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Galileo's Work on Swiftest Descent from a Circle and How He Almost Proved the Circle Itself Was the Minimum Time Path

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In Proposition 36 (Third Day) of his *Two New Sciences*[1] Galileo proved that descent to the bottom from any point on the lower quadrant of a vertical circle was swifter by a longer two-chord path than it was by the direct one-chord path. In the Scholium[2] to this proposition Galileo proved (not rigorously as we shall see) that the swiftest descent utilizing the circle constraint was descent via the circle itself. By the 'circle constraint' we mean that any path to the bottom must consist of a sequence of planes (circle chords), with every chord beginning and ending on the circle (with the circle itself as the limiting case of an infinite set of infinitesimal chords). In this paper we will sometimes refer to the inclined plane segments (the circle chords) as planes, and sometimes as line segments. The reason for this is to remind ourselves that we are dealing with a historical paper and that historical accuracy requires us to remember that Galileo was concerned with motion along inclined planes. In going from the older translation of the *Two New Sciences* by Crew and de Salvio to the newer and more historically exact translation of Drake, we find for example, that in the opening statement of Proposition 36, Crew and de Salvio render it as "If from the lowest point of a vertical circle, a chord is drawn . . . ," whereas Drake renders it as "From the lowest point of a vertical circle, let an inclined plane be raised . . . " The study of Proposition 36 and its Scholium is the subject of this paper. This study provides us with some very interesting information on Galileo's geometrical methods.

I. THE PROBLEM OF PROPOSITION 36. Figure 1 is a diagram for what Galileo set out to prove in Proposition 36. The arc DC is the arc of a circle not exceeding a quadrant of the circle. Point B is an arbitrary point on the arc. According to Drake, as early as 1602 Galileo knew "the fact that a body descends more swiftly along conjugate chords of a circular arc than along its chord, though the latter path is the shorter"[3].

[1]Galileo, *Two New Sciences*, published in Holland in 1638 in Italian. We use the English translation by Stillman Drake (University of Wisconsin Press, 1974). Proposition 36 can be found on pp. 211–212. We draw upon the translation by H. Crew and A. de Salvio (1954 Dover reprint of the original 1914 book published by the Macmillan Company) for the figures, since they are larger in Crew and de Salvio than they are in Drake.

[2]Ref. 1, pp. 212–213.

[3]Ref. 1, footnote 21 on p. 164. It is not clear what Drake meant by "the fact." It is unlikely that Galileo had actually performed any experiments to determine this "fact," so probably Drake meant that Galileo had done the proof ultimately contained in Proposition 36, or had done a typical calculation, as early as the year 1602.

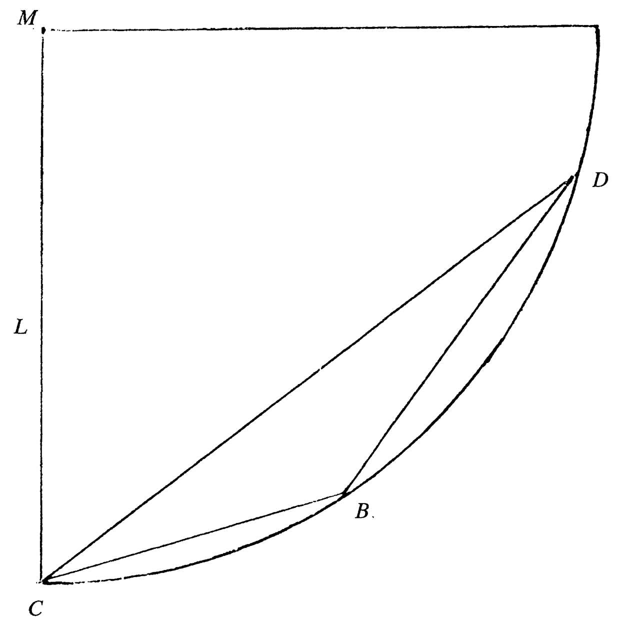

Figure 1. A diagram for the problem Proposition 36.

