ME256 Variational Methods and Structural Optimization Jan. 2005

Final Examination

Points: 30 Time: 120 minutes

Question 1 (10 points)

A beam is to be optimally designed to for a given volume of material V^* so that $\int_0^L \{w(x)\}^2 \, dx$ is minimized where $w(x)$ is the transverse displacement of the beam under a given load $q(x)$. Assume a rectangular cross-section with a fixed width, b , but variable depth, $t(x)$.

a) Write the complete statement of the problem as a constrained variational problem. Use the weak form of the governing equation.
b) Ignoring the upper and lower bounds on the design variable, write the necessary conditions for the problem.
c) Identify the adjoint equation.
d) Write the update formula for the design variable if we want to use the optimality criteria method.
e) Outline the procedure for the numerical implementation of the optimality criteria method based on the update formula. Be sure to keep in mind the upper and lower bounds on the design variables.
f) How does the update formula change if the width of the beam is made variable by keeping the depth constant throughout the beam?

Question 2 (10 points)

Verify if $\left\{ \frac{4}{3} \quad \frac{4}{3} \right\}^T$ and $\{1.2 \quad 1.4\}^T$ are local minima for the following constrained optimization problem.

$$\underset{x_1, x_2}{\text{Minimize}} \; f = x_1^2 + x_2^2 - 2x_1 - 2x_2 + 2$$

Subject to

$$-2x_1 - x_2 + 4 \leq 0$$
$$-x_1 - 2x_2 + 4 \leq 0$$

Question 3 (10 points)

Solve the following problem using the dual method. Show all the steps clearly.

$$\underset{x_1, x_2}{\text{Minimize}} \; f = (x_1 - 3)^2 + (x_2 - 3)^2$$

Subject to

$$2x_1 + x_2 - 2 \leq 0$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

ME256 Variational Methods and Structural Optimization June 30, 2006

Final Examination

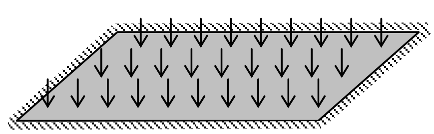
Points: 30 Time: 120 minutes

Question 1 (10 points)
The cross-section profile along the longitudinal axis of a cantilever beam of length L is to be optimally designed without exceeding the given volume of material V^* so that the mean compliance (i.e., $\int_0^L q(x)\, w(x)\, dx$) is minimized where $w(x)$ is the transverse displacement of the beam under a given load $q(x) = q_0$. The design variable is $A(x)$. Assume that the shape of the cross-section of the beam is such that $I(x) = \alpha A(x)$ where α is a constant. There are upper and lower bounds on the area of cross-section (i.e., $A_l \le A(x) \le A_u$) and it is given that $V^* < A_u L$. Denote the Young's modulus by E.

a) Write the complete statement of the problem as a constrained variational problem. Indicate the Lagrange multipliers corresponding to all the constraints by using lower case Greek symbols for local (point-wise) constraints and upper case Greek symbols for global constraints.
b) Write all the necessary conditions for the problem.
c) Prove that the volume constraint must be active in this problem.
d) Partition the span of the beam to identify in which parts $A(x)$ might reach its upper/lower bound.
e) Sketch the optimal area of cross-section and outline the steps to solve this problem *analytically* (<u>not</u> iteratively by using an update formula based on the optimality criterion).

Question 2 (10 points)
The governing equation for a rectangular membrane in the xy plane that is held fixed at its boundary, has a uniform tension T throughout, and is subject to a load $q(x, y)$, is given by $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{q}{T} = 0$ where $w(x, y)$ is the transverse deflection of the membrane.

a) Recalling the principle of minimum potential energy, deduce the expression for the potential energy of the membrane.
b) Justify the expression for the potential energy by noting that a membrane has no bending stiffness and that the deflections are small.
c) Next, using the Hamilton's principle, write the equation of motion for the transient behavior of the membrane if q is also a function of time.



Question 3 (10 points)
Consider the damped transient vibration of an axially deforming fixed-free bar subject to a time-varying load $p(x,t)$ per unit length of the bar. It is intended that the displacement (given by $u(x,t)$ in general) at time T at the free end is to be minimized for a given volume of material V^*. This leads to the following statement of the optimization problem.

$$\underset{A(x)}{\text{Minimize}}\ u(L,T)$$

Subject to

$$\lambda(x,t): \quad p + (EAu')' - b\dot{u} - \rho A\ddot{u} = 0$$

$$\Lambda: \quad \int_0^L A\,dx - V^* \le 0$$

where b is the constant damping coefficient per unit length, $A(x)$ is the design variable (the area of the cross-section of the bar), and the dot represents a derivative with respect to time and prime the derivative with respect to x.

a) Write the Lagrangian and then its variations with respect to $\lambda(x,t)$, $u(x,t)$ and $A(x)$.
b) Identify the adjoint equation. Does it have the same form as the governing equation? If it is not, what is its implication in the numerical solution to this problem?
c) Set up the formula for iteratively updating $A(x)$ using the optimality criteria method and outline the steps in the numerical algorithm.

The following fact may be useful to you for solving one of the problems in this examination.

If $z = f(x,y)$ is a single-valued continuously differential function of x and y, the area of a portion of the surface represented by this function is given by $\iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$ where D is the domain in the xy plane onto which the surface z projects.

