ME256 Variational Methods and Structural Optimization Jan. 2005

Final Examination

Points: 30 Time: 120 minutes

Question 1 (10 points)

A beam is to be optimally designed to for a given volume of material V^* so that $\int_0^L \{w(x)\}^2\, dx$ is minimized where $w(x)$ is the transverse displacement of the beam under a given load $q(x)$. Assume a rectangular cross-section with a fixed width, b , but variable depth, $t(x)$.

a) Write the complete statement of the problem as a constrained variational problem. Use the weak form of the governing equation.
b) Ignoring the upper and lower bounds on the design variable, write the necessary conditions for the problem.
c) Identify the adjoint equation.
d) Write the update formula for the design variable if we want to use the optimality criteria method.
e) Outline the procedure for the numerical implementation of the optimality criteria method based on the update formula. Be sure to keep in mind the upper and lower bounds on the design variables.
f) How does the update formula change if the width of the beam is made variable by keeping the depth constant throughout the beam?

Question 2 (10 points)

Verify if $\left\{ 4/3 \quad 4/3 \right\}^T$ and $\left\{ 1.2 \quad 1.4 \right\}^T$ are local minima for the following constrained optimization problem.

$$\underset{x_1, x_2}{\text{Minimize}} \; f = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$

Subject to

$$-2x_1 - x_2 + 4 \le 0$$
$$-x_1 - 2x_2 + 4 \le 0$$

Question 3 (10 points)

Solve the following problem using the dual method. Show all the steps clearly.

$$\underset{x_1, x_2}{\text{Minimize}} \; f = (x_1 - 3)^2 + (x_2 - 3)^2$$

Subject to

$$2x_1 + x_2 - 2 \le 0$$
$$-x_1 \le 0$$
$$-x_2 \le 0$$

Final Examination

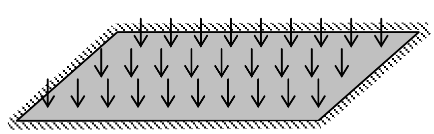
Points: 30 Time: 120 minutes

Question 1 (10 points)
The cross-section profile along the longitudinal axis of a cantilever beam of length L is to be optimally designed without exceeding the given volume of material V^* so that the mean compliance (i.e., $\int_0^L q(x)\, w(x)\, dx$) is minimized where $w(x)$ is the transverse displacement of the beam under a given load $q(x) = q_0$. The design variable is $A(x)$. Assume that the shape of the cross-section of the beam is such that $I(x) = \alpha A(x)$ where α is a constant. There are upper and lower bounds on the area of cross-section (i.e., $A_l \le A(x) \le A_u$) and it is given that $V^* < A_u L$. Denote the Young's modulus by E.

a) Write the complete statement of the problem as a constrained variational problem. Indicate the Lagrange multipliers corresponding to all the constraints by using lower case Greek symbols for local (point-wise) constraints and upper case Greek symbols for global constraints.
b) Write all the necessary conditions for the problem.
c) Prove that the volume constraint must be active in this problem.
d) Partition the span of the beam to identify in which parts $A(x)$ might reach its upper/lower bound.
e) Sketch the optimal area of cross-section and outline the steps to solve this problem *analytically* (<u>not</u> iteratively by using an update formula based on the optimality criterion).

Question 2 (10 points)
The governing equation for a rectangular membrane in the xy plane that is held fixed at its boundary, has a uniform tension T throughout, and is subject to a load $q(x, y)$, is given by $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{q}{T} = 0$ where $w(x, y)$ is the transverse deflection of the membrane.

a) Recalling the principle of minimum potential energy, deduce the expression for the potential energy of the membrane.
b) Justify the expression for the potential energy by noting that a membrane has no bending stiffness and that the deflections are small.
c) Next, using the Hamilton's principle, write the equation of motion for the transient behavior of the membrane if q is also a function of time.



