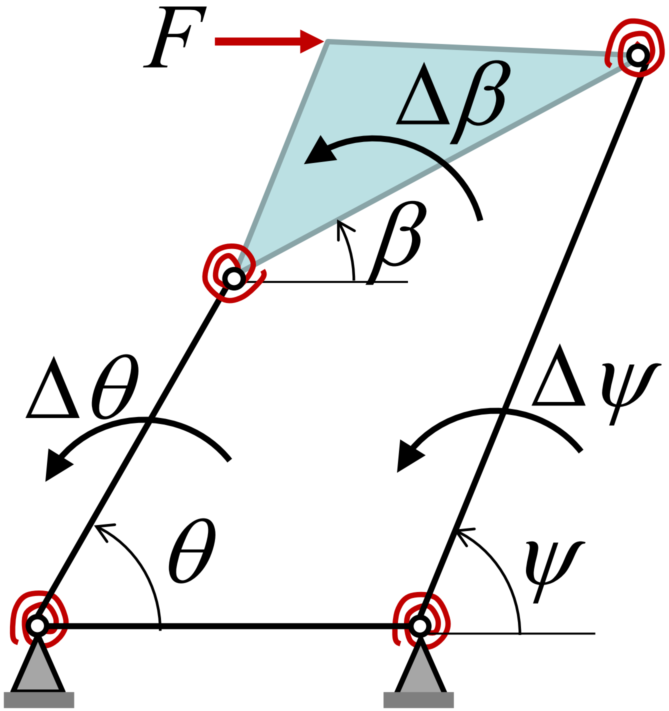
$$\beta_{updated} = \beta_{current} + \frac{d\beta}{d\theta}\Delta\theta = \beta_{current} + \frac{l_2 \sin(\psi - \theta)}{l_3 \sin(\beta - \psi)}\Delta\theta$$

$$\psi_{updated} = \psi_{current} + \frac{d\psi}{d\theta}\Delta\theta = \psi_{current} + \frac{l_2 \sin(\beta - \theta)}{l_4 \sin(\beta - \psi)}\Delta\theta$$



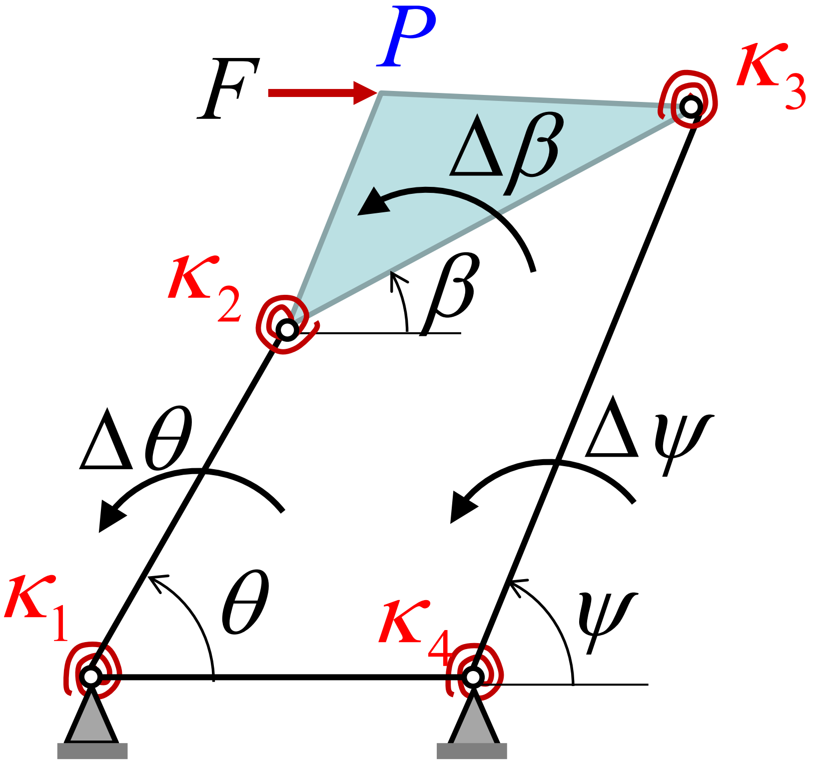
But how do we know $\Delta\theta$?

Elastostatic analysis



Use F to get $\Delta\theta$

Elastostatic analysis



Use F to get $\Delta\theta$

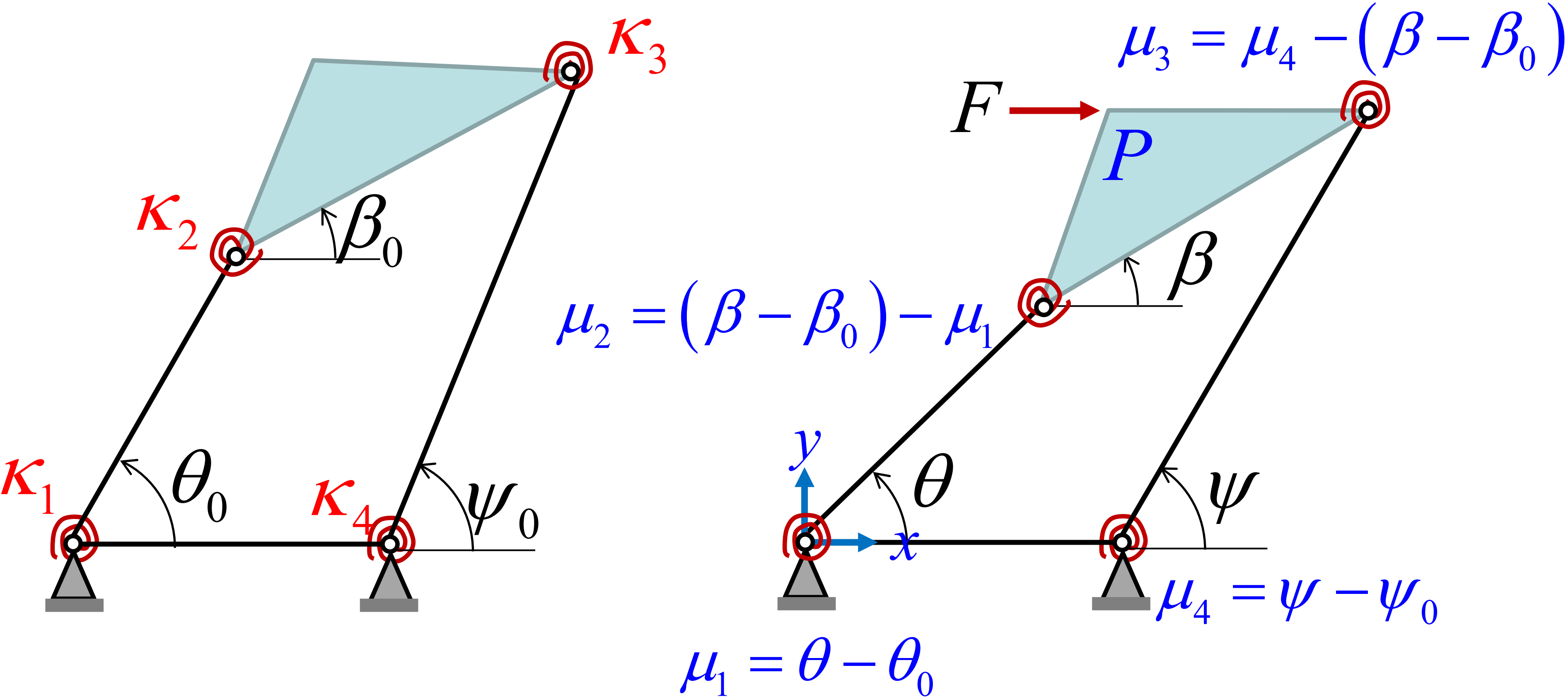
Energy method

Minimize potential energy.

