Lecture 1

Classification of Optimization Problems and the Place of Calculus of Variations in it

ME256 Indian Institute of Science

Variational Methods and Structural Optimization

G. K. Ananthasuresh

Professor, Mechanical Engineering, Indian Institute of Science, Banagalore

suresh@mecheng.iisc.ernet.in

Outline of the lecture

Understanding the spirit of optimization

What makes up an optimization problem?

Types of optimization problems

Finite-variable optimization vs. Calculus of variations

What we will learn:

What is optimization? Philosophically and mathematically?

What distinguishes one type of optimization problem from another?

The similarities and differences between finite-variable optimization and calculus of variations.

An example of a calculus of variation and how it can be tuned into a finite-variable optimization through discretization.

Optimization and its spirit

Optimization is achieving the best with the available resources while satisfying the constraints.

- We optimize in our daily lives. Nature seems to have optimized almost everything.
  - It is about surviving with what one has and getting the best.
  - Optimization is a way of life!
- Optimization has a lot to do with optimism.
  - Optimists view the proverbial glass half full and not half empty.
  - Given any number of obstacles (i.e., constraints), optimists try to make the best out of the situation.

Optimization hinders evolution!

- It is a witty way of saying the essence of optimization.
- Eventually, everything gets optimized.
  - The fittest and the best survive. The fittest would have used resources in the most effective manner and would have surmounted or circumvented the obstacles.
- But when we optimize at this instant, evolution is not necessary as we have already achieved the best already.
  - So, evolution is hindered.

Everything can be optimized.

- It is not an exaggeration; indeed everything can be optimized.
- It is simply a question of knowing what is the best, what the issues are, and how we can achieve it.

What do we need to optimize?

Objective function

- This is the most important thing in optimization; we need to know what we want to improve to the extreme.
- Extreme can be a maximum or minimum, depending on the identified objective.
  - It is a cost, we minimize; and if it is profit, we maximize.
  - In structures, if it weight, we minimize; and if it is strength we maximize, most often.
- We call it an objective function because it must depend on some variables in order to optimize.
  - The objective function should be **a function of optimization variables**.

Optimization variables

- These are the variables to which we try to assign suitable values to optimize the objective function.

Constraints

- Quantities that should be within bounds while we optimize the objective function.
- Constraints too are functions of optimization variables.

Formulae or methods to quantify and compute objective function and constraints

- We should have either mathematical expressions (formulae) or numerical methods to compute the values of the objective function and constraints once we know the values of the optimization variables.

There will be subsidiary variables too. (more later: see Slide 8)

There will also be constraints that govern those subsidiary variables.

Subsidiary variables are called state variables in the context of structural optimization.

There must be conflict in order to optimize!

It is no fun to maximize $f = 2x$!

- You would simply say that x should be ∞.
- This objective function is unbounded: you can make x as large as you want to maximize the objective function.

It is more fun to maximize $f = 2x$ subject to $x^2 + y^2 - 4 \leq 0$.

- Now there is an upper limit on x.
- Even if you have $y = 0$, x is bounded from above at 2. So, $f = 2 * 2 = 4$ at best.
- It will be more fun if y is also constrained in some other way to prevent $2x$ from becoming too large.

Basically, there must be conflict between the objective function and at least one constraint.

- This should be so with respect to each optimization variable.

Conflicts can be between two (or more) constraints too.

Conflict can be within a single function, i.e., the objective function.