Question 3 (10 points)
Consider the damped transient vibration of an axially deforming fixed-free bar subject to a time-varying load $p(x,t)$ per unit length of the bar. It is intended that the displacement (given by $u(x,t)$ in general) at time T at the free end is to be minimized for a given volume of material V^*. This leads to the following statement of the optimization problem.

$$\underset{A(x)}{\text{Minimize}}\ u(L,T)$$

Subject to

$$\lambda(x,t):\quad p+(EAu')'-b\dot{u}-\rho A\ddot{u}=0$$

$$\Lambda:\quad \int_0^L A\,dx - V^* \le 0$$

where b is the constant damping coefficient per unit length, $A(x)$ is the design variable (the area of the cross-section of the bar), and the dot represents a derivative with respect to time and prime the derivative with respect to x.

a) Write the Lagrangian and then its variations with respect to $\lambda(x,t)$, $u(x,t)$ and $A(x)$.
b) Identify the adjoint equation. Does it have the same form as the governing equation? If it is not, what is its implication in the numerical solution to this problem?
c) Set up the formula for iteratively updating $A(x)$ using the optimality criteria method and outline the steps in the numerical algorithm.

The following fact may be useful to you for solving one of the problems in this examination.

If $z = f(x,y)$ is a single-valued continuously differential function of x and y, the area of a portion of the surface represented by this function is given by $\iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$ where D is the domain in the xy plane onto which the surface z projects.

Final Examination

Points: 30 | Open book and open notes | Time: 2 Hours

Question 1 (10 points)

We want to minimize the volume of an axially deforming *fixed-free* bar of length L and Young's modulus E which is subject to a point load $(-F)$ at $x = 0.2L$ and another point load F at $x = 0.8L$. The strain energy of the bar should not exceed SE^*, and its deflection at $x = 0.6L$ should not exceed Δ.

a) Write the *complete* problem statement in the standard form by taking the area of cross-section along the length of the bar as the design variable.
b) Solve the above problem to obtain an *analytical expression* for the area of cross-section in terms of the Lagrange multipliers.
c) If it is a fixed-fixed bar, how would you formulate the problem and then solve it numerically?

Question 2 (10 points)

a) Write down the convex, separable approximation of the following problem using appropriate linear or reciprocal linear approximation at $\{x_1 \quad x_2 \quad x_3\}^T = \{1 \quad -1 \quad -2\}^T$.
b) Using your approximation, obtain the dual problem in terms of the Lagrange multiplier.

$$\underset{\bar{x}}{\text{Minimize}} \quad f = 20x_2x_3 + 30x_1x_3 + 15x_1x_2$$

Subject to

$$g = 125 - x_1x_2x_3 \le 0$$

Question 3 (10 points)

A multi-physics topology optimization problem for electro-statically actuated microstructure is formulated as follows in 2D. Do *all* the analytical steps that are necessary to set up the numerical update scheme to solve this problem using the optimality criteria method.

$$\underset{\gamma(x,y)}{\text{Minimize}} \quad \int_\Omega \phi(\mathbf{u}, V)\, d\Omega$$

Subject to

$$\mu(x,y): \quad \nabla \cdot (\sigma \nabla V) = 0$$

$$\Gamma: \quad \int_\Omega \left\{ \boldsymbol{\varepsilon}(\mathbf{u})^T \mathbf{Y} \boldsymbol{\varepsilon}(\mathbf{v}) - \mathbf{f}^T \mathbf{u} \right\} d\Omega = 0 \quad \text{where} \quad \mathbf{f} = -\rho \nabla V - \frac{1}{2} \left\| \nabla V \right\|^2 \nabla e$$

$$\Lambda: \quad \int_\Omega \gamma \, d\Omega - V^* \le 0$$

Data :

$\sigma_0, \mathbf{Y}_0, V^*, e_0, \rho_0$; and the geometry of Ω is specified in 2D.