Potential energy = strain energy + work potential

PE = SE + WP

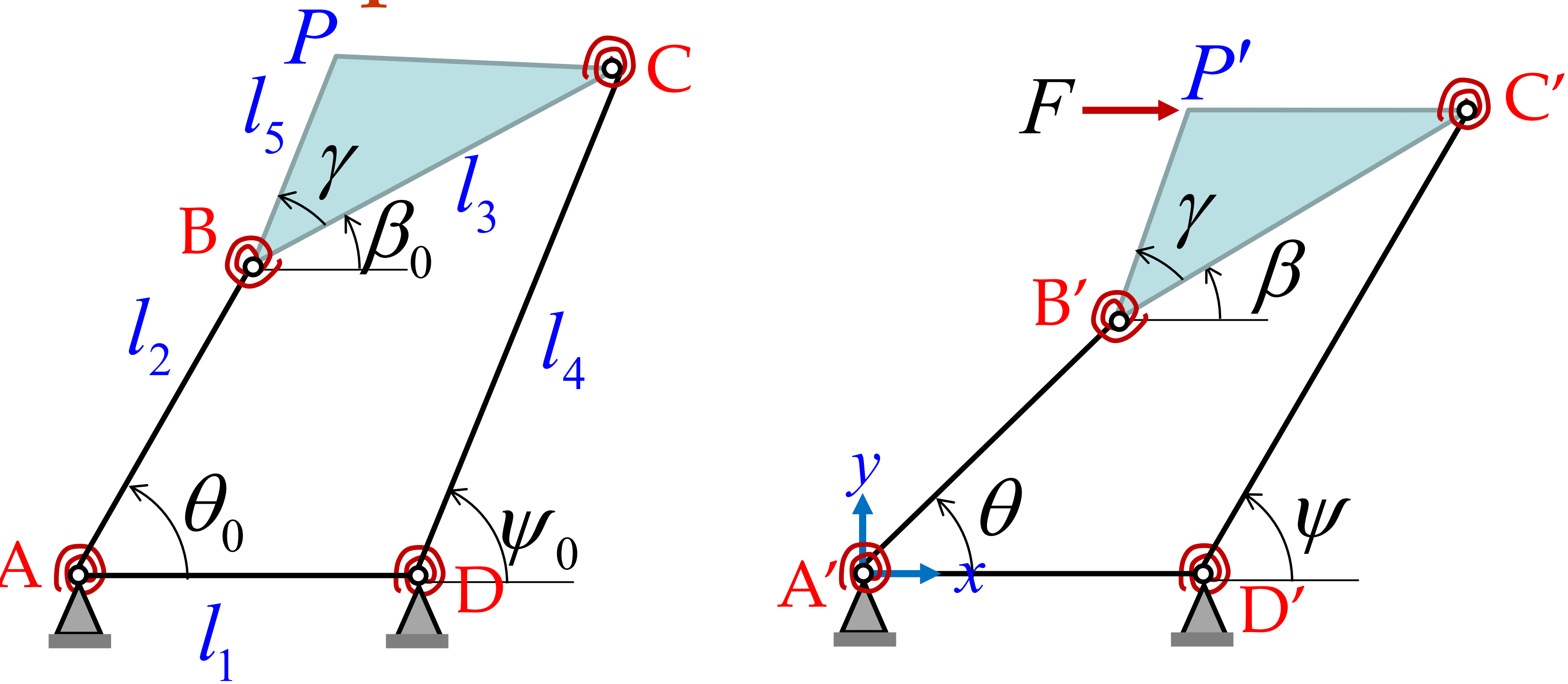
Original and moved configurations



$$WP = -F\Delta P_x$$

$$SE = \frac{1}{2}\sum_{i=1}^{4}\kappa_i \mu_i^{\ 2}$$

Work potential



$$P_x = l_2 \cos\theta_0 + l_5 \cos(\beta_0 + \gamma)$$

$$WP = -F\Delta P_x$$

$$P_x' = l_2 \cos\theta + l_5 \cos(\beta + \gamma)$$

$$\Delta P_x = P_x' - P_x$$

Potential energy

$$SE = \frac{1}{2}\sum_{i=1}^{4}\kappa_i \mu_i^{\,2}$$

$$\mu_1 = \theta - \theta_0$$

$$\mu_2 = (\beta - \beta_0) - \mu_1$$

$$\mu_3 = \mu_4 - (\beta - \beta_0)$$

$$WP = -F\Delta P_x$$

$$\mu_4 = \psi - \psi_0$$

$$PE = SE + WP = \frac{1}{2}\sum_{i=1}^{4}\kappa_i \mu_i^{\,2} - F\Delta P_x$$

Minimization of potential energy

$$PE = SE + WP = \frac{1}{2}\sum_{i=1}^{4}\kappa_i \mu_i^{\,2} - F\,\Delta P_x$$

$$\frac{dPE}{d\theta} = \sum_{i=1}^{4}\kappa_i \mu_i \frac{d\,\mu_i}{d\theta} - F\frac{d\,\Delta P_x}{d\theta} = 0$$

} Necessary condition

Derivatives

$$\mu_1 = \theta - \theta_0$$

$$\frac{d\mu_1}{d\theta} = 1$$

$$\mu_2 = \left(\beta - \beta_0\right) - \mu_1$$

$$\frac{d\mu_2}{d\theta} = \frac{d\beta}{d\theta} - 1$$

$$\mu_3 = \mu_4 - \left(\beta - \beta_0\right)$$

$$\frac{d\mu_3}{d\theta} = \frac{d\mu_4}{d\theta} - \frac{d\beta}{d\theta}$$

$$\mu_4 = \psi - \psi_0$$

$$\frac{d\mu_4}{d\theta} = \frac{d\psi}{d\theta}$$

Recall that

$$\frac{d\beta}{d\theta} = \frac{l_2 \sin\left(\psi - \theta\right)}{l_3 \sin\left(\beta - \psi\right)}$$

$$\frac{d\psi}{d\theta} = \frac{l_2 \sin\left(\beta - \theta\right)}{l_4 \sin\left(\beta - \psi\right)}$$

Derivatives (contd.)

$$P_x = l_2 \cos\theta_0 + l_5 \cos(\beta_0 + \gamma)$$

$$P_x' = l_2 \cos\theta + l_5 \cos(\beta + \gamma)$$

$$\Delta P_x = P_x' - P_x$$

$$\frac{d\Delta P_x}{d\theta} = -l_2 \sin\theta - l_5 \sin(\beta + \gamma)\frac{d\beta}{d\theta}$$

Elastic equilibrium equation

$$\frac{dPE}{d\theta} = \sum_{i=1}^{4} \kappa_i \mu_i \frac{d\mu_i}{d\theta} - F\frac{d\Delta P_x}{d\theta} = 0$$

$$\kappa_1\mu_1 + \kappa_2\mu_2\left(\frac{d\beta}{d\theta} - 1\right) + \kappa_3\mu_3\left(\frac{d\psi}{d\theta} - \frac{d\beta}{d\theta}\right) + \kappa_3\mu_4\frac{d\psi}{d\theta}$$

$$-F\left\{-l_2\sin\theta - l_5\sin(\beta+\gamma)\frac{d\beta}{d\theta}\right\} = 0$$

We need to solve this numerically to find θ for given F

Principle of virtual work

$$\kappa_1\mu_1 + \kappa_2\mu_2\left(\frac{d\beta}{d\theta} - 1\right) + \kappa_3\mu_3\left(\frac{d\psi}{d\theta} - \frac{d\beta}{d\theta}\right) + \kappa_3\mu_4\frac{d\psi}{d\theta}$$

$$-F\left\{-l_2\sin\theta - l_5\sin(\beta+\gamma)\frac{d\beta}{d\theta}\right\} = 0$$

Force equilibrium

$$\left\{\kappa_1\mu_1 + \kappa_2\mu_2\left(\frac{d\beta}{d\theta} - 1\right) + \kappa_3\mu_3\left(\frac{d\psi}{d\theta} - \frac{d\beta}{d\theta}\right) + \kappa_3\mu_4\frac{d\psi}{d\theta}\right\}\delta\theta$$

$$= F\left\{-l_2\sin\theta - l_5\sin(\beta+\gamma)\frac{d\beta}{d\theta}\right\}\delta\theta = 0$$

