ME256 Variational Methods and Structural Optimization June 30, 2006

Final Examination

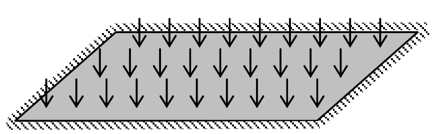
Points: 30 Time: 120 minutes

Question 1 (10 points)
The cross-section profile along the longitudinal axis of a cantilever beam of length L is to be optimally designed without exceeding the given volume of material V^* so that the mean compliance (i.e., $\int_0^L q(x)\, w(x)\, dx$) is minimized where $w(x)$ is the transverse displacement of the beam under a given load $q(x) = q_0$. The design variable is $A(x)$. Assume that the shape of the cross-section of the beam is such that $I(x) = \alpha A(x)$ where α is a constant. There are upper and lower bounds on the area of cross-section (i.e., $A_l \le A(x) \le A_u$) and it is given that $V^* < A_u L$. Denote the Young's modulus by E.

a) Write the complete statement of the problem as a constrained variational problem. Indicate the Lagrange multipliers corresponding to all the constraints by using lower case Greek symbols for local (point-wise) constraints and upper case Greek symbols for global constraints.
b) Write all the necessary conditions for the problem.
c) Prove that the volume constraint must be active in this problem.
d) Partition the span of the beam to identify in which parts $A(x)$ might reach its upper/lower bound.
e) Sketch the optimal area of cross-section and outline the steps to solve this problem *analytically* (<u>not</u> iteratively by using an update formula based on the optimality criterion).

Question 2 (10 points)
The governing equation for a rectangular membrane in the xy plane that is held fixed at its boundary, has a uniform tension T throughout, and is subject to a load $q(x, y)$, is given by $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{q}{T} = 0$ where $w(x, y)$ is the transverse deflection of the membrane.

a) Recalling the principle of minimum potential energy, deduce the expression for the potential energy of the membrane.
b) Justify the expression for the potential energy by noting that a membrane has no bending stiffness and that the deflections are small.
c) Next, using the Hamilton's principle, write the equation of motion for the transient behavior of the membrane if q is also a function of time.



Question 3 (10 points)

Consider the damped transient vibration of an axially deforming fixed-free bar subject to a time-varying load $p(x,t)$ per unit length of the bar. It is intended that the displacement (given by $u(x,t)$ in general) at time T at the free end is to be minimized for a given volume of material V^*. This leads to the following statement of the optimization problem.

$$\underset{A(x)}{\text{Minimize}}\ u(L,T)$$

Subject to

$$\lambda(x,t): \quad p + (EAu')' - b\dot{u} - \rho A\ddot{u} = 0$$

$$\Lambda: \quad \int_0^L A\, dx - V^* \le 0$$

where b is the constant damping coefficient per unit length, $A(x)$ is the design variable (the area of the cross-section of the bar), and the dot represents a derivative with respect to time and prime the derivative with respect to x.

a) Write the Lagrangian and then its variations with respect to $\lambda(x,t)$, $u(x,t)$ and $A(x)$.
b) Identify the adjoint equation. Does it have the same form as the governing equation? If it is not, what is its implication in the numerical solution to this problem?
c) Set up the formula for iteratively updating $A(x)$ using the optimality criteria method and outline the steps in the numerical algorithm.

The following fact may be useful to you for solving one of the problems in this examination.

If $z = f(x, y)$ is a single-valued continuously differential function of x and y, the area of a portion of the surface represented by this function is given by $\iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$ where D is the domain in the xy plane onto which the surface z projects.