If the modern mind looks at Galileo's problem there is an inclination to seek an algebraic solution. Clearly, the acceleration along *DB* is greater than that along *DC*; whereas the acceleration along *BC* is less than that along *DC*. So the broken path has, first an advantage over the straight path, and then a disadvantage. Since the problem is closely related to the problem of pendulum motion down arc *DBC*, we denote the radius of the circle as L. The time for the direct path *DC* is

$$t_D = 2\sqrt{L/g} \tag{1}$$

This descent time is independent of the position of initial point D on the quadrant. Indeed, the point D can be any point on the circle, as shown by Galileo in his Law of Chords (*Two New Sciences*, Third Day, Proposition 6 on accelerated motion). Thus, (1) gives the descent time to the bottom from any point on the circle. If we use the topmost point on the circle the descent distance is $2L$, the final speed at C is $2\sqrt{gL}$, the average speed is $\sqrt{gL}$, and time = distance/average speed = $2L\sqrt{gL} = 2\sqrt{L/g}$.

The descent time by the broken path *DB-BC* is a function of the location of the point B on arc *DBC*. Consider Figure 2, which assumes point D is at the top of the quadrant. The speed of the particle at the arbitrary point B is $\sqrt{2gL\sin\theta_1}$ and its speed at the bottom is $\sqrt{2gL}$. The time t_B along the broken path is

$$t_B = \frac{2L\sin(\theta_1/2)}{\sqrt{2gL\sin\theta_1}/2} + \frac{2L\sin(\theta_2/2)}{\left(\sqrt{2gL\sin\theta_1} + \sqrt{2gL}\right)/2}$$

$$= 2\sqrt{2}\sqrt{L/g}\left\{\frac{\sin(\theta_1/2)}{\sqrt{\sin\theta_1}} + \frac{\sin(\theta_2/2)}{\left(\sqrt{\sin\theta_1} + 1\right)}\right\} \tag{2}$$

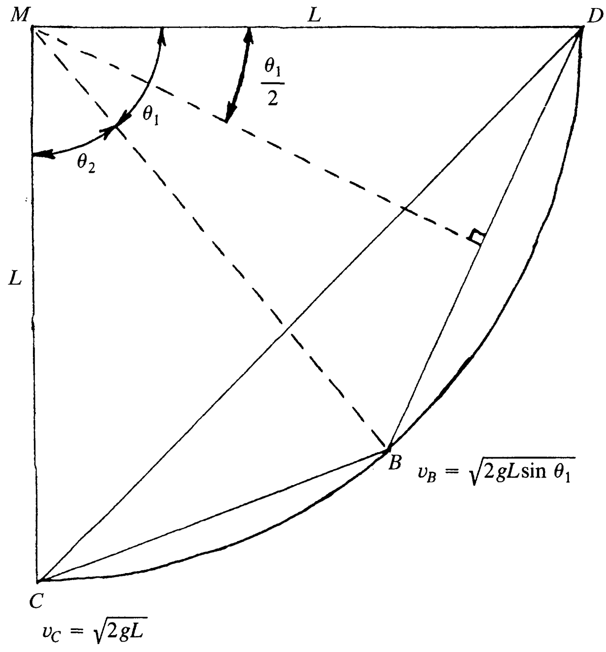

Figure 2. Diagram for the calculation of the time t_B along the broken path *DB-BC*.

To simplify our notation we set $K = \sqrt{L/g}$ so, for example, the time t_D of (1) is $t_D = 2K$. It is not immediately apparent whether t_B is less than, or greater than t_D. So one can try a sample value, say $\theta_1 = \theta_2 = 45°$. This yields $t_B = 1.875K$, so in this case, the descent is swifter via the broken path. It is also of some interest to compare this value of t_B with the time t_C for descent down the circular path *DBC*. This latter time[4] is $1.854K$. Since Galileo found in his Scholium to Proposition 36 that the swiftest descent from D is via the circle, it is of great interest to note that this two-segment descent time is only some 1% longer than the minimum descent time via the circle. This seems remarkable, considering that the circle path is an infinite set of infinitesimal segments, yet even the very crude approximation using two segments and a random θ_1 brings one only some 1% away from the minimal result. One is naturally curious to find out whether one can approach the circular decent time even more closely by using some other value of θ_1.