Internal virtual work = external virtual work

Force-displacement relationship

$$\kappa_1\mu_1 + \kappa_2\mu_2\left(\frac{d\beta}{d\theta} - 1\right) + \kappa_3\mu_3\left(\frac{d\psi}{d\theta} - \frac{d\beta}{d\theta}\right) + \kappa_3\mu_4\frac{d\psi}{d\theta}$$

$$-F\left\{-l_2\sin\theta - l_5\sin(\beta+\gamma)\frac{d\beta}{d\theta}\right\} = 0$$

$$\frac{\kappa_1\mu_1 + \kappa_2\mu_2\left(\frac{d\beta}{d\theta} - 1\right) + \kappa_3\mu_3\left(\frac{d\psi}{d\theta} - \frac{d\beta}{d\theta}\right) + \kappa_3\mu_4\frac{d\psi}{d\theta}}{\left\{-l_2\sin\theta - l_5\sin(\beta+\gamma)\frac{d\beta}{d\theta}\right\}} = F$$

$$f_{\text{int}}(\theta) = F$$

Solving the force-equilibrium equation: tangent stiffness

$$f_{\text{int}}(\theta) = F$$

→ Iterative method

→ Incremental method

$$f_{\text{int}}(\theta_p) + \left.\frac{df_{\text{int}}}{d\theta}\right|_{\theta_p} \Delta\theta = F_p + \Delta F$$

$$f_{\text{int}}(\theta_p) + \left.\frac{df_{\text{int}}}{d\theta}\right|_{\theta_p} \Delta\theta = F_p + \Delta F \Rightarrow \left.\frac{df_{\text{int}}}{d\theta}\right|_{\theta_p} \Delta\theta = \Delta F$$

$$\Delta\theta = \frac{\Delta F}{\left.\frac{df_{\text{int}}}{d\theta}\right|_{\theta_p}} = \frac{\Delta F}{K_t|_{\theta_p}}$$

Tangent stiffness matrix

Tangent stiffness

$$f_{\text{int}} = \frac{\kappa_1\mu_1 + \kappa_2\mu_2\left(\frac{d\beta}{d\theta} - 1\right) + \kappa_3\mu_3\left(\frac{d\psi}{d\theta} - \frac{d\beta}{d\theta}\right) + \kappa_3\mu_4\frac{d\psi}{d\theta}}{\left\{-l_2\sin\theta - l_5\sin(\beta+\gamma)\frac{d\beta}{d\theta}\right\}} = \frac{N}{D}$$

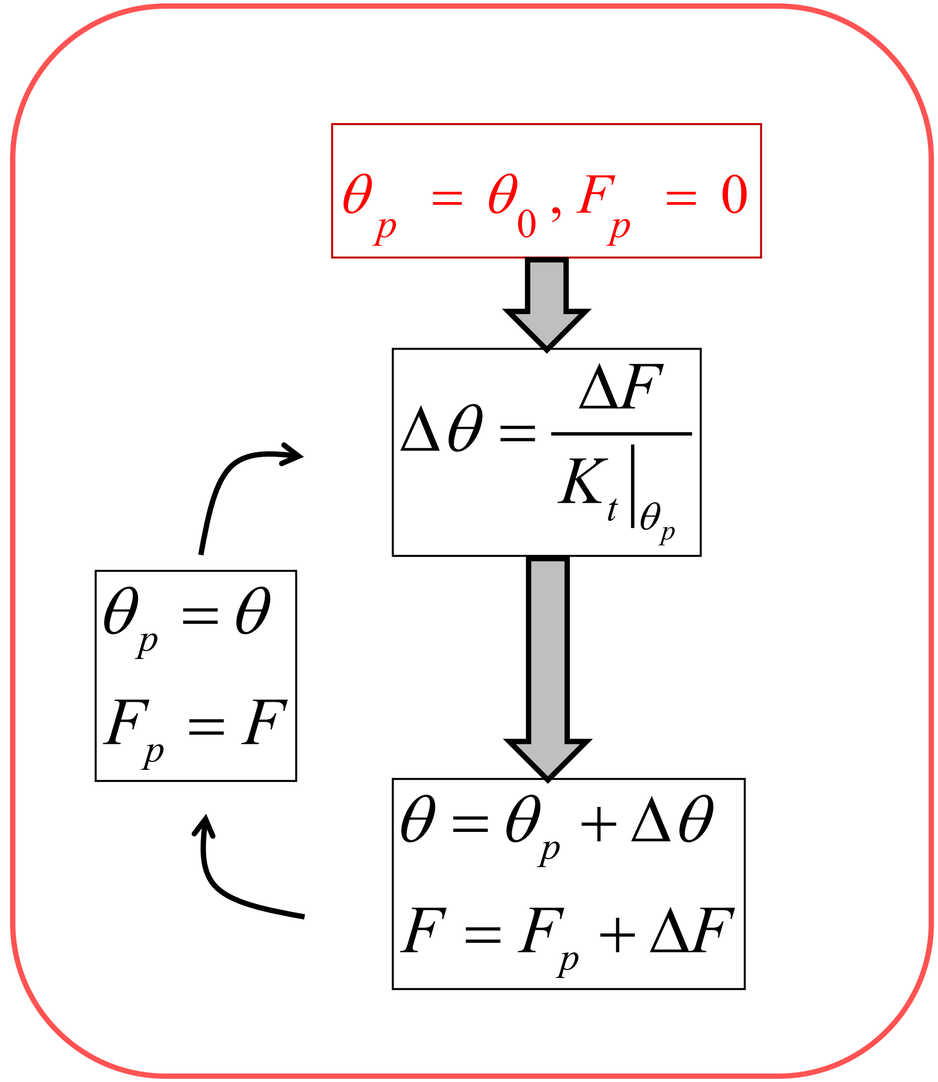
$$\frac{df_{\text{int}}}{d\theta} = K_t = \frac{D\frac{dN}{d\theta} - N\frac{dD}{d\theta}}{D^2}$$

$$\frac{dN}{d\theta} = \kappa_1\frac{d\mu_1}{d\theta} + \kappa_2\frac{d\mu_2}{d\theta}\left(\frac{d\beta}{d\theta} - 1\right) + \kappa_2\mu_2\frac{d^2\beta}{d\theta^2} + \kappa_3\frac{d\mu_3}{d\theta}\left(\frac{d\psi}{d\theta} - \frac{d\beta}{d\theta}\right) +$$
$$+\kappa_3\mu_3\left(\frac{d^2\psi}{d\theta^2} - \frac{d^2\beta}{d\theta^2}\right) + \kappa_3\frac{d\mu_4}{d\theta}\frac{d\psi}{d\theta} + \kappa_3\mu_4\frac{d^2\psi}{d\theta^2}$$

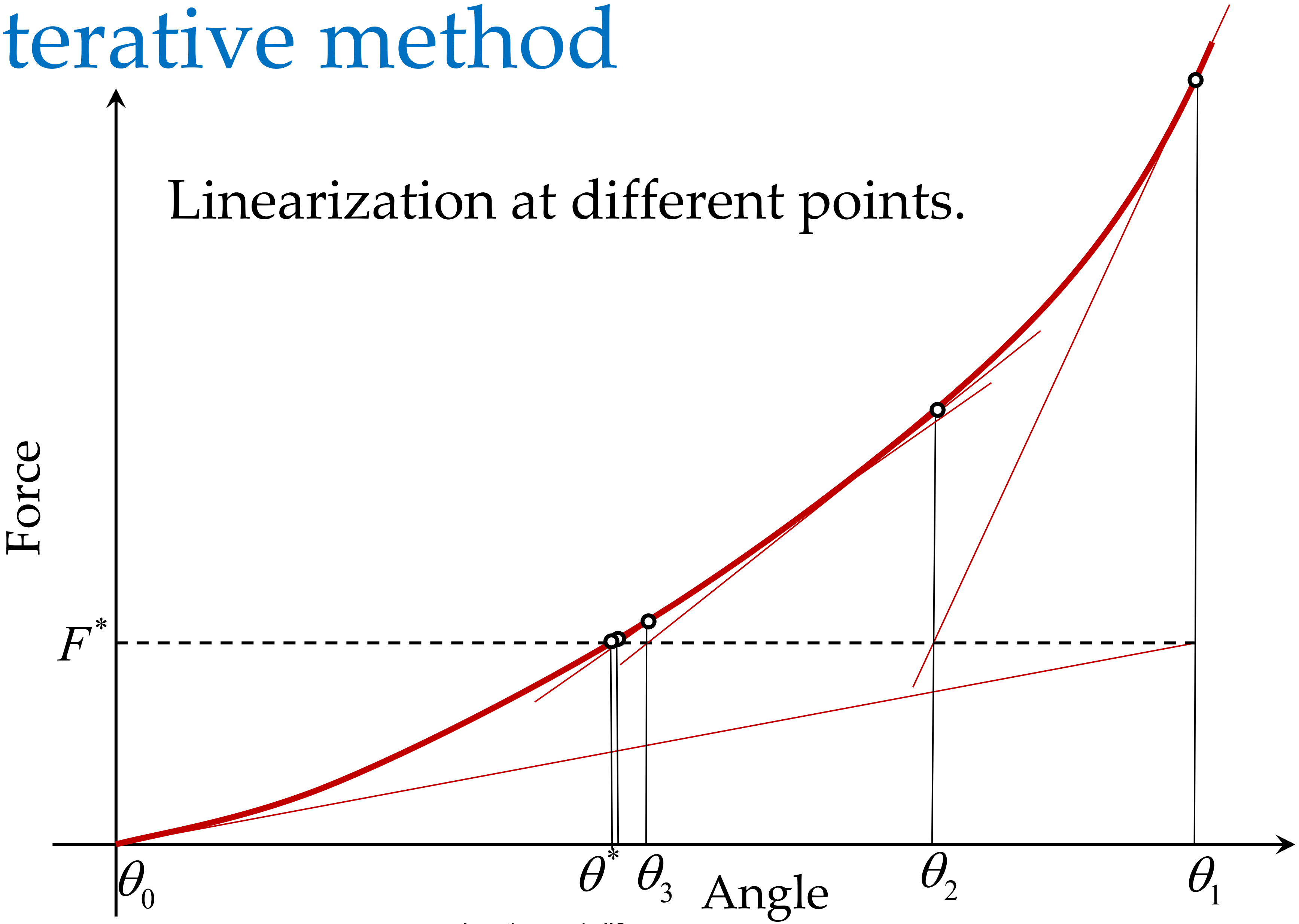
$$\frac{dD}{d\theta} = -l_2\cos\theta - l_5\cos(\beta+\gamma)\frac{d\beta}{d\theta} - l_5\sin(\beta+\gamma)\frac{d^2\beta}{d\theta^2}$$