Other relationships and dependencies :

$$\sigma = \gamma^3 \sigma_0; \; V(x,y); \; \boldsymbol{\varepsilon}(\mathbf{u}) = \left\{ \frac{\partial u_1}{\partial x} \quad \frac{\partial u_2}{\partial y} \quad \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \right\}^T; \; \mathbf{u} = \{ u_1(x,y) \quad u_2(x,y) \}^T$$

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \left\{ \frac{\partial v_1}{\partial x} \quad \frac{\partial v_2}{\partial y} \quad \frac{1}{2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) \right\}^T; \; \mathbf{v} = \{ v_1(x,y) \quad v_2(x,y) \}^T$$

$\mathbf{Y} = \gamma^3 \mathbf{Y}_0$; $\mathbf{Y}_0$ = stress-strain matrix in 2D ;

$e = \gamma^3 e_0$; $\rho = \gamma^3 \rho_0$; $\phi(\mathbf{u}, V)$

ME256 Variational Methods and Structural Optimization April 21, 2008

Final Examination

Points: 30 Time: 120 minutes

Question 1 (10 points)

In the topology optimization of compliant mechanisms, the ratio MSE/SE is maximized. The expressions for MSE (mutual strain energy) and SE (strain energy) are given by the following.

$$MSE = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{v})^T \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u})\, d\Omega \qquad \text{and} \qquad SE = \int_{\Omega} \frac{1}{2}\boldsymbol{\varepsilon}(\mathbf{u})^T \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u})\, d\Omega$$

where $\boldsymbol{\varepsilon}$ is the strain, $\mathbf{D}$ the stress-strain relationship, $\mathbf{u}$ the displacement vector under one loading (say body force $\mathbf{p}_u$), and $\mathbf{v}$ is the displacement vector under a second loading (say, body force $\mathbf{p}_v$). Derive the gradient of MSE/SE with respect to a design variable b. Note that $\mathbf{D}$ is an explicit function of b.

Question 2 (6 points)

In order to have a weak form for the governing equation of a damped spring-mass oscillator problem in x, the following functional is suggested where x and y are functions of t.

$$J = \int_0^T 0.5\left\{m\,x\,\ddot{y} + m\,\ddot{x}\,y - b\,x\,\dot{y} + b\,\dot{x}\,y + 2\,k\,x\,y\right\} dt$$

- Verify if it leads to the strong form of the governing equation of a damped spring-mass oscillator by equating the first variations of J with respect to x and y to zero.
- Do you see any drawback with this?

Question 3 (14 points)

Consider the damped transient vibration of an axially deforming fixed-free bar subject to a time-varying load $p(x,t)$ per unit length of the bar. It is intended that the displacement (given by $u(x,t)$ in general) at time T at the free end is to be minimized for a given volume of material V^*. This leads to the following statement of the optimization problem.

$$\underset{A(x)}{\text{Minimize}}\ u(L,T)$$

$$\text{Subject to}$$

$$\lambda(x,t): \quad p + (EAu')' - b\dot{u} - \rho A\ddot{u} = 0$$

$$\Lambda: \quad \int_0^L A\,dx - V^* \le 0$$

where b is the constant damping coefficient per unit length, $A(x)$ is the design variable (the area of the cross-section of the bar), and the dot represents a derivative with respect to time and prime the derivative with respect to x.

a) Write the Lagrangian and then its variations with respect to $\lambda(x,t)$, $u(x,t)$ and $A(x)$.

b) Identify the adjoint equation. Does it have the same form as the governing equation? If it is not, what is its implication in the numerical solution to this problem?

c) Set up the formula for iteratively updating $A(x)$ using the optimality criteria method and outline the steps in the numerical algorithm.

ME256 Variational Methods and Structural Optimization April 21, 2009

Final Examination

Points: 30 Time: 90 minutes

Question 1 (10 points)

Imagine yourself as the designer who wants to determine the cross-section dimensions of a rectangular beam along its length such that its maximum deflection over its length is minimized for given distributed transverse load, volume of material, and material properties.

a) Formulate the above statement as a calculus of variations problem and write a clear statement mathematically showing the objective function, design variables, constraints, and the given data.
b) Derive the necessary conditions for the optimum. Simplify the equations as much as possible.
c) Clearly list the number of unknowns and equations and ensure that the problem is solvable, analytically or numerically.

Question 2 (8 points)

$$\underset{\mathbf{x}}{\text{Min}} \quad \frac{\mathbf{v}^T \mathbf{K} \mathbf{v}}{\mathbf{u}^T \mathbf{K} \mathbf{u}}$$

Subject to

$$\mathbf{K}\mathbf{u} = \mathbf{f}_1$$

$$\mathbf{K}\mathbf{v} = \mathbf{f}_2$$

a. Write the optimality condition for the above problem stated in the discretized finite element framework.
b. Do you see any physical (energetical) interpretation of the optimality condition?