Figure 3 is a graph of descent time versus the angle θ_1. The minimum descent time of approximately $1.863K$ is achieved for an angle of about 25°. This value is only about .5% greater than the minimum value of $1.854K$ for descent via the circular quadrant.

II. GALILEO'S PROOF OF PROPOSITION 36. It would appear from examination of (1) and (2) that an algebraic proof showing that $t_B < t_D$ for all values of θ_1

[4]See the table on p. 269 of H. Erlichson, "Galileo's Pendulums and Planes," *Annals of Science 51*, 263–272 (1994).

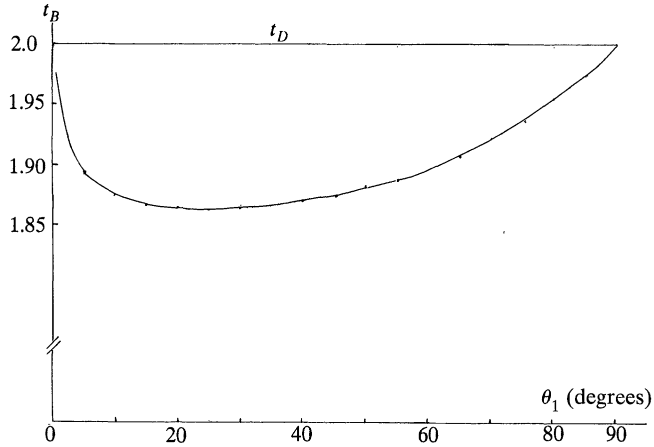

Figure 3. Descent time versus angle θ_1.

and for all starting positions along the quadrant would not be easy to achieve. For example, when D is at the full quadrant position one would have to show that.

$$\sqrt{2}\left[\frac{\sin(\theta_1/2)}{\sqrt{\sin\theta_1}} + \frac{\sin(45° - \theta_1/2)}{\sqrt{\sin\theta_1} + 1}\right] \leqslant 1.$$

Since an algebraic proof would admittedly be difficult, we ask "How did Galileo proceed?"

Figure 4 is Galileo's diagram for his Proposition 36. The key to his solution is his use of the law of chords in the circle *DFBN*. By the Law of Chords the descent time from rest at D to F equals the descent time from rest at D to B. Thus, the

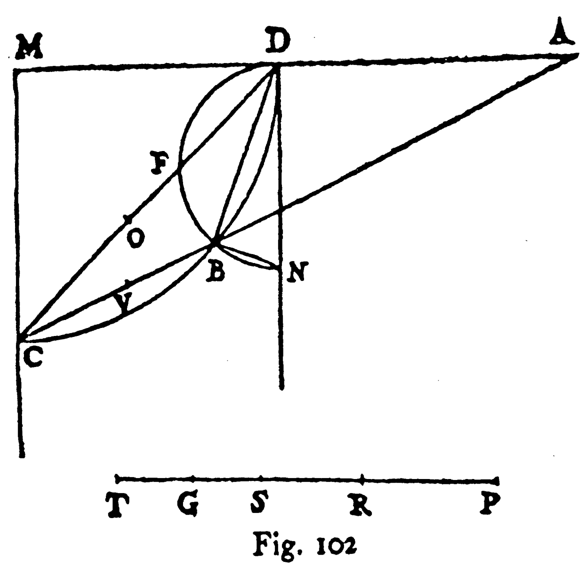

Figure 4. Galileo's diagram for his Proposition 36 (taken from Crew and de Salvio, Figure 102 on p. 237).