Solve in small increments.

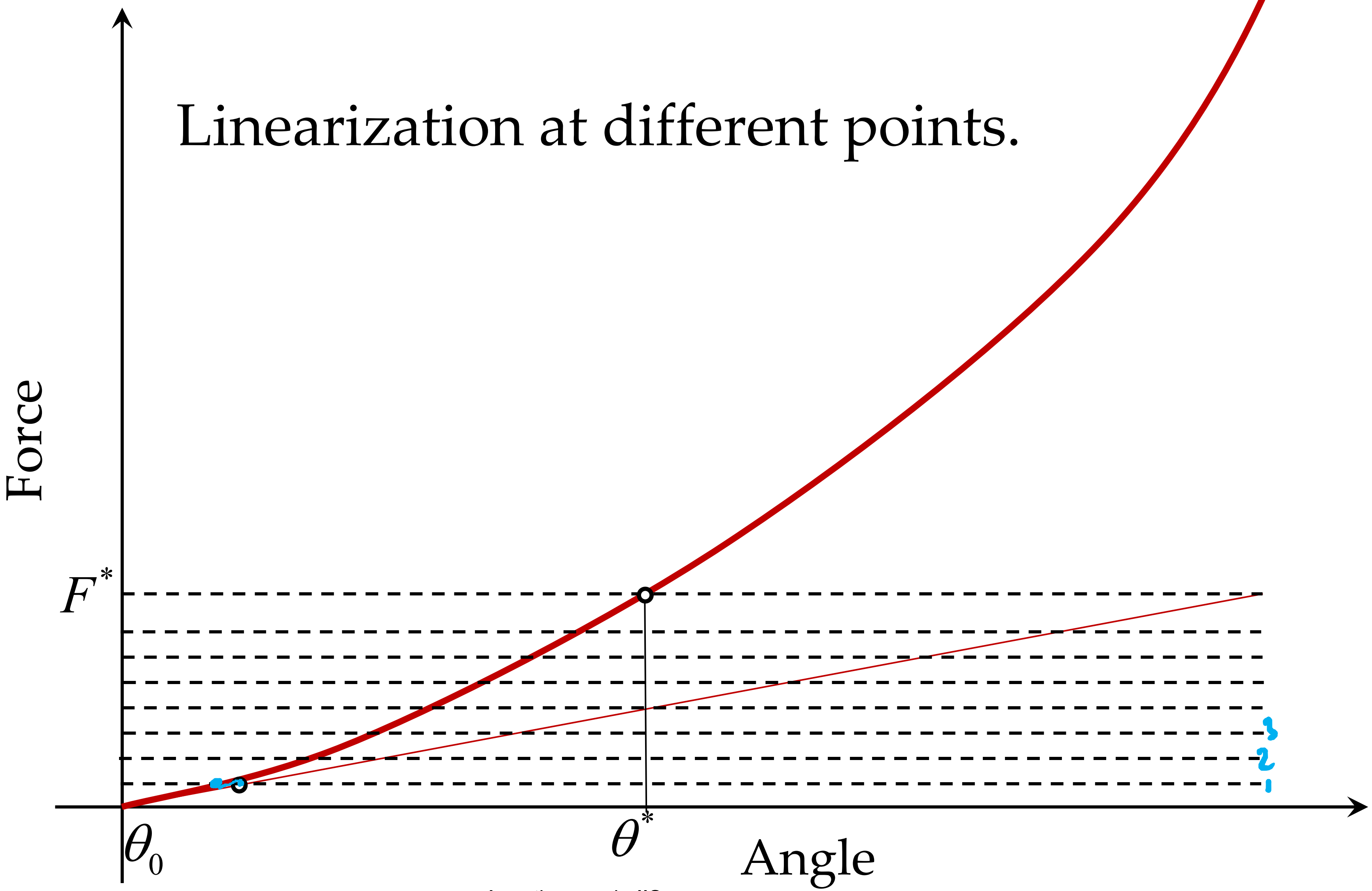
$$\Delta\theta = \frac{\Delta F}{\left.\dfrac{df_{\text{int}}}{d\theta}\right|_{\theta_p}} = \frac{\Delta F}{\left.K_t\right|_{\theta_p}}$$