Question 3 (12 points)

The governing equation for the free vibration of a plate is given as follows.

$$\rho \frac{\partial^2 w}{\partial t^2} + D \nabla^4 w = 0$$

where $w(x, y, t)$ is the transverse displacement of the plate at (x, y) in the 2D domain of the plate at time t, ρ and D are mass per unit length and the plate modulus, and $\nabla^4 w = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$. Let $w(x, y, t) = \phi(x, y)\, q(t)$.

Derive the Rayleigh quotient, which when minimized with respect to the transverse displacement $\phi(x, y)$, will give the first natural frequency of the plate.

NOTE: *First, clearly write the steps you would follow and then proceed with the derivation so that you are not lost in algebra or calculus.*

ME256 Variational Methods and Structural Optimization April 27, 2010

Final Examination

Points: 30 Time: 120 minutes

NOTE: *First, clearly write the steps you would follow and then proceed with the derivations so that you are not lost in algebra or calculus.*

Question 1 (12 points)

A cantilever beam of length L is subject to a transverse load of $q(x)$. We desire that it should have a deflection w^* at $x = x^*$ where $0 < x^* < L$. By assuming that $I(x) = \alpha b(x)$, we want find $b(x)$ to minimize the volume of material used by the cantilever.

(a) Write the above structural optimization statement in the standard mathematical form showing the objective function, design variable, constraints and their corresponding Lagrange multipliers, and the data.

(b) Obtain the necessary equations to solve the problem. But do not attempt to solve these equations.

(c) By recognizing that the cantilever beam considered here is statically determinate, re-cast the problem and give the complete analytical solution.

Question 2 (8 points)

$$\underset{\mathbf{x}}{\text{Min}} \quad f = \mathbf{u}^T \mathbf{v}$$

Subject to

$$\boldsymbol{\lambda}_1 : \quad \mathbf{K}_1 \mathbf{u} = \mathbf{f}_1$$

$$\boldsymbol{\lambda}_2 : \quad \mathbf{K}_2 \mathbf{v} = \mathbf{f}_2$$

$$\Lambda : \quad \mathbf{a}^T \mathbf{x} - c^* \le 0$$

$$\text{Data} : \mathbf{K}_1(\mathbf{x}), \mathbf{K}_2(\mathbf{x}), \mathbf{f}_1, \mathbf{f}_2, \mathbf{a}, c^*$$

Obtain $\left({df}/{d\mathbf{x}} \right)$ using the direct method; and then obtain the same using the adjoint method.

Question 3 (10 points)

The governing equation for the transverse free vibration of an axially loaded beam is given as follows. Assume that the axial load N is constant throughout the beam.

$$\rho \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} - N \frac{\partial^2 w}{\partial x^2} = 0$$

where $w(x,t)$ is the transverse displacement of the beam at x and at time t, ρ the mass per unit length, and EI the beam modulus. Let $w(x,t) = \phi(x) q(t)$. Derive the Rayleigh quotient for this problem, which when extremized with respect to the transverse displacement $\phi(x)$, will give the first natural frequency of the beam.

ME256 **Variational Methods and Structural Optimization** **April 21, ~~2010~~ 2011**

Final Examination

Points: 30 Time: 120 minutes

NOTE: *First, clearly write the steps you would follow and then proceed with the derivations so that you are not lost in algebra or calculus.*

Question 1 (10 points)

Start with the definitions of the potential energy and the kinetic energy of an unloaded straight, fixed-fixed beam and derive the Rayleigh quotient for computing the natural frequencies of a beam. Show all your steps in detail for the following.

a. Governing equation for free vibration
b. Definition of the natural frequency by separation of variables method
c. Prove that natural frequencies are all positive.
d. State and prove the minimum characterization theorem of the eigenvalue problem for the beam.
e. Finally, define the Rayleigh quotient and argue why that works.

