1. Minimum of a function, f(x)

Global minimum

x^* is global minimizer of $f(x)$ if $f(x^*) \le f(x)\ \forall\ x$ in a feasible interval of x .

This is just a definition, not a condition.

Local minimum

x^* is a local minimizer of $f(x)$ if $f(x^*) \le f(x)$ in a small neighborhood of x^* in the feasible interval of x

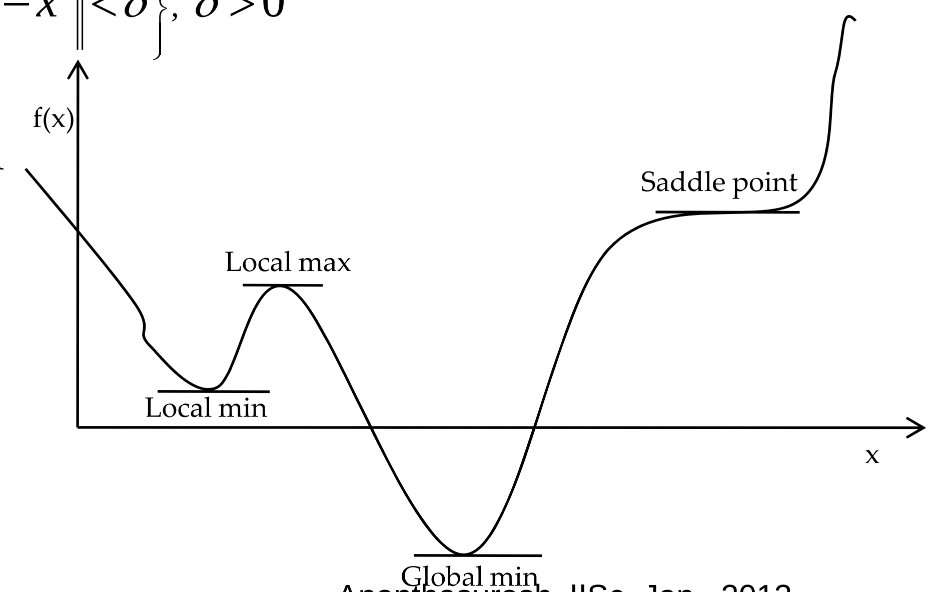
N = small neighborhood = $\left\{ x \middle| x \in S \text{ with} \left\| x - x^* \right\| < \delta \right\},\ \delta > 0$

This is also a definition, not a condition

Condition for a local minimum

Necessary condition: $\left.\dfrac{df}{dx}\right|_{x^*} = 0$

Sufficient condition: $\left.\dfrac{d^2 f}{dx^2}\right|_{x^*} > 0$

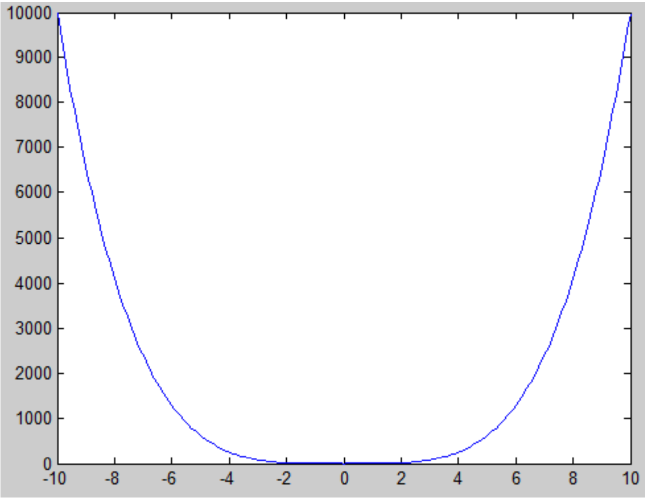


What happens if $\frac{d^2 f(x^*)}{dx^2} = 0$?

Look at $\frac{d^3 f(x^*)}{dx^3}$.

$$\frac{d^3 f(x^*)}{dx^3} = 0 \quad \text{Necessary condition}$$

$$\frac{d^4 f(x^*)}{dx^4} > 0 \quad \text{Sufficient condition}$$



x^4 vs x

E.g. $f(x) = x^4$

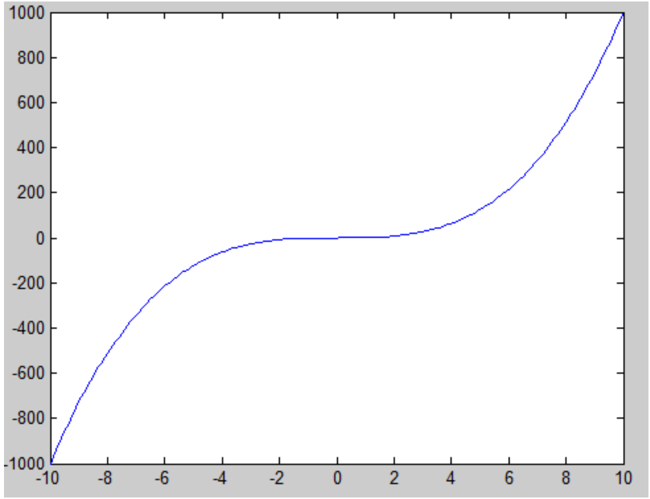
$$f'(x) = 4x^3 \quad \Rightarrow \quad x^* = 0$$

$$f''(x) = 12x^2 = 0$$

$$f'''(x) = 24x = 0 \quad \Rightarrow x^* = 0$$

$$f^{iv}(x) = 24 > 0$$

} $x^* = 0$ is a minimizer



x^3 vs x

E.g. $f(x) = x^3$

$$f'(x) = 3x^2 \quad \Rightarrow \quad x^* = 0$$

$$f''(x) = 6x = 0$$

$$f'''(x) = 6 \neq 0$$

} Necessary condition is satisfied, but not sufficient condition $\Rightarrow$ Saddle point