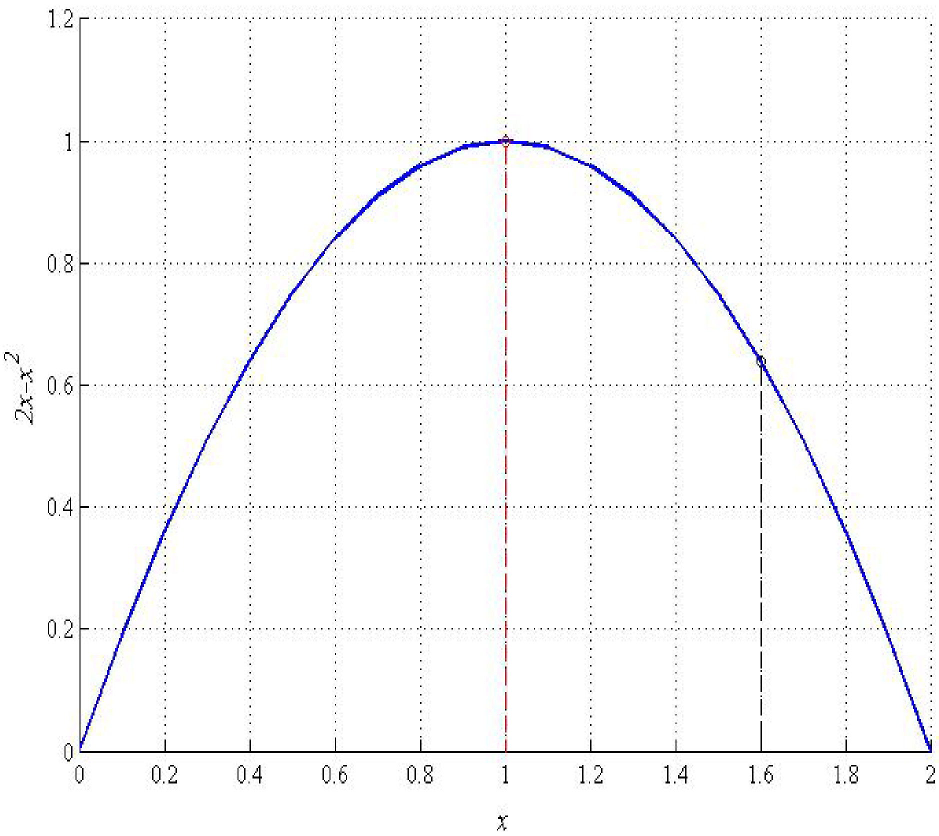
- Consider $f = 2x - x^2$.
- Now, at different points (i.e., at different values of x), as you move to the right or left, you have no conflict; you simply in the direction of increasing f.
- But then you come to a point ($x = 1$) at which you have a conflict; there, f does not increase whether you move to the right or left. Then, you stop; you have reached your maximum.
- **Here, the conflict is in the two terms within f.**
  - $2x$ says you should make x as large as possible.
  - x^2 says that you should make x as small as possible because it is subtracted.

Conflict within a function: non-monotonicity

Conflict of the terms within an objective function

$$2x - x^2$$

- While $2x$ increases with x and makes f larger, $(-x^2)$ decreases f with increasing x.
- Such as function is called non-monotonous.
- Monotonicity, i.e., a situation when a function increases with increasing x and decreases with decreasing x, does not give an optimum.
  - As we say, in real life, that monotonicity is boring? So, it is in optimization.



Conflicting monotonicities in objective and constraints

- If the objective function is monotonous and a constraint is also monotonous, then their monotonicities should be opposite.
- That is, when x increases, one should increase and the other decrease, and vice versa.
- Conflicting monotonicites can be in two different constraints also.

Anyway, the main point is that conflict is crucial to optimization.

Is there conflict in structural optimization?

Sure, there is.

- If you want to make a stiff structure for given loading, you need more material; more material increases the weight and cost.
- So, there is conflict if you want to design the stiffest structure with least amount of material.

What if we want to make a lightest structure with high natural frequency?

- Light structures have less inertia and less stiffness too, at least in general. This will mean that their frequencies will be low.
- So, there is conflict.

Suppose that you want to make a flexible structure that is very strong.

- Flexible structures deform and it may seem that they are weak when strains are large in them.
- So, there is conflict too.

Imagine a structure that is subject to multiple loading conditions.

- Making a structure stiff under one loading may cause it less stiff in another loading.
- So, there will be conflict.

Imagine more situations of designing structures. There will be enough conflict!

Optimization problem statement

Minimize Objective function

Optimization variables

Subject to

Constraints

Limits on variables

Minimize $f(x_1, x_2)$

x_1, x_2

Subject to

$$h(x_1, x_2) = 0$$

$$g(x_1, x_2) \leq 0$$

$$x_1^l \leq x_1 \leq x_1^u$$

$$x_2^l \leq x_2 \leq x_2^u$$

This is a typical constrained optimization problem statement. Make it a habit to write in this format.

Structural optimization problem statement

Minimize Objective(optimization variables, state variables)

Optimization variables

Related to the geometrical features

Displacements, temperature, electric field, etc.

