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Johann Bernoulli's brachistochrone solution using Fermat's principle of least time

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Abstract. Johann Bernoulli's brachistochrone problem is now three hundred years old. Bernoulli's solution to the problem he had proposed used the optical analogy of Fermat's least-time principle. In this analogy a light ray travels between two points in a vertical plane in a medium of continuously varying index of refraction. This solution and connected material are explored in this paper.

1. Introduction

It is now three hundred years since Johann Bernoulli challenged the world with his brachistochrone problem. The statement of the problem is deceptively simple. We quote the English translation of Bernoulli's words from Struik's excellent book:

> 'Let two points A and B be given in a vertical plane. To find the curve that a point M, moving on a path AMB, must follow that, starting from A, it reaches B in the shortest time under its own gravity.' [1, p 392]

This challenge to the world was originally published in the *Acta Eruditorum* for June 1696.

Bernoulli's own solution to the problem was published in the *Acta Eruditorum* for May 1697. This solution utilized Fermat's optical priciple of least time. Fermat's least-time principle is equivalent to the optical law of refraction. The brachistochrone problem is considered to be one of the foundational problems of the calculus of variations. The standard method in that field now uses the Euler–Lagrange equation. Bernoulli's 1697 solution, which did not use the Euler–Lagrange equation, is of great interest to both mathematicians and physicists. It is discussed by both Struik [1, p 392] and Goldstine [2, p 30]. In this paper we also include a discussion of Galileo's earlier least-time problem which has sometimes been confused with Bernoulli's brachistochrone problem.

2. Galileo's problem of the swiftest descent to the bottom of a circle

In an earlier paper [3] we discussed the fastest descent problem of Galileo as found in the Scholium to Proposition 36 of his *Two New Sciences* [4, pp 212–3]†. There we remarked on a

† There is also an earlier translation by Crew and de Salvio [5].

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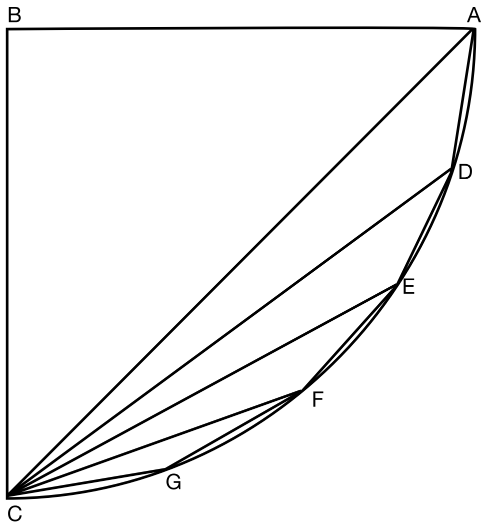

Figure 1. Galileo's diagram for his Scholium to Proposition 36 (re-drawn from the figure appearing in Crew and de Salvio [5, p 239]).

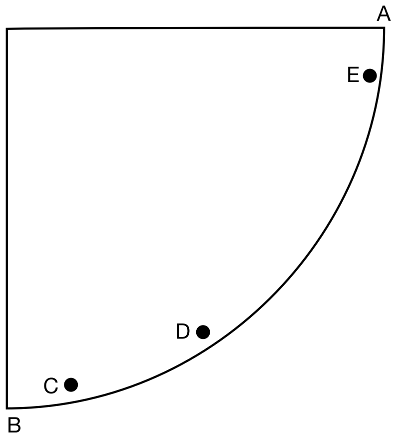

Figure 2. Galileo's least-time problem, re-drawn from the figure appearing in his *Dialogue Concerning the Two Chief World Systems* [6, p 451]).

minor 'incompleteness' in Galileo's proof. At the end of the paper a challenge was issued to fill in the incompleteness. To date nobody has come forward to complete Galileo's proof.

It is important to distinguish between Galileo's Scholium problem and the similar, but very different, Bernoulli brachistochrone problem. Figure 1 shows Galileo's diagram for his Scholium problem. In this problem one is limited to a single plane, or sequence of connected planes which have their end points on a quarter circle, or on an arc no greater than a quadrant of the circle. One seeks the sequence of planes which will yield the minimum time for a particle to slide frictionlessly from the upper point on the circle down to the bottom of the circle. It is assumed that the transition from one inclined plane to another occurs smoothly and with no loss of time.

In Bernoulli's brachistochrone problem one has two points at different elevations and one seeks the minimum-time curve for a particle to slide frictionlessly from the higher point to the lower point. The well known answer to the Bernoulli problem is the unique cycloid extending from the higher point to the lower point. The less well known answer to the Galileo problem is the infinite sequence of planes extending from the starting point to the bottom of the circle, i.e. the circle arc itself extending from the upper point to the bottom of the circle.