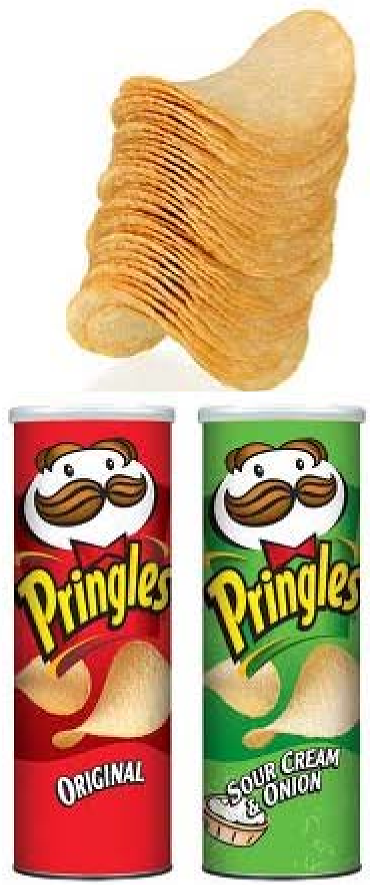
Ananthasuresh, IISc, Jan., 2013

2. Minimization in several variables

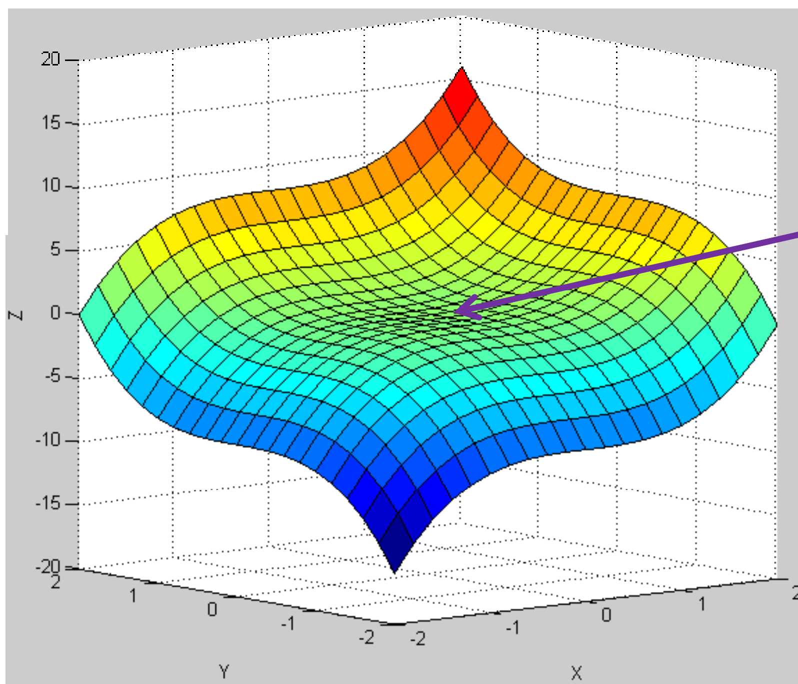
$$\frac{\partial f}{\partial x_i} = 0 \qquad\qquad i = 1, 2, \;..................n$$

Taylor Series :

$$f(x^*) = f(\overline{x}^*) + \sum_{i=1}^{n} \frac{\partial f(\overline{x}^*)}{\partial x_i}(x_i - x_{i0}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f(\overline{x}^*)}{\partial x_i \partial x_j}(x_i - x_{i0})(x_j - x_{j0}) + O(3)$$



$$+ \overline{\nabla} f^T(\overline{x}^*)\delta\,\overline{x} + \frac{1}{2}\partial\,\overline{x}^T\,\overline{H}(\overline{x}^*)\partial\,\overline{x} + O(3)$$



A special saddle point

plot of $z = x^3 + y^3$

where,

$$\bar{\nabla} f(\bar{x}) = \begin{Bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{Bmatrix}$$

Gradient of f w.r.t $\bar{x}$

$$\bar{\bar{H}}(\bar{x}) = \begin{Bmatrix} \dfrac{d^2 f}{dx_1^{\;2}} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{d^2 f}{dx_2^{\;2}} & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_n} \end{Bmatrix}$$

Hessian

(1st order) necessary condition: $\bar{\nabla} f(x^*) = 0$

(2nd order) sufficienct condition: $\underbrace{\frac{1}{2}\delta \bar{x}^{*T} H^* \delta \bar{x}^* > 0}_{qudratic\ form}$

H is positive definite

- All Eigen values are positive
- All principal minors are positive
- Pivots are positive in the row-echelon form

Quadratic form $\delta \bar{x}^{*T} \bar{H}^* \delta \bar{x}^*$	$\bar{H}$	Eigen values of $\bar{H}$	Nature of x^*
Positive	PD	All are positive	Local min
Negative	ND	All are negative	Local max
Non-negative	PSD	Some zero, others positive	Probably valley with flats.
Non-positive	NSD	Some zero, others negative	Probably ridge or
Any sign	Indefinite	Mixed signs	Saddle point