Subject to

Weight, cost, size, etc.

Constraints on state variables

Constraints on resources

Constraints on performance

Limits on variables

Governing differential equations

Stiffness, strength, frequency, etc.

Data

Material properties, loads, etc.

This is a typical structural optimization problem statement. Make it a habit to write in this format. Including the data.

Types of optimization problems

There are many, many types of optimization problems.

The types arise because of…

- How many objective functions you have.
- Types of objective function and constraints.
- Types of variables.
- Nature of optimization we want to do.
  - Global
  - Local

We will examine important ones, one at a time.

We do this at the outset just so we understand what calculus of variations is.

First note that… calculus of variations is also optimization.

In fact, the theory of calculus of variations got developed much before the "usual" optimization theory got developed.

Classification based on the objective function

One or more objective functions

- Working with a single objective is easy.
  - Even in life!
  - Most of the theory of optimization is focused on dealing with a single objective function.
- Multi-objective optimization is hard.
  - Weights can be given but then how do you give the weights when you do not know which objective is more important for you?
  - The best thing to do is to move the less important objective to a constraint.
- Pareto optimum
  - Pareto optimum concept is an important concept in multi-objective optimization.
  - Pareto optimum is one where you can improve on objective function without hurting another.
  - Often, Pareto optimum will be a set; that is there will be many Pareto optima.
    - Pareto optimum set can be continuous or discontinuous.
    - Generating the entire Pareto set is difficult in practice.

Global or local (more later)

- Local optimum is one in a small vicinity of the optimum point.
  - It is like you are the smartest in your class. You are a local maximum.
- Global optimum is one that considers the entire domain of the objective function.
  - If your school beauty is a local maximum, Miss Universe if the global optimum.

Classification based on constraints

Without or with constraints

- If there are no constraints, it becomes an **unconstrained optimization** problem.
- With constraints, it is **constrained optimization** problem.

Equality or inequality

- Constraints can be equalities.
  - Governing equations are usually equality constraints in structural optimization.
- They can be inequalities
  - Inequalities arise mostly due to resource and performance constraints in structural optimization.

Constraints reduce the permissible values of the optimization variables.

- Constraints constrain the space of optimization variables.

Feasible space

- The space of optimization variables where all constraints are satisfied is called the **feasible space**.
- In constrained optimization, we need to search for the optimum of the objective function only in the feasible space.
- Constructing feasible space is often impractical but we can certainly search within it.

Classification based on objective and constraints

Linear programing

- Both objective function and constraints are linear.

Quadratic programming

- Objective function is quadratic and constraints are linear.

Nonlinear programming

- Both objective and constraints are nonlinear.

Geometric programming

- Objective and constraints are posynomials.
- Posynomials are polynomials with positive coefficients.
  - Positive + Polynomial = posynomial! A portmanteaus word.
  - Exponents in posynomials can be real numbers, positive or negative, in the context of geometric programming.

Convex optimization

- The objective function is convex and so is the feasible sapce.

Non-smooth optimization

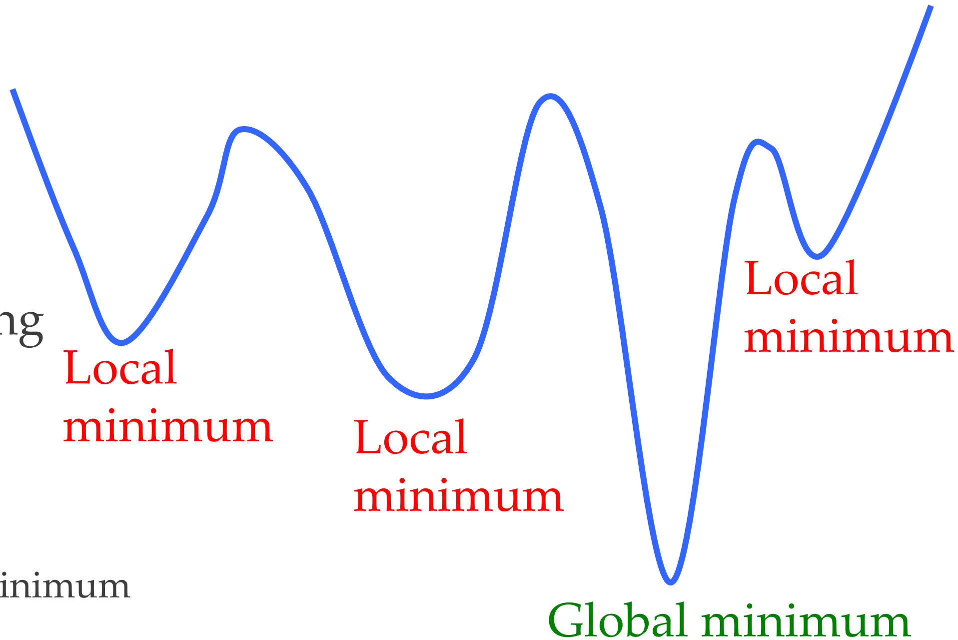
- Where either objective function or the constraints, or both, are not differentiable.