The Galileo problem is mentioned in both of his major works. In the *Two New Sciences* it is the topic of his Scholium to Proposition 36 of the Third Day. Drake's footnote to the Scholium is misleading and may be part of the reason that scholars have confused Galileo's problem with Bernoulli's problem. In footnote 48 to Galileo's statement of the Scholium, Drake wrote:

> 'All that could properly be deduced was that the shortest descent is along some kind of curve. The curve is in fact only approximately circular, and was later shown to be cycloidal' [4, p 213].

As we have pointed out, Galileo's problem included the 'circular constraint', i.e. the constraint that the end-points of the inclined planes be on the vertical circle, so that a cycloidal arc was not a possible solution to Galileo's minimum time to the bottom problem.

In the *Dialogue Concerning the Two Chief World Systems* [6] the least time to the bottom of the vertical circle is mentioned on p 451. Referring to his figure which shows the lower quadrant of a vertical circle, here shown as our figure 2, Galileo said:

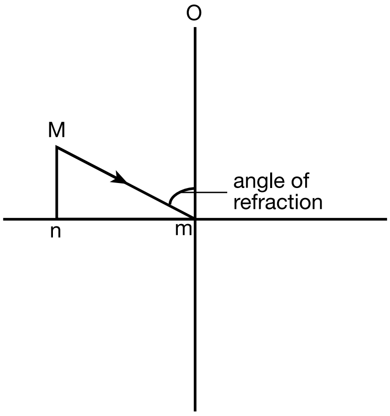

Figure 3. Refraction of a light ray.

> 'The motions of bodies falling along the arcs of the quadrant AB are made in shorter times than those made along the chords of the same arcs, so that the fastest motion, made in the shortest time, by a movable body going from the point A to the point B will be along the circumference ADB and will not be that which is made along the straight line AB, although that is the shortest of all the lines which can be drawn between the points A and B.'

Unfortunately, if this quotation is not seen in the context of the circular constraint Galileo's problem is liable to be misread as Bernoulli's brachistochrone problem. Among prominent writers who have made this misidentification, we mention Herman Goldstine [2, p 30]†.

3. The optical analogy

The fundamental law of refraction is most often referred to as Snell's Law in physics texts. In its most common form it is written as

$$n_1 \sin i = n_2 \sin r$$

where i is the angle of incidence and r is the angle of refraction. The n's are the indices of refraction of the two media, with the index being defined as the speed of light in vacuum over the speed in the medium, i.e.

$$n = \frac{c}{v}.$$

Snell's law can therefore be rewritten as

$$\frac{\sin i}{v_{\mathrm{i}}} = \frac{\sin r}{v_{\mathrm{r}}}.$$

The angle of incidence is defined as the angle between the incident ray of light and the normal, and the angle of refraction is defined as the angle between the refracted ray of light and the normal. The optical analogy to Bernoulli's problem is a ray of light penetrating through an infinite sequence of horizontal, differentially thin, optical media of decreasing optical density. The ray of light will follow a curved path. As the speed in the medium increases the angle of refraction increases. In figure 3 we show a ray of light Mm and its angle of refraction. Fermat's principle states that a light ray will take the path between two points which minimizes the time to go between the two points. Fermat's principle is a mathematically elegant way of stating Snell's law of refraction. We can now understand Johann Bernoulli's optical–mechanical solution of his brachistochrone problem.

† We also note this misidentification in a lecture given in 1985 by C Truesdell and published in [7].

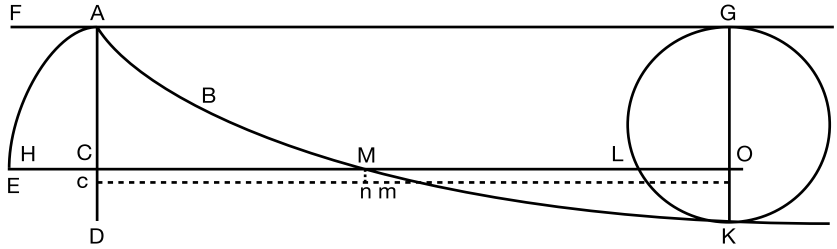

Figure 4. Johann Bernoulli's diagram for the brachistochrone problem (re-drawn from the figure appearing in [1, p 394].

The angle of refraction continually increases in this optical analogy. The constantly increasing speed of the particle sliding down the brachistochrone curve corresponds to a light ray proceeding along that curved optical path as it penetrates through an optical medium of ever-decreasing optical density.

4. Johann Bernoulli's diagram

We are now ready to present the main diagram from Johann Bernoulli's solution to his brachistochrone challenge problem. The challenge to the world was issued in the *Acta Eruditorium* of June 1696. Figure 4, Bernoulli's diagram, shows the medium FGD and the luminous point A.