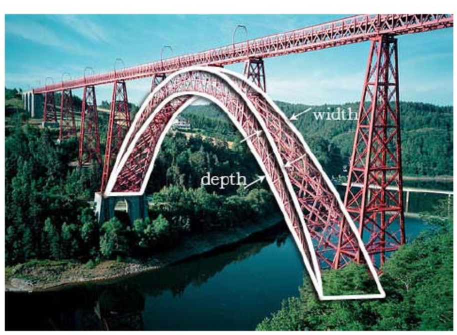
Question 2 (10 points)

You are given a fixed-fixed bar of area profile $A(x)$, Young's modulus E, axial load $p(x)$, and length L.

a. At which value of x would you change $A(x)$ by a specified small amount in order to have the most change in the strain energy of the bar?
b. At which value of x would you change $A(x)$ by a specified small amount in order to have the most change in the maximum stress of the bar?

Question 3 (10 points)

Shown in the figure below is the Maria-Pia bridge located in Portugal and designed by Gustave Eiffel. Notice that the arch's width profile is widest at both ends and is the narrowest at the midpoint. The depth of the arch is the other way around: thickest at the midpoint and thinnest (actually zero) at both the ends. By using calculus of variations formulations, argue why this arch might be optimal or not. State your arguments, optimization problem statements, necessary conditions and their interpretation, assumed boundary conditions and their implication, etc.; in a nutshell, make your thought process clear through cogent arguments and mathematical reasoning.



ME256 Variational Methods and Structural Optimization April 25, 2012

Final Examination

Points: 25 Time: 120 minutes

NOTE: *First, clearly write the steps you would follow and then proceed with the derivations so that deserving credit can be given even if you are lost in algebra or calculus.*

Question 1 (8 points)

Consider the following optimization problem where V^* is a scalar, $\mathbf{A},\mathbf{f},\mathbf{L}$ are $n\times 1$ vectors, and $\mathbf{K}$ is a $n\times n$ matrix.

$$\underset{\mathbf{A}}{\text{Minimize}}\ \mathbf{u}^T\mathbf{K}\mathbf{u}$$

Subject to

$$\Gamma: \quad \mathbf{K}\mathbf{u}-\mathbf{f}=0$$

$$\Lambda: \quad \mathbf{A}^T\mathbf{L}-V^* \le 0$$

$$\text{Data}: \mathbf{K}(\mathbf{A}),\mathbf{f},\mathbf{L},V^*$$

Write the equations to solve for $\mathbf{A}$ by noting that $\mathbf{K}$ is a known function of elements of $\mathbf{A}$.

Question 2 (10 points)

A structural designer wants to minimize the longitudinal normal stress in a transversely loaded fixed-fixed beam of length, L, and given volume, V^*. For this purpose, the following objective can be used as a measure of the overall stress in the beam.

$$S=\int_0^L \sigma^2(x)\,dx$$

where $\sigma(x)$ is the longitudinal normal stress. Let the transverse loading be $q(x)$ per unit length and Young's modulus, E, be uniform throughout the beam.

a. Pose the optimization problem in the framework of calculus of variations by assuming circular cross-section whose radius, $r(x)$, is the design variable.
b. Derive the sensitivity of S with respect to $r(x)$.

Question 3 (7 points)

Consider the eigenvalue problem given by $\left(r\phi'\right)'+\left(\mu+\lambda\sigma\right)\phi=0$ where $r(x)$, $\mu(x)$, and $\sigma(x)$ are known functions of x. Derive the Rayleigh quotient for this problem, which when minimized with respect to $\phi(x)$, will give the lowest eigenvalue, λ_1.

ME256 Variational Methods and Structural Optimization April 27, 2013

Final Examination

Points: 25 Time: 120 minutes

NOTE: *First, clearly write the steps you would follow and then proceed with the derivations so that deserving credit can be given even if you are lost in algebra or calculus.*

Question 1 (4 + 4 = 8 points)

Considering the equilibrium equation of a discretized elastic body $\mathbf{Ku} = \mathbf{f}$, where $\mathbf{K}$ and $\mathbf{f}$ are functions of b, obtain the gradient of the strain energy $0.5\mathbf{u}^T\mathbf{Ku}$ with respect to b using direct and adjoint methods.