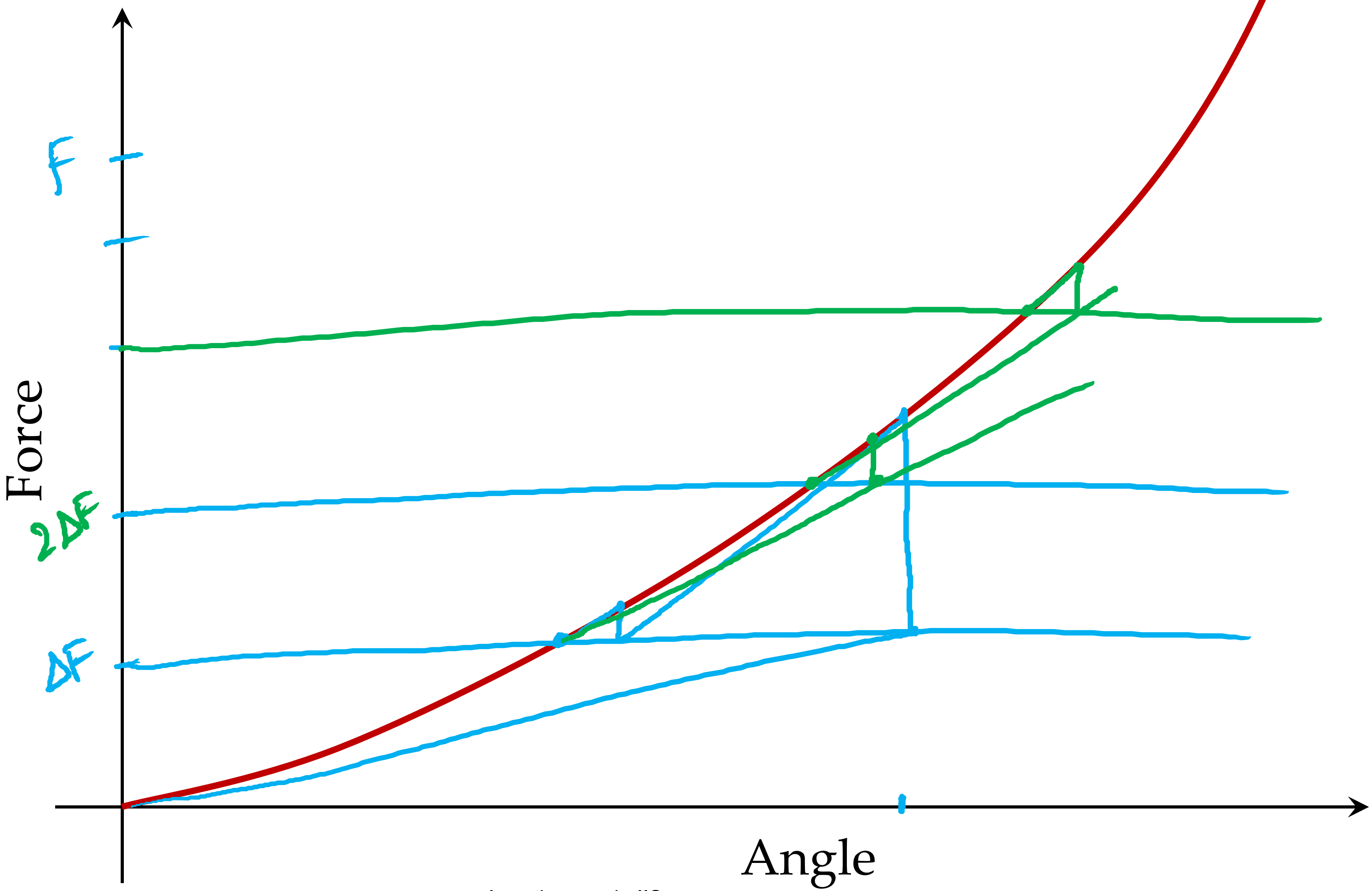
Graphical explanation:
Iterative method



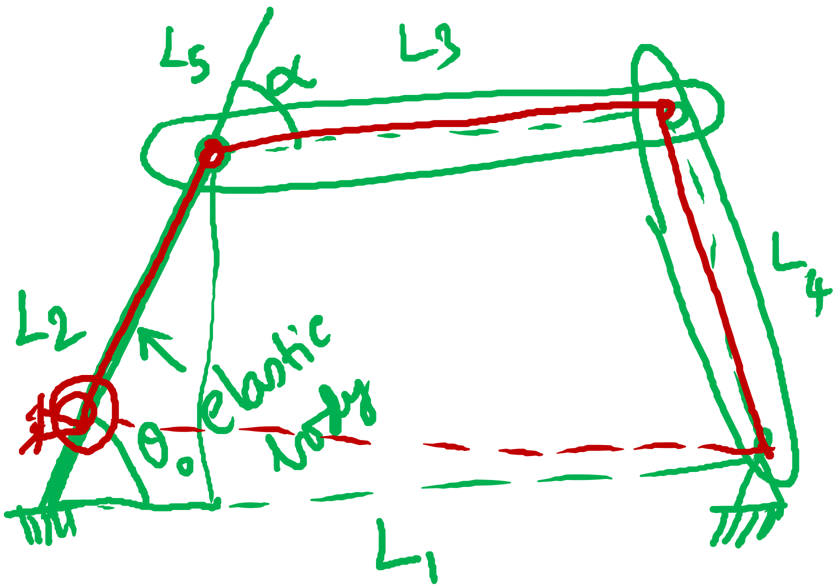
Graphical explanation:
Incremental method



Incremental and iterative method

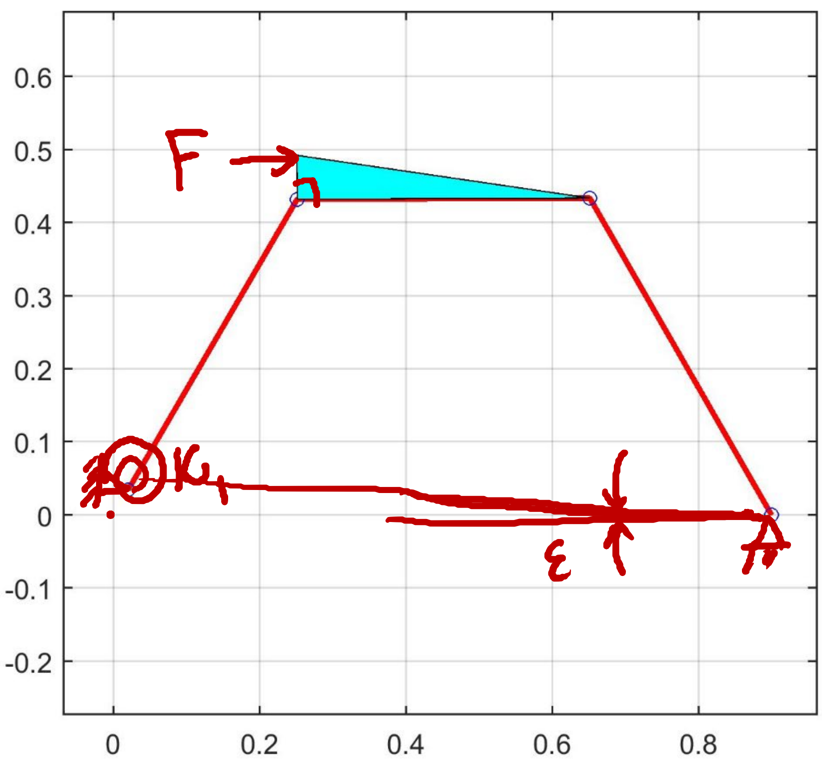


An example



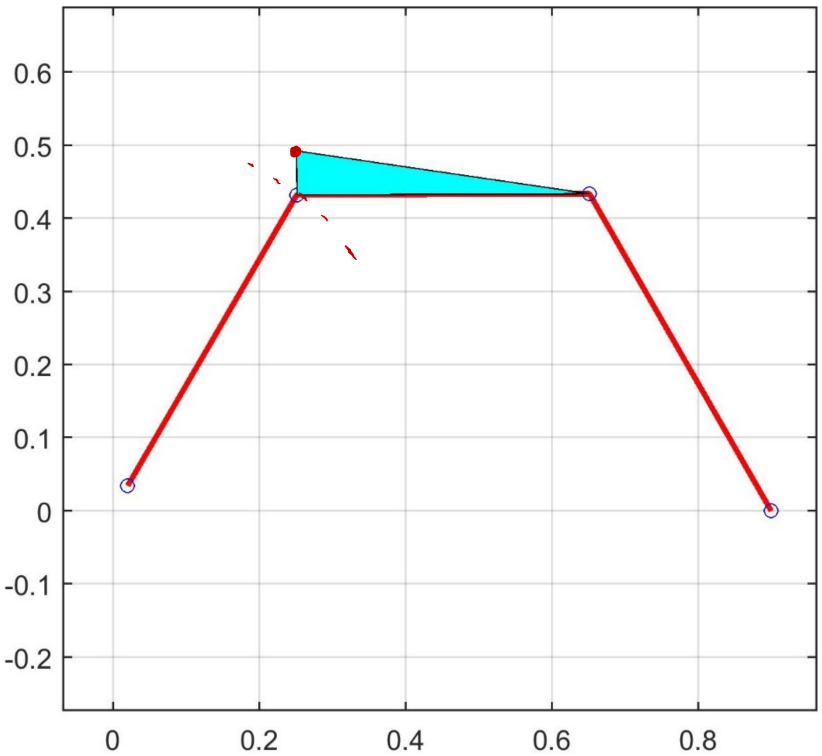

```
% Geometric information
gamma = 0.92;
L1 = 0.9; L10 = L1;
L2 = 0.5; L20 = L2;
L3 = 0.4;
L4 = 0.5;
theta0 = atan(0.43/0.25);
L5 = 0.06;
alpha = 90*pi/180;

% Elastostatic information
E = 2.1E9;
I = 1E-2*(1E-2)^3/12;
kappa1 = 2.25*E*I/L20;
kappa2 = 0;
kappa3 = 0;
kappa4 = 0;
```