Final Examination

Points: 30 Open book and open notes Time: 2 Hours

Question 1 (10 points)

We want to minimize the volume of an axially deforming *fixed-free* bar of length L and Young's modulus E which is subject to a point load $(-F)$ at $x = 0.2L$ and another point load F at $x = 0.8L$. The strain energy of the bar should not exceed SE^*, and its deflection at $x = 0.6L$ should not exceed Δ.

a) Write the *complete* problem statement in the standard form by taking the area of cross-section along the length of the bar as the design variable.
b) Solve the above problem to obtain an *analytical expression* for the area of cross-section in terms of the Lagrange multipliers.
c) If it is a fixed-fixed bar, how would you formulate the problem and then solve it numerically?

Question 2 (10 points)

a) Write down the convex, separable approximation of the following problem using appropriate linear or reciprocal linear approximation at $\{x_1 \quad x_2 \quad x_3\}^T = \{1 \quad -1 \quad -2\}^T$.
b) Using your approximation, obtain the dual problem in terms of the Lagrange multiplier.

$$\underset{\bar{x}}{\text{Minimize}} \quad f = 20x_2x_3 + 30x_1x_3 + 15x_1x_2$$

Subject to

$$g = 125 - x_1x_2x_3 \le 0$$

Question 3 (10 points)

A multi-physics topology optimization problem for electro-statically actuated microstructure is formulated as follows in 2D. Do *all* the analytical steps that are necessary to set up the numerical update scheme to solve this problem using the optimality criteria method.

$$\underset{\gamma(x,y)}{\text{Minimize}} \quad \int_\Omega \phi(\mathbf{u}, V)\, d\Omega$$

Subject to

$$\mu(x,y): \quad \nabla \cdot (\sigma \nabla V) = 0$$

$$\Gamma: \quad \int_\Omega \left\{ \boldsymbol{\varepsilon}(\mathbf{u})^T \mathbf{Y} \boldsymbol{\varepsilon}(\mathbf{v}) - \mathbf{f}^T \mathbf{u} \right\} d\Omega = 0 \quad \text{where} \quad \mathbf{f} = -\rho \nabla V - \frac{1}{2} \left\| \nabla V \right\|^2 \nabla e$$

$$\Lambda: \quad \int_\Omega \gamma\, d\Omega - V^* \le 0$$

Data :

$\sigma_0, \mathbf{Y}_0, V^*, e_0, \rho_0$; and the geometry of Ω is specified in 2D.

Other relationships and dependencies :

$$\sigma = \gamma^3 \sigma_0; \; V(x,y); \; \boldsymbol{\varepsilon}(\mathbf{u}) = \left\{ \frac{\partial u_1}{\partial x} \quad \frac{\partial u_2}{\partial y} \quad \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \right\}^T; \; \mathbf{u} = \{ u_1(x,y) \quad u_2(x,y) \}^T$$

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \left\{ \frac{\partial v_1}{\partial x} \quad \frac{\partial v_2}{\partial y} \quad \frac{1}{2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) \right\}^T; \; \mathbf{v} = \{ v_1(x,y) \quad v_2(x,y) \}^T$$

$\mathbf{Y} = \gamma^3 \mathbf{Y}_0; \; \mathbf{Y}_0 =$ stress-strain matrix in 2D ;

$e = \gamma^3 e_0; \; \rho = \gamma^3 \rho_0; \; \phi(\mathbf{u}, V)$

ME256 Variational Methods and Structural Optimization April 21, 2008

Final Examination

Points: 30 Time: 120 minutes

Question 1 (10 points)

In the topology optimization of compliant mechanisms, the ratio MSE/SE is maximized. The expressions for MSE (mutual strain energy) and SE (strain energy) are given by the following.

$$MSE = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{v})^T \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u})\, d\Omega \qquad \text{and} \qquad SE = \int_{\Omega} \frac{1}{2}\boldsymbol{\varepsilon}(\mathbf{u})^T \mathbf{D}\boldsymbol{\varepsilon}(\mathbf{u})\, d\Omega$$

where $\boldsymbol{\varepsilon}$ is the strain, $\mathbf{D}$ the stress-strain relationship, $\mathbf{u}$ the displacement vector under one loading (say body force $\mathbf{p}_u$), and $\mathbf{v}$ is the displacement vector under a second loading (say, body force $\mathbf{p}_v$). Derive the gradient of MSE/SE with respect to a design variable b. Note that $\mathbf{D}$ is an explicit function of b.

Question 2 (6 points)

In order to have a weak form for the governing equation of a damped spring-mass oscillator problem in x, the following functional is suggested where x and y are functions of t.

$$J = \int_0^T 0.5\left\{m\,x\,\ddot{y} + m\,\ddot{x}\,y - b\,x\,\dot{y} + b\,\dot{x}\,y + 2\,k\,x\,y\right\} dt$$

- Verify if it leads to the strong form of the governing equation of a damped spring-mass oscillator by equating the first variations of J with respect to x and y to zero.
- Do you see any drawback with this?

Question 3 (14 points)

Consider the damped transient vibration of an axially deforming fixed-free bar subject to a time-varying load $p(x,t)$ per unit length of the bar. It is intended that the displacement (given by $u(x,t)$ in general) at time T at the free end is to be minimized for a given volume of material V^*. This leads to the following statement of the optimization problem.

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$$\text{Subject to}$$

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where b is the constant damping coefficient per unit length, $A(x)$ is the design variable (the area of the cross-section of the bar), and the dot represents a derivative with respect to time and prime the derivative with respect to x.

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