Question 2 (7 points)

State and solve the problem of maximizing the first critical buckling load of a column using Rayleigh quotient. Assume that the moment of inertia is linearly proportional to the area of cross-section.

Question 3 (10 points)

Shown in Fig. 1 is a fixed-fixed beam of length L under transverse load $q(x)$ with a pointed rigid obstacle directly below the midpoint of the beam at a distance g_0. Assume circular cross-section of radius $r(x)$. Denote Young's modulus with E.

1. Formulate a calculus of variations problem in order to extremize the contact force at the midpoint of the beam after it touches the apex of the rigid obstacle.
2. Write the necessary conditions.
3. State whether this is a minimization or a maximization problem.

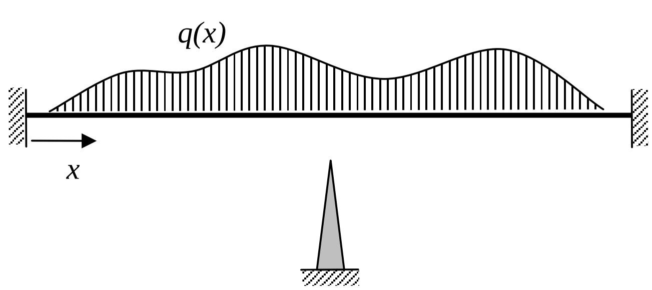


Figure 1. A beam under transverse load with a pointed rigid obstacle beneath

Final Examination

Points: 25 Open-book and Open-notes Time: 120 minutes

NOTE: *First, clearly write the steps you would follow and then proceed with the derivations so that deserving credit can be given even if you are lost in algebra or calculus.*

Question 1 (10 points)

Consider a fixed-fixed beam subject to transverse loads applied in two orthogonal planes as shown in Fig. 1(a). A modern architect is not agreeable to change the shape or size of the rectangular cross-section but is agreeable to rotate the cross-section. That is, $\phi(x)$, which is indicated in Fig. 1(b), can be designed as desired. Formulate and solve a calculus of variations problem for such a beam to maximize the mean compliance for given volume of material, V^*. Use usual symbols for the other quantities.

a. Write a clear and complete statement of the problem showing the objective function, design variable, state variables, governing equations, resource constraint, and data.
b. Deduce the necessary conditions for a minimum and also write other equations that enable us solve for all unknown quantities.
c. Set up the optimality criteria procedure.

Note that when the cross-section is rotated about the longitudinal axis by an angle $\phi(x)$, the second area moment of inertia is given by ($I_1 \cos^2\phi + I_2 \sin^2\phi$) where I_1 and I_2 are the principal moments of inertia of the cross-section. Figure 1(b) shows how the optimal beam might look as the cross-section rotates along the axis of the beam.

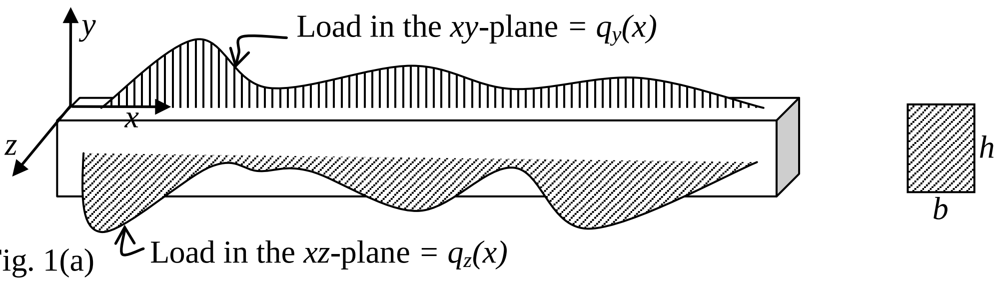


Fig. 1(a)

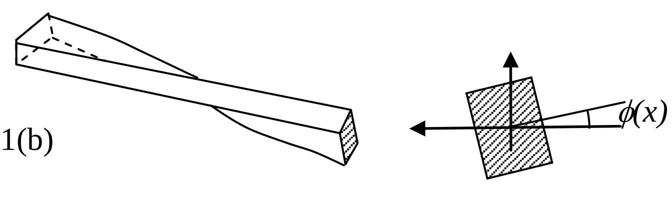


Fig. 1(b)