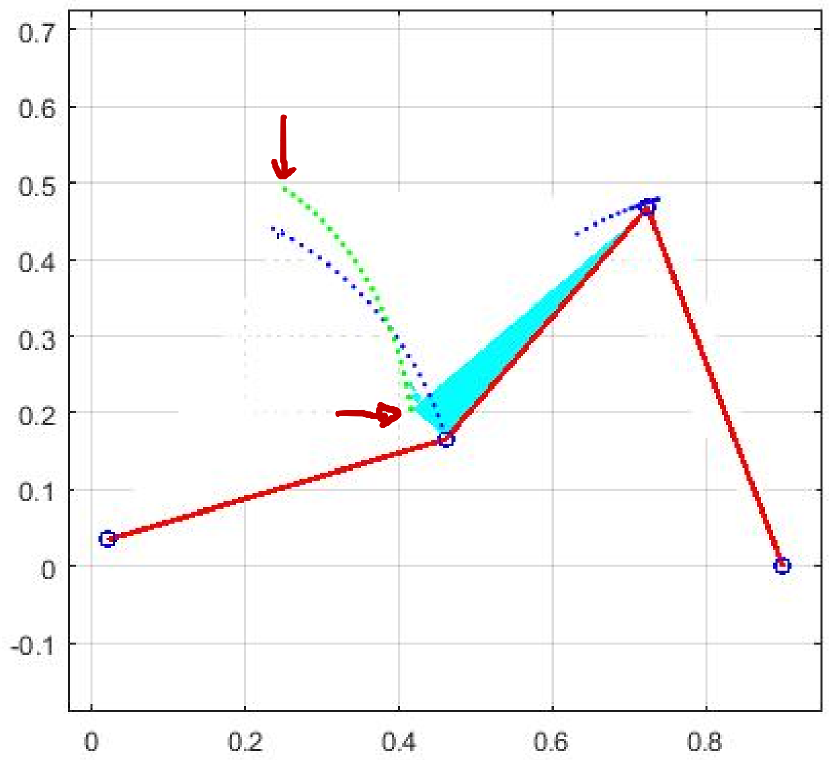
Constructing the PRB model



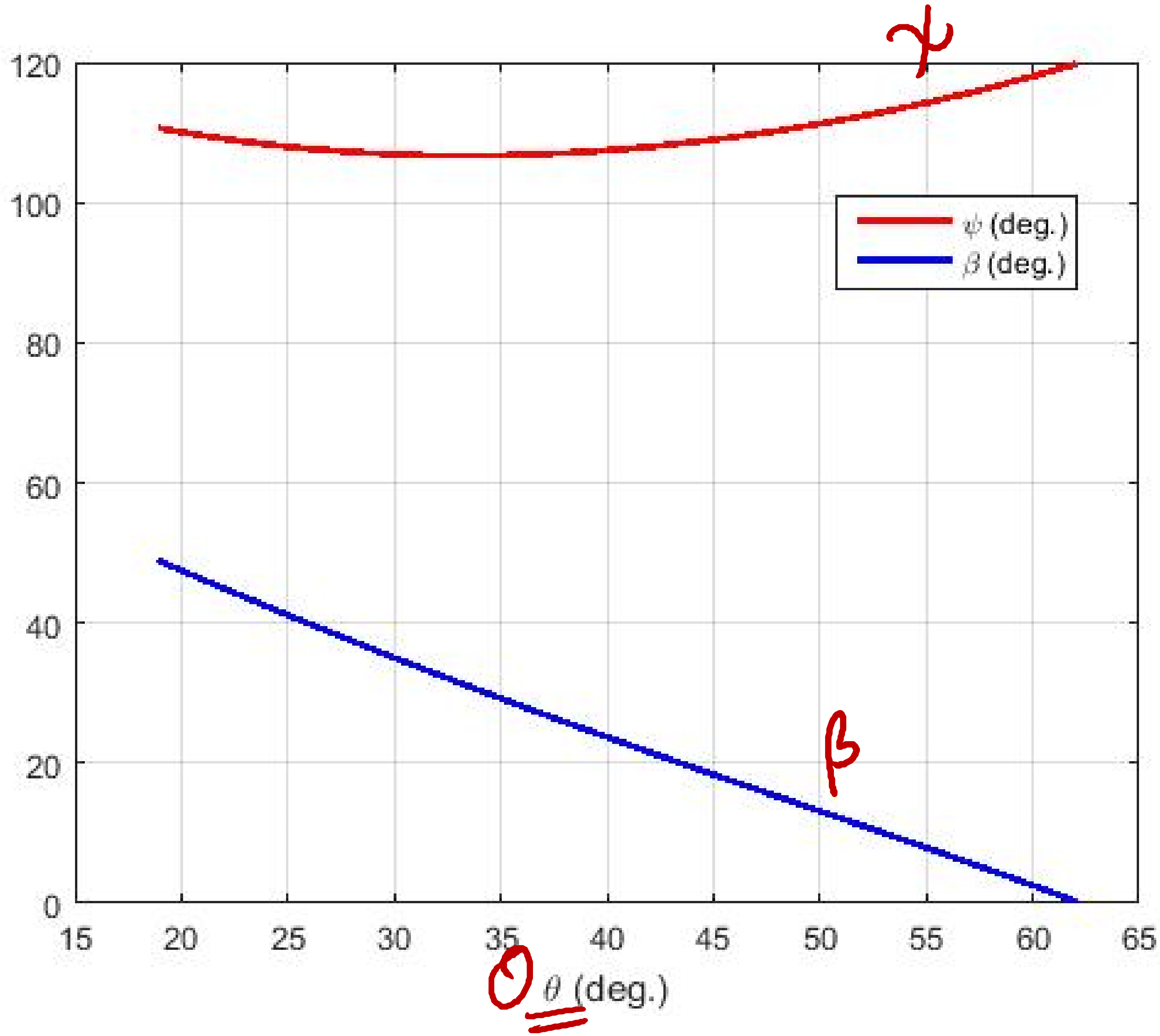

```
% Compute the new initial angle, theta0; and
% compute the new lengths in the pseudo rigid-body model
s = sqrt( L1^2 + L2^2 - 2*L1*L2*cos(theta0) );
Ax = L2*(1-gamma)*cos(theta0);
Ay = L2*(1-gamma)*sin(theta0);
L2 = L2*gamma;
L1 = sqrt( (Ax-L1)^2 + Ay^2 );
theta0 = acos( (L2^2 + L1^2 - s^2) / (2*L2*L1) );
epsilon = acos( (L10^2 + L1^2 - (L20*(1-gamma))^2) / ...
    (2*L10*L1) );
```