proof of the proposition hinges on comparing the additional time from F to C with the additional time from B to C. In his earlier [Third] Lemma[5] Galileo had shown that BC was shorter than FC. This would have clinched Galileo's proof of Proposition 36 because point B is below point F, hence the average from B to C (after descent from D) is greater than the average speed from F to C (after descent from D). Since BC is shorter than FC, and BC is covered at a greater average speed than FC, the time to cover BC (after descent from D) is less than the time to cover FC (after descent from D). This would have been a quick and easy way for Galileo to complete his proof of Proposition 36. Instead, Galileo used a significantly longer proof[6] involving points O and V (see Figure 4) where DO is the mean proportional between DF and DC, and AV is the mean proportional between AB and AC. [Note: The *mean proportional* d between two nonnegative numbers a and b is their geometric mean $d = \sqrt{ab}$]. We do not here review this longer proof by Galileo since it is clearly laid out by Galileo.

We choose instead to analyze Galileo's proof of his crucial [Third} Lemma, needed both for his proof of Proposition 36, and for our suggested shorter proof. Galileo's proof of this lemma contains hidden assumptions that would very likely not be obvious, even to a sophisticated reader. Since the {Third] Lemma depends on the {first] Lemma we first discuss the [First] Lemma.

Galileo's diagram for his [First] Lemma is here shown as our Figure 5. Galileo's proof of this lemma is straightforward. The line CD is perpendicular to diameter AB. Point E is any point on the circle arc AFB. Galileo considers two cases: in one case E is below F, and in the other it is above F. When the line CF is drawn perpendicular to AB, point D inside the circle is associated with the upper point E, and point D outside the circle is associated with the lower point E. Galileo wants to demonstrate that in either case BF is the mean proportional between BD

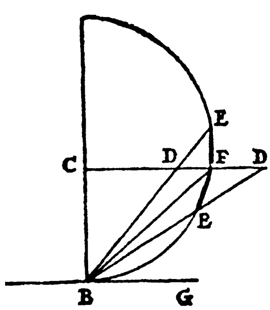

Fig. 99

Figure 5. Diagram for Galileo's [First] Lemma (taken from Crew and de Salvio, p. 235, Fig. 99).

and BE. Line BG is tangent to the circle at B. Angle EFB is measured by half of arc BE, and angle GBD is measured by half of the same arc BE, hence angle EFB equals angle GBD. Since CD is parallel to BG, angle CDB = angle DBG, whence angle EFB = angle CDB. Triangles FDB and FEB are therefore similar (one common angle and angle EFB = angle CDB). Thus, $BD/BF = BF/BE$, i.e., BF is the mean proportional between BD and BE.

[5]Ref. 1, pp. 210–211. The lemmas were not numbered by Galileo. We use the [First], [Second], [Third] scheme of Drake.

[6]Ref. 1, pp. 211–212.

Galileo uses his [First] Lemma in his proof of his [Third] Lemma, but he omits an explanation of why his figure meets the conditions for his [First] Lemma. This is an important omission, and we here investigate it.

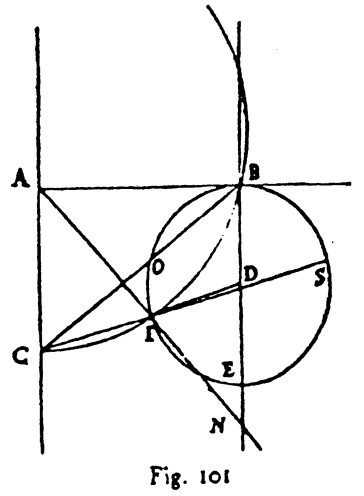

Figure 6. Galileo's diagram for the [Third] Lemma for the case where arc *BIC* is less than a quadrant (taken from Crew and de Salvio, Figure 101, p. 236).