And many more types!

Classification based on the nature of optimization

Global or local

- Local when we are happy with a local minimum.
  - That is smallest in the vicinity of a point.
- Global when we want to find the smallest among all minima.
- There are no easy methods to find the global minimum.
  - Only special types of problems allow finding the global minimum



Deterministic or stochastic

- Deterministic means that you have the same thing any number of times you try it.
- Stochastic means that there is also an element of randomness in addition to deterministic nature.
- An optimization problem can be stochastic if the variables involved are stochastic.
- But, one can use non-deterministic methods to solve a deterministic problem.
  - Genetic algorithms, Simulated annealing methods, Monte Carlo search, etc.

Classification based on types of variables

Continuous or discrete

Discrete

- Binary
  - Only 0 or 1 are allowed.
- Integer
- Discrete sets
  - Bearing sizes, screw threads, etc.; you cannot have whatever you want. They will be certain pre-specified values.

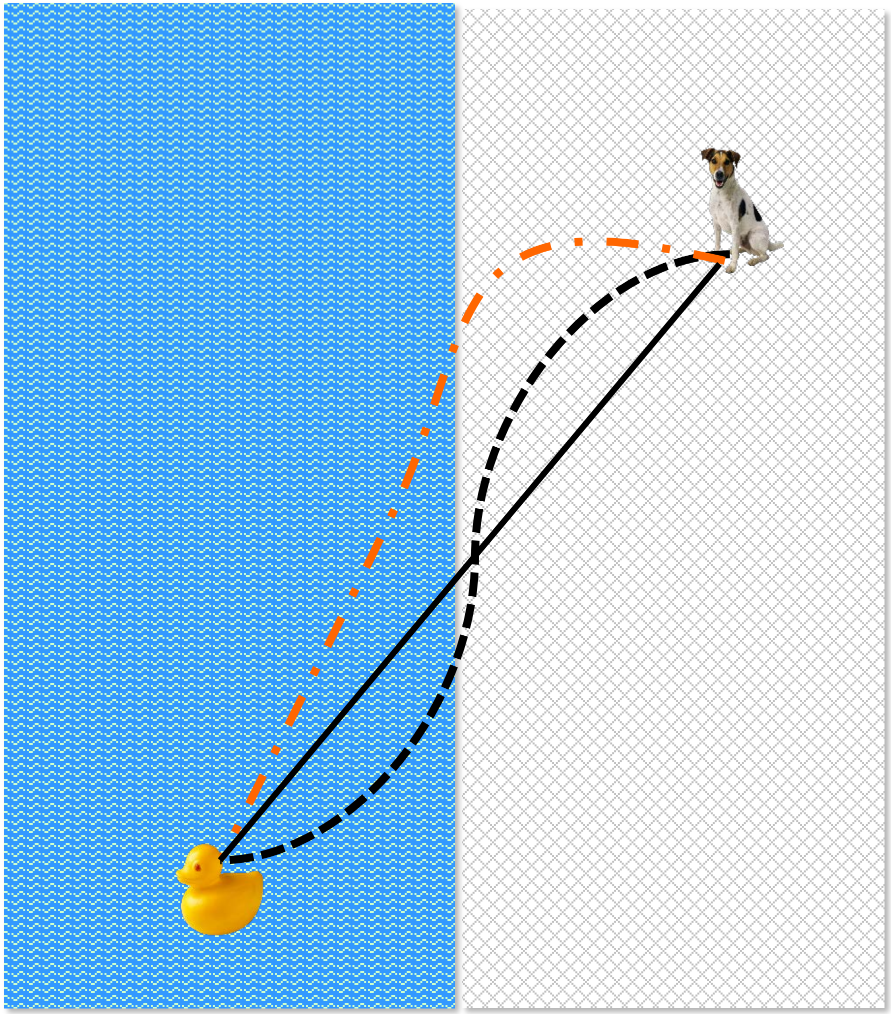
Deterministic or stochastic

- Discussed in the previous slide
- Uncertainties bring about stochastic nature in structural optimization.