The least-time light path from A to K is the brachistochrone solution. We draw the reader's attention to the infinitesimal 'triangle of refraction' Mmn. The angle between Mm and the vertical is the angle of refraction. The brachistochrone curve is the curve ABMK, which Bernoulli will find to be a cycloid. The analogy to the particle picking up speed as it descends the cycloid is the increasing speed of the light particle as it enters regions of lower optical density. As the index of refraction decreases the speed v of the light particle increases. The curve AHC on the left in Bernoulli's diagram represents this increase in speed as the mechanical–light particle descends. From the mechanics of freely falling bodies it is well known that the square of the speed is proportional to the distance of fall. In Bernoulli's problem this is implemented as

$$v^2 = \text{constant} \times x$$

where v is the velocity and x is the distance of fall. In terms of figure 4, we have

$$v = \text{velocity} = \text{CH}$$

$$x = \text{distance of fall} = \text{AC}.$$

The distance CM is denoted by y. Thus, x and y are the rectangular coordinates of the particle with respect to the starting point at A.

$$y = \text{horizontal coordinate} = \text{CM}$$

$$x = \text{vertical coordinate} = \text{AC}.$$

The key to the solution is the law of refraction. We have seen that $\sin r/v = \text{constant}$ or $\sin r = kv$, i.e. the sine of the angle of refraction is proportional to the light speed in the medium. Referring to the differential triangle Mmn in Bernoulli's figure, we see that

$$\text{Mn} = \mathrm{d}x \qquad \text{nm} = \mathrm{d}y \qquad \text{Mm} = \mathrm{d}z = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}.$$

Substituting this into the law of refraction, we have

$$\sin r = kv$$
$$\frac{\mathrm{d}y}{\mathrm{d}z} = kv$$
$$\mathrm{d}y = kv\,\mathrm{d}z = kv\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}.$$

If we replace the constant k by Bernoulli's constant $1/a$ and solve for $\mathrm{d}y$, we obtain

$$\mathrm{d}y = \frac{v\,\mathrm{d}x}{\sqrt{a^2 - v^2}}.$$

This is the differential equation for the brachistochrone curve ABM. Bernoulli triumphantly declared:

> 'In this way I have solved at one stroke two important problems – an optical and a mechanical one – and have achieved more than I demanded from others: I have shown that the two problems, taken from entirely separate fields of mathematics, have the same character.' [1, p 394]

The optical least-time path can be found by specifying the speed of light through the various media. The mechanical problem is the brachistochrone problem and the least-time path is found by specifying the varying speed of the particle along the path. As we mentioned above, the speed v along the path, starting from descent at point A, is given by $v^2 = \text{constant} \times x$. Bernoulli writes the value a for the constant and substitutes $v^2 = ax$ in his 'optical' least-time differential equation. One finds

$$\mathrm{d}y = \mathrm{d}x\sqrt{\frac{x}{a-x}}.$$

Bernoulli expands the square root as follows:

$$\sqrt{\frac{x}{a-x}} = \frac{1}{2}\,\frac{a\,\mathrm{d}x}{\sqrt{ax - x^2}} - \frac{1}{2}\,\frac{a\,\mathrm{d}x - 2x\,\mathrm{d}x}{\sqrt{ax - x^2}}.$$

Integrating, we find

$$y = a\sin^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax - x^2}.$$

To show that this is a cycloid one can use the trigonometric substitution

$$x = \tfrac{1}{2}a(1 - \cos\theta).$$

This yields

$$y = a\sin^{-1}(\sin\tfrac{1}{2}\theta) - a\sin\tfrac{1}{2}\theta\sqrt{\tfrac{1}{2}(1 + \cos\theta)}$$
$$= \tfrac{1}{2}a\theta - \tfrac{1}{2}a\sin\theta.$$

If we now make the further substitution

$$r = \tfrac{1}{2}a$$

we get

$$y = r\theta - r\sin\theta.$$

This, coupled with the original substitution

$$x = r(1 - \cos\theta)$$

allows us to recognize Bernoulli's brachistochrone as a standard cycloid generated by a wheel of radius r (or diameter GK) which rolls to the right without slipping along the line FAG in Bernoulli's diagram.

5. Conclusion

We have reviewed Johann Bernoulli's 1697 solution of his least-time problem and remarked upon the connection with the earlier problem of Galileo's Scholium to Proposition 36. Bernoulli was impressed that the solution of the least-time problem was the same as the cycloid of Huygens. Huygens had been involved with the cycloid because he was interested in building a perfect clock. It was known to Huygens that for a pendulum descending a circular arc the time to the bottom was a function of the size of the circular arc. Only in the limit of the small-angle approximation could a pendulum be considered isochronous. Huygens found out that the curve along which a pendulum would have to descend to be exactly isochronous was a cycloid. Bernoulli saw this identity between Huygens' equal vibration time for his clock (the tautochrone) and the solution for the least time of fall between two points (the brachistochrone) as an example of the simplicity of Nature.

Acknowledgment

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