Figure 6 is Galileo's diagram for the most general case where arc *BIC* is less than a quadrant. We are interested in explaining why line *SO* (not drawn in Galileo's diagram) is parallel to line *AB*. Without this, one is not entitled to apply

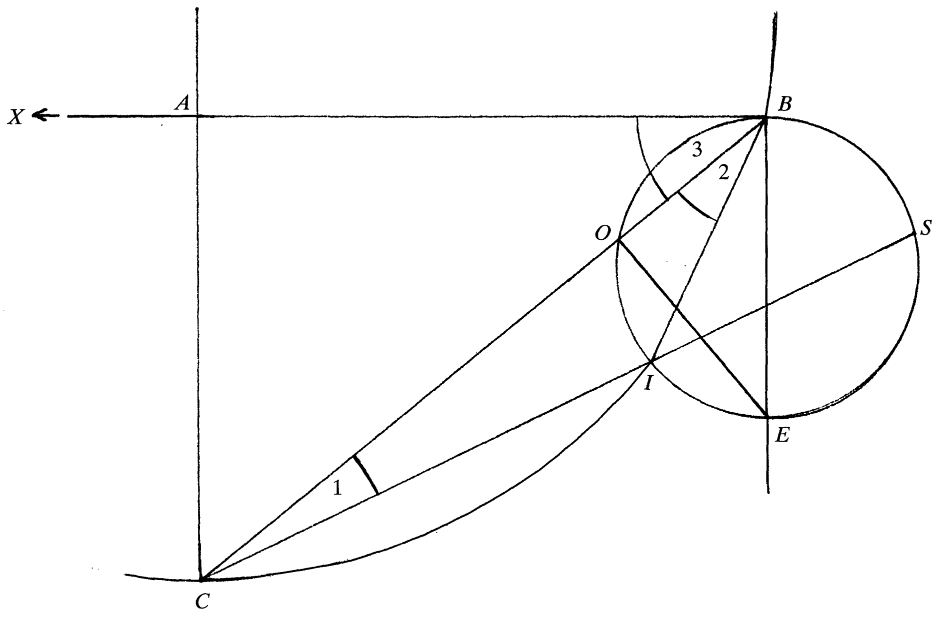

Figure 7. Diagram for the proof that points *O* and *S* lie in the same distance below line *AB*.

the [First] Lemma to show that $(CI)(CS) = (CO)(CB)$. Note carefully that the line *COB* in Figure 6 corresponds to the upper line *BDE* in Figure 5, and that the line *CIS* in Figure 6 corresponds to the lower line *BED* in Figure 5. If we distinguish upper points by a 'u' subscript and lower points by an 'ℓ' subscript, Galileo has shown in Lemma [1] that $BD_u \cdot BE_u = BD_\ell \cdot BE_\ell$, i.e., that in Figure 6, $(CI) \cdot (CS) = (CO) \cdot (CB)$. In Figure 7 we show the essentials for establishing that points S and O are the same distance below line *AB*. This would be the case if one could show that arc *BO* equals arc *BS*. We label angle *BCI* as angle 1, angle *CBI* as angle 2, and angle *ABC* as angle 3. Angle *BIS* equals angle 1 plus angle 2 because *BIS* is an exterior angle of triangle *CBI*. If we can establish that angle *BIS* (which is measured by arc *BS*) is equal to angle 3 (which is measured by arc *BO*) then we would have that arc *BS* equals arc *BO*, i.e., we want to show that angle 1 plus angle 2 equals angle 3. Now, angle 1 is measured by arc *BI* and angle 2 is measured by arc *CI*, hence angle 1 plus angle 2 is measured by arc *BIC*. Angle 3 is measured by arc *AXC*, which is equal to arc *BIC*. Hence, angle 1 plus angle 2 equals angle 3, arc *BS* equals arc *BO*, and points O and S lie on a line that is parallel to *AB*. Galileo, no doubt, assumed that this was readily verified, a clear indication that he had a very strong familiarity with Euclidean geometry.

III. FASTEST DESCENT—GALILEO'S UNPROVEN ASSUMPTION. In an important Scholium to Proposition 36, Galileo attempted to establish that the swiftest descent, given the constraint of using points on the circular arc, was via the circular arc itself. He said "*From the things demonstrated, it appears that one can deduce that the swiftest movement of all from one terminus to the other is not the shortest line of all, which is the straight line* [*AC*], *but through the circular arc*"[7] (see Figure 8).