Question 2 (8 points)

A slender beam of uniform cross-section, fixed at both the ends to a rigid frame is being dragged in the transverse plane in ocean waters in an under-water vehicle, as shown in Fig, 2. By neglecting the inertia of the beam, the equation of motion, called the *hydrodynamic beam equation*, can be written as

$$EI\frac{\partial^4 w}{\partial x^4} + c\frac{\partial w}{\partial t} = 0$$

Solve this using the technique of separation of variables and set up the eigenvalue problem. State the constrained minimization problem using which eigenvalues can be solved. And then derive the Rayleigh quotient for this problem.

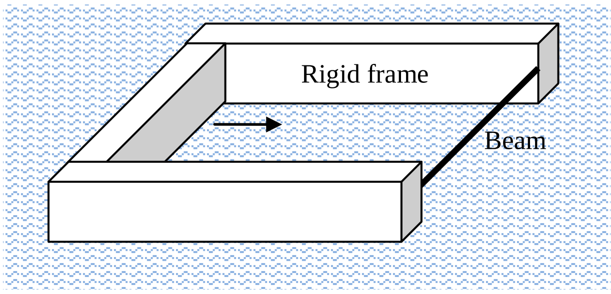


Fig. 2

Question 3 (7 points)

A designer wants to minimize the ~~cross-~~product of displacement vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ that result from a structure subject to two load vectors $\mathbf{f}_1$ and $\mathbf{f}_2$. Assume that the stiffness matrix and the loading vectors are functions of a design variable vector $\mathbf{x}$. That is, $\mathbf{K}(\mathbf{x})\mathbf{u}_1 = \mathbf{f}_1(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})\mathbf{u}_2 = \mathbf{f}_2(\mathbf{x})$. Obtain an expression for the gradient of $(\mathbf{u}_1^T\mathbf{u}_2)$ with respect to $\mathbf{x}$.

ME256 Variational Methods and Structural Optimization April 23, 2015

Final Examination

Points: 25 Open-book and Open-notes Time: 120 minutes

NOTE: *First, clearly write the steps you would follow and then proceed with the derivations so that deserving credit can be given even if you are lost in algebra or calculus.*

Question 1 (10 points)

Consider a simply supported beam model of a bridge subject to an as-yet undetermined transverse load $q(x)$. The beam has constant rectangular cross-section $b \times d$ all along its length L. Its material has an Young's modulus of E.

Pose an optimization problem in the framework of calculus of variations to maximize the strain energy of the beam by using $q(x)$ as the unknown function.

Include a constraint that $\int_0^L q(x)dx - Q^* \leq 0$.

Use Euler-Lagrange equations to write the necessary conditions.

Solve the equations and comment on the nature of the solution to the problem you posed.

Question 2 (5 points)

Given a differential equation $\nabla^2 \phi + Q = 0$ where $\phi(x, y, z)$ is a function in the 3D space and Q is a constant, write a calculus of variations problem whose Euler-Lagrange equation leads to the given differential equation.

Question 3 (10 points)

A designer of compliant mechanisms wants to maximize the ratio of the mutual strain energy defined as $\mathbf{u}^T\mathbf{K}\mathbf{v}$ and the strain energy defined as $(1/2)\mathbf{u}^T\mathbf{K}\mathbf{u}$, where $\mathbf{K}$ is the stiffness matrix of a meshed elastic body, $\mathbf{K}\mathbf{u} = \mathbf{f}_1$ and $\mathbf{K}\mathbf{v} = \mathbf{f}_2$ with $\mathbf{f}_1$ and $\mathbf{f}_2$ denoting two different loading conditions.

If $\mathbf{K}$ is a function of the design variable vector $\mathbf{x}$ and $\mathbf{f}_1$ and $\mathbf{f}_2$ aren't, compute the design sensitivity of $(\mathbf{u}^T\mathbf{K}\mathbf{v}) / (1/2\,\mathbf{u}^T\mathbf{K}\mathbf{u})$ using the direct <u>and</u> adjoint methods.