Finite variables or functions themselves!

- Variables are finite if they are like $x_1, x_2, x_3, \ldots, x_n$.
- What if variables are not finite?
- What if variables are **functions themselves**?
- This is what brings us to calculus of variations

Consider this optimization problem:



A dog is sitting next to a swimming pool and his owner threw a rubber duck into the pool. The dog can **run on the pool-tiles twice as fast as it can swim** in water.

What path should the dog take to touch the duck in the **shortest time**?

It is clearly an optimization problem.

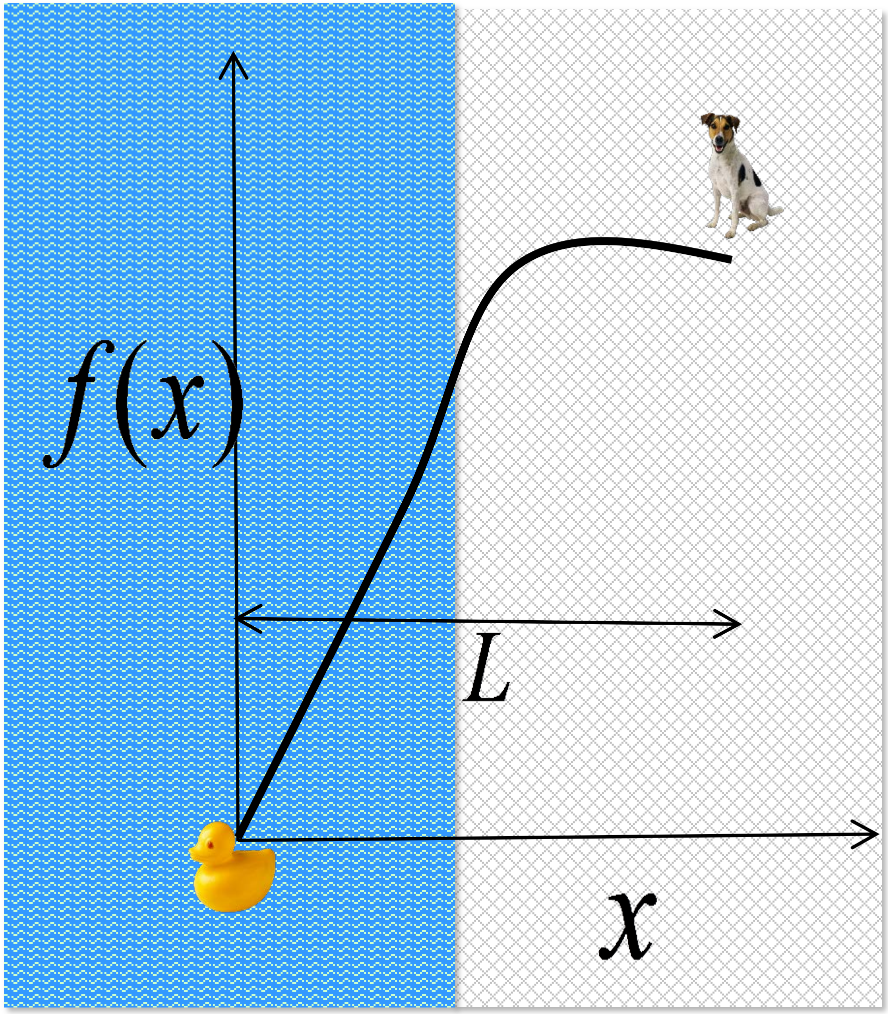
Straight line is not the solution! We need shortest-time path and not the shortest path.

What are the optimization variables here?

The variable here is the continuous function that represents the path of the curve to be taken by the dog.

It is a calculus of variations problem!