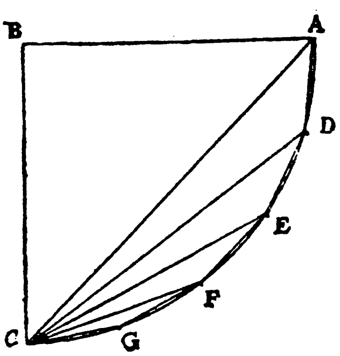

Fig. 103

Figure 8. Galileo's diagram for his Scholium to Proposition 36 (taken from Crew and de Salvio, Figure 103 on p. 239).

Drake commented that "All that could properly be deduced was that the shortest descent is along some kind of curve. The curve is in fact only approximately circular, and was later shown to be cycloidal".[8] We believe that Drake's comment was inappropriate because we think that Galileo was limiting himself to descent paths that used points on the circle. Note, however, that Drake found no incom-

[7]Ref. 1, pp. 212–213.
[8]Ref. 1, p. 213.

pleteness in Galileo's demonstration whereas, as we will shortly show, Galileo simply assumed the truth of what was perhaps the most crucial step in his proof. Drake was not alone in not noting the incompleteness of Galileo's proof, nor in realizing that Galileo was limiting himself to points on the circle. Dijksterhuis, referring to multiple chord descent down the circular quadrant $AC \ldots B$, said "He is able to show that the time of descent from A to B grows smaller as the number of parts of this broken line-segment increases. However, in his formulation of the proposition he had stated that the quickest descent from A to B takes place along the circular arc AC, and this conclusion of course is not warranted by the result obtained".[9].

To establish that the circle itself was the swiftest path, Galileo divided the quadrant into five equal arcs, as shown in his figure, which is here reproduced as our Figure 8. He had already established that "*movement through the two* [*lines*] *AD-DC is finished more quickly than through AC alone*." But then Galileo made a crucial assumption when he said "*Yet it seems true that from rest at A, descent is finished more quickly through the two DE-EC than through CD only*".[10] Now, Galileo had established that from *rest* at D descent is swifter through DE followed by EC, as against descent through DC only, but *he had not established this when the particle was already moving at point D.* Thus his "yet it seems true ..." is an important unproven assumption. Given this assumption it does indeed follow that descent through the five segments shown in Figure 8 is swifter than descent through any lesser number of the segments shown, and that increasing the quadrant subdivision into a greater and greater number of segments continues to decrease the descent

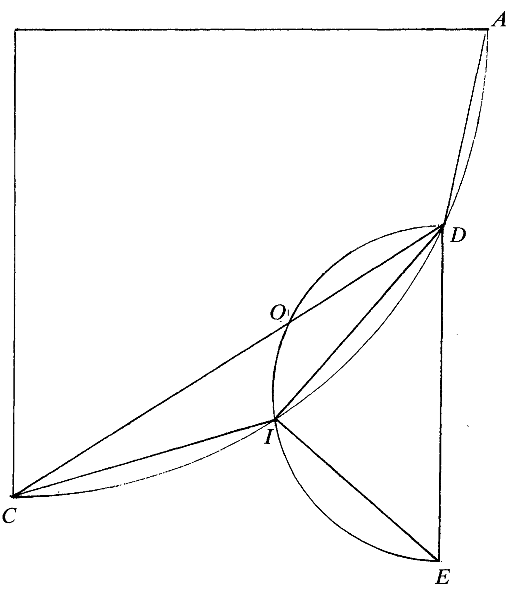

Figure 9. Diagram for comparing descent time along DC with that along DI-IC, starting from rest at A.

[9] E. J. Dijksterhuis, *The Mechanization of the World Picture*, translated by C. Dikshoorn from the Dutch work *De Mechaniserring van het Wereldbeeld* (Amsterdam, 1950). This translation published by Oxford University Press, 1961; quoted material is on p. 346.

[10] Ref. 1, p. 213.