Initial and final configuration

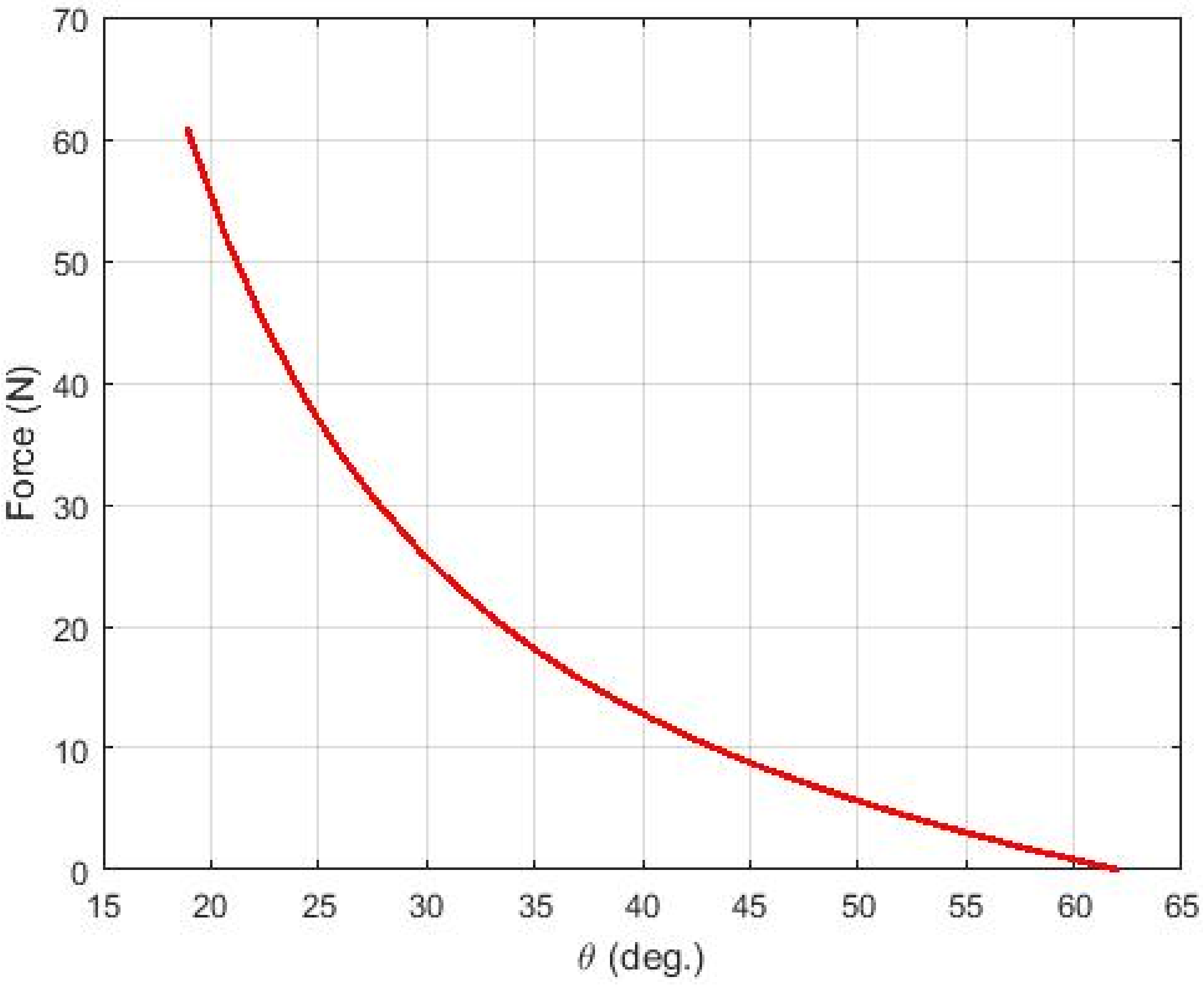




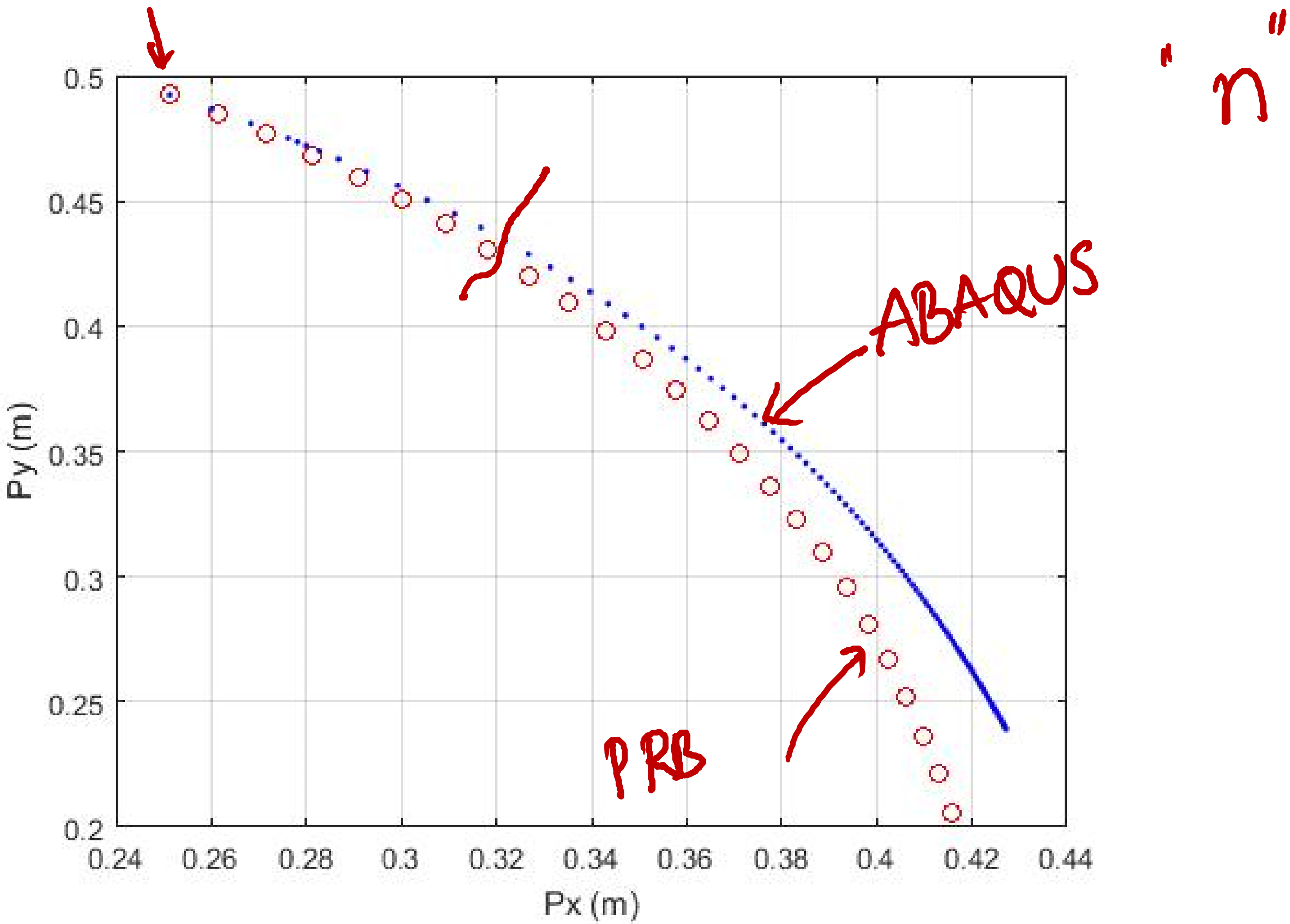
Output and coupler angles



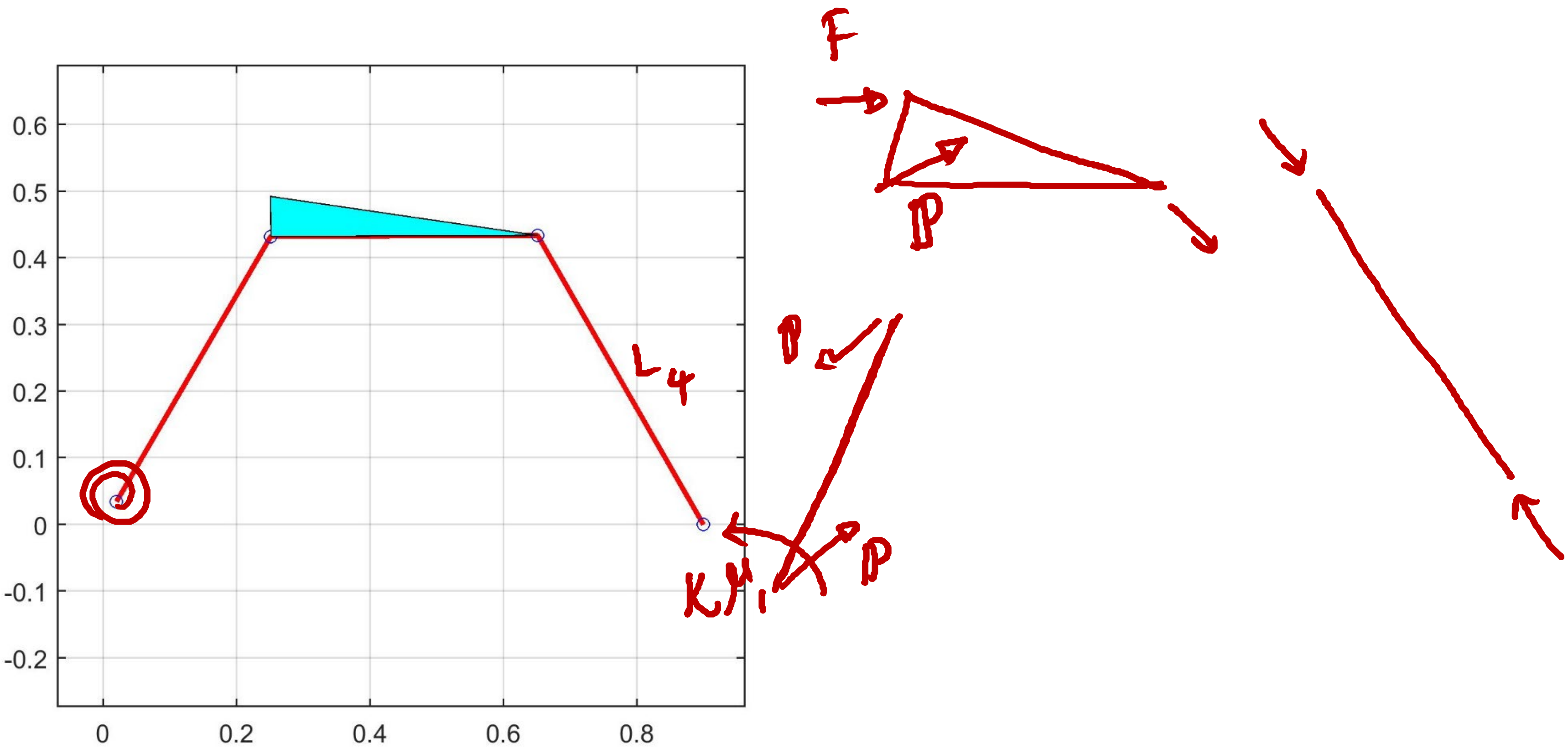
Force vs. theta



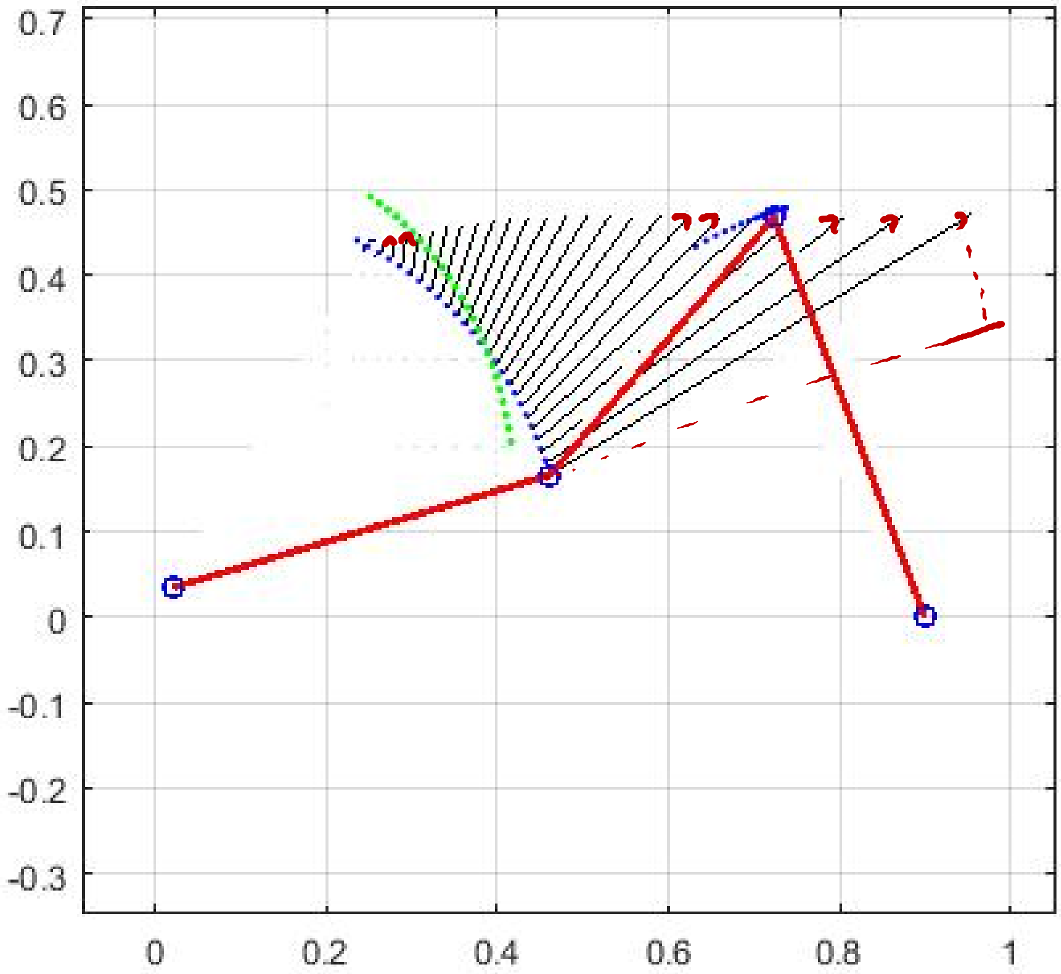
Compare PRB and FEA results



Force on the cantilever



Force on the cantilever



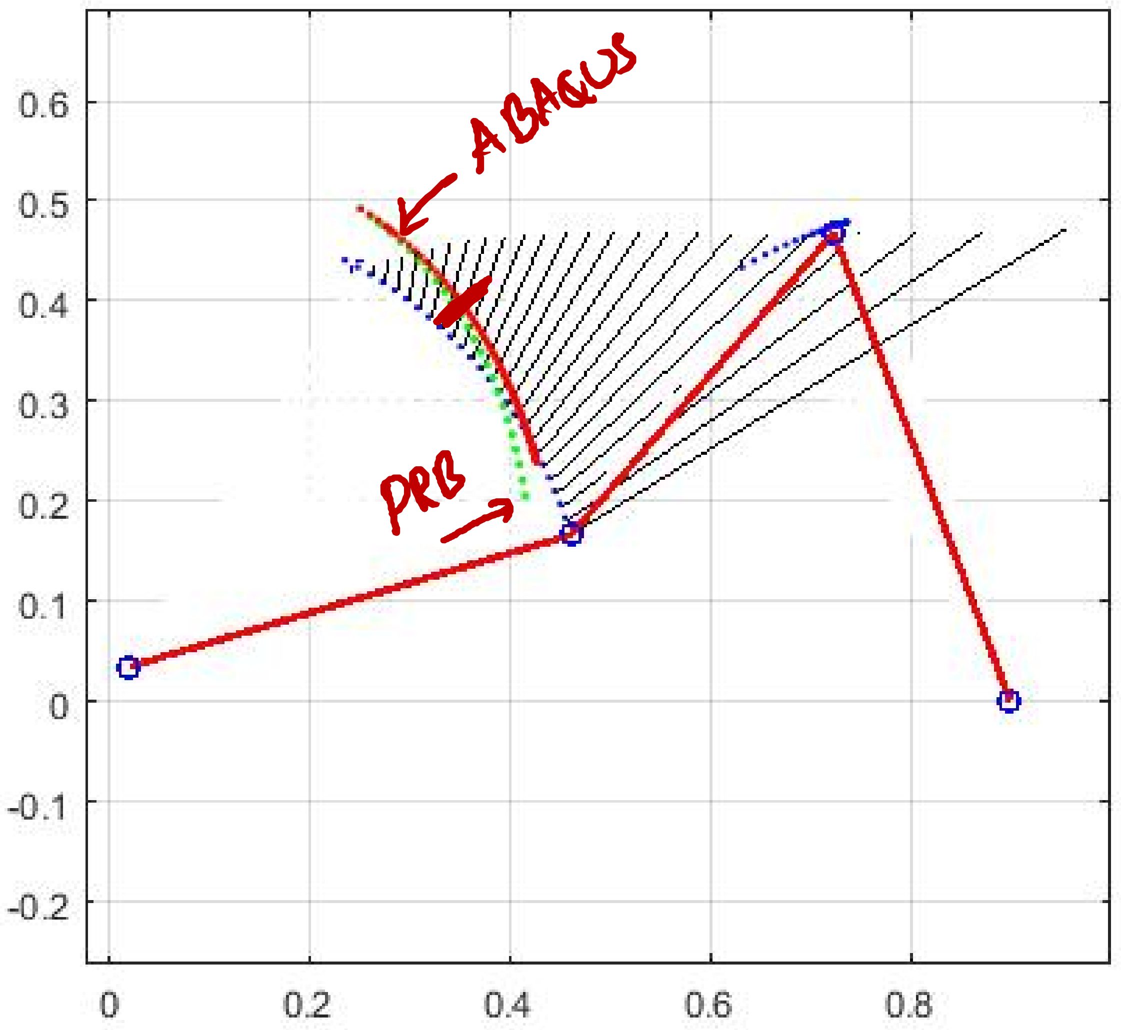
PRB parameters with n-ranges

$$\kappa = \gamma K_{\Theta} \frac{EI}{L}$$

$$\gamma = \begin{cases} 0.841655 - 0.0067807n + 0.000438n^2 & (0.5 < n < 10.0) \\ 0.852144 - 0.0182867n & (-1.8316 < n < 0.5) \\ 0.912364 + 0.0145928n & (-5 < n < -1.8316) \end{cases}$$

$$K_{\Theta} = \begin{cases} 3.024112 + 0.121290n + 0.003169n^2 & (-5 < n \le -2.5) \\ 1.967647 - 2.616021n - 3.738166n^2 - 2.649437n^3 - 0.891906n^4 & \\ \quad -0.113063n^5 & (-2.5 < n \le -1) \\ 2.654855 - 0.509896 \times 10^{-1}n + 0.126749 \times 10^{-1}n^2 & \\ \quad -0.142039 \times 10^{-2}n^3 + 0.584525 \times 10^{-4}n^4 & (-1 < n \le 10) \end{cases}$$

Compare PRB and FEA paths



Main points

- A pseudo rigid-body (PRM) model of a compliant mechanism can be analyzed using
  - Kinematic update equations
  - Static equilibrium equation

Further reading

- Howell, L. L., *Compliant Mechanisms,* John Wiley, 2001.
- Matlab code in the supplementary files of this lecture