Calculus of variations problem



Speed of the dog = $v(x)$

$$v(x) = \frac{ds}{dt} = \frac{\sqrt{dx^2 + df^2}}{dt}$$

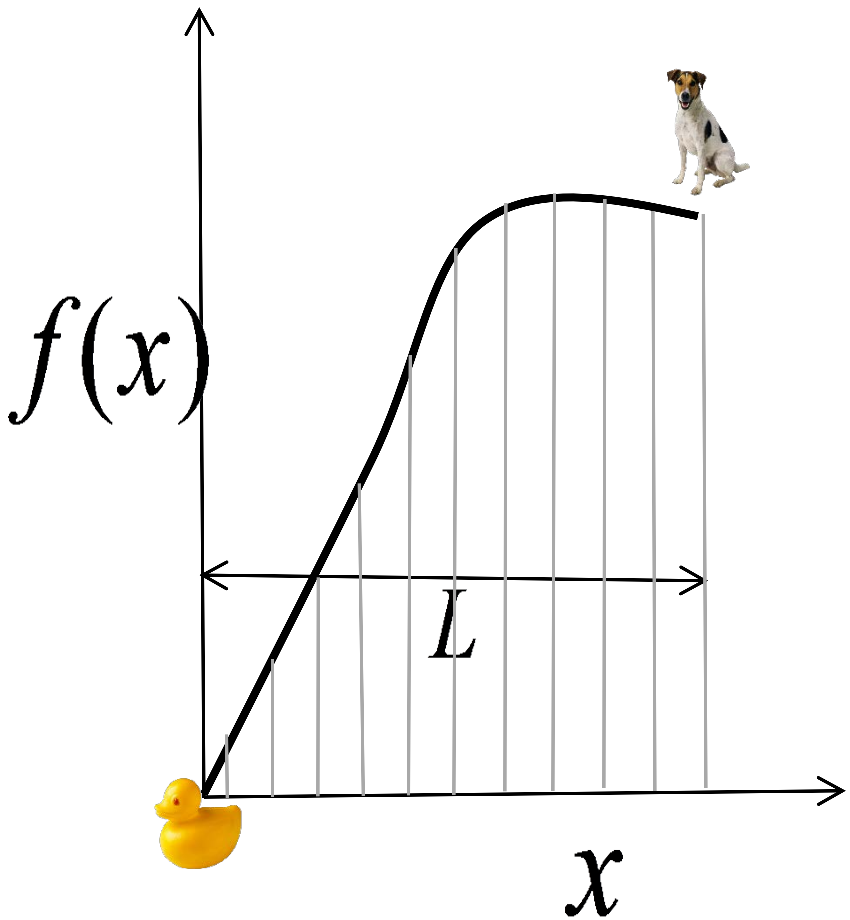
$$\Rightarrow dt = \frac{\sqrt{dx^2 + df^2}}{v(x)}$$

So, $T = \int_0^L \frac{\sqrt{1 + \left(\frac{df}{dx}\right)^2}}{v(x)} dx$

$$\underset{f(x)}{\text{Minimize}}\; T = \int_0^L \frac{\sqrt{1 + \left(\frac{df}{dx}\right)^2}}{v(x)} dx$$

The variable is the function, $f(x)$. The objective function depends on the derivative of this function.