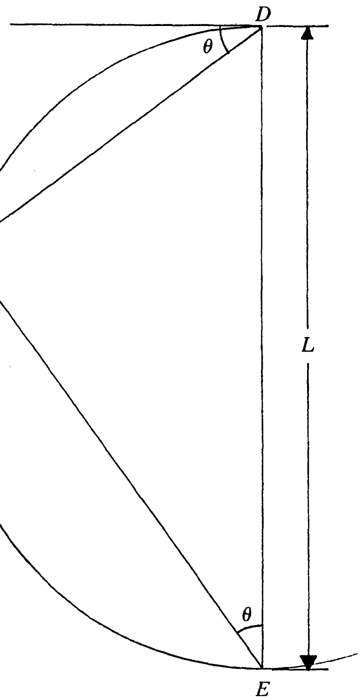

Figure 10. The descent time along *DI*, starting with speed v_0 at *D*, increases with increasing θ.

time. This permits Galileo to conclude that "*motion between two selected points, A and C, is finished the more quickly, the more closely we approach the circumference through inscribed polygons*".[11]

But now we must ask "How difficult would it have been for Galileo to go one step further and add to Proposition 36 a proof that, starting from some finite speed, descent using two segments is swifter than using one segment alone"? Alas, this difference of a finite speed at point *D* would completely destroy Galileo's method of using an equal time circle. To see this, consider Figure 9, in which we try to apply Galileo's method to the situation where the particle starts from rest at *A* and arrives at *D* with speed v_0. As before, we can construct circle *DOIE*. As before, descent along *CI* is swifter than along the longer leg *CO*. But alas it is no longer true that path *DI* takes the same time as path *DO*. Indeed, now with a starting velocity of v_0 at point *D*, one readily finds that traversing *DI takes longer than traversing DO*, which destroys the possibility of an easy extension of Galileo's method of proof. To see that this is so consider Figure 10 where θ is a variable slope angle and the speed of the particle at *D* is v_0. If *L* is the diameter of the circle then the length *DI* equals $L \sin \theta$. The speed of the particle at *I* is

$$v_I = \sqrt{v_0{}^2 + 2gL\sin^2\theta}$$

[11] *Ibid.*

and the descent time along DI is

$$t_{DI} = \frac{L \sin \theta}{\frac{v_I + v_0}{2}}.$$

Thus, t_{DI} is no longer independent of θ as it was when $v_0 = 0$. In fact t_{DI} increases with increasing θ as can be seen by differentiating t_{DI} with respect to θ, and observing that $dt_{DI}/d\theta$ is positive, i.e., t_{DI} increases with increasing θ.

We are inclined to hypothesize that Galileo knew full well that he did not have a complete proof, and that he also knew that it would be quite difficult to prove his "yet it seems that...". If the reader has any doubts about the truth of Galileo's unproven assumption we compare in the table below the descent time using five equal planes with the descent time along the circle[12] for some selected values of circular descent arc.

circular descent arc in degrees	descent time along the circular arc ($K = \sqrt{L/g}$)	descent time using 5 equal planes
90	1.8541 K	1.85609 K
80	1.7868 K	1.79574 K
70	1.7312 K	1.74553 K
60	1.6858 K	1.70411 K
50	1.6490 K	1.67047 K
40	1.6200 K	1.64384 K
30	1.5981 K	1.62367 K
20	1.5828 K	1.60954 K

III. THE CHALLENGE—TO PROVE GALILEO'S UNPROVEN ASSUMPTION. The readers of the MONTHLY are invited to try to prove the unproven assumption of Galileo contained in his statement "Yet it seems true that from rest at A, descent is finished more quickly through the two DE-EC than through CD only," preferably by methods available to Galileo. As we have already seen the law of chords technique used by Galileo does not work in this case. This is not an easy problem, especially if one is limited to the mathematics known at Galileo's time. We make the historical guess that is why Galileo decided to avoid a proof and use the phrase "yet it seems true...". We only add that we have been unable to devise such a proof.

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[12] For descent times along the circle see Ref. 4.