Discretization of the "function" variable



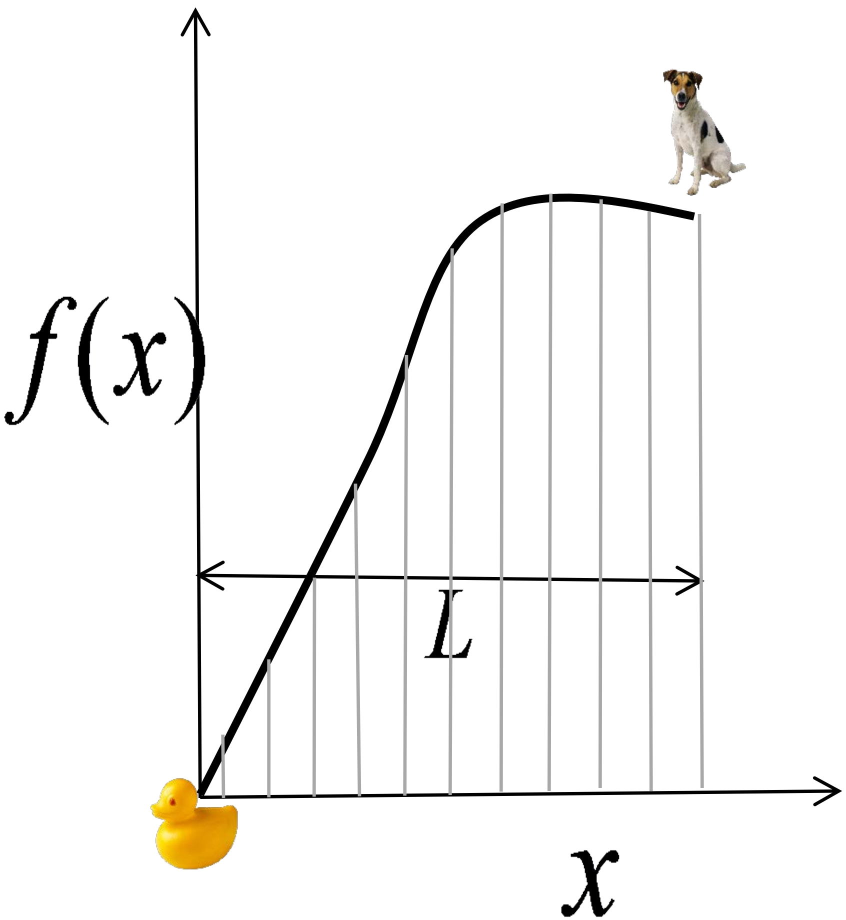
Imagine that the span of length L is discretized into small intervals. Then, $f(x)$ can also be imagined as different heights.
Let us denote the heights with

$$f_1, f_2, f_3, \ldots, f_n$$

It now, becomes a **finite-variable optimization problem**.

But then, we have to take a very, very fine intervals to get the smooth curve, $f(x)$.

"function" variable becoming "finite" variables…



Calculus of variations

$$\underset{f(x)}{\text{Minimize}}\ T = \int_0^L \frac{\sqrt{1+\left(\dfrac{df}{dx}\right)^2}}{v(x)}\,dx$$

Finite-variable optimization

$$\underset{f_1,\, f_2,\, f_3,\, \ldots,\, f_n}{\text{Minimize}}\ T(f_1, f_2, f_3, \ldots, f_n)$$

Finite-variable optimization vs. calculus of variations

Variables are finite in number. Each variable may be continuous (i.e., a real number) or discrete (as in binary, integer, etc.).

Objective function and constraints will be functions of the finite number of variables.

Variable is an unknown function.

There can be many such functions that are unknown. That is, finite number of functions can be variables.

Objective function and constraints will be **functions of functions.**

Such a thing is called a ***functional***. More later.

We need to know the nature of "function" variables and the "functionals".
We will review the basic notion of function spaces later.
They form the basis for calculus of variations.

Calculus of variations is analytical… not computational

At the outset, it is useful to note that calculus of variations is analytical in the sense that everything will be in symbols and not numbers. Hence, it is not computational.

Calculus of variations, i.e., optimization with functions as variables, gives us differential equations to solve for those unknown functions.

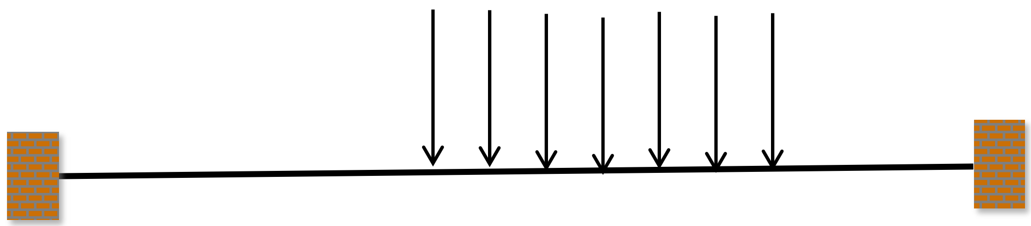
Calculus of variations also gives us boundary conditions along with the differential equations.

It does not tell us how to compute a solution. It just gives equations using which we can compute the unknown function using other methods.

Many problems in geometry, physics, chemistry, mathematics, engineering, economics, etc., can be posed as calculus of variations.

Calculus of variations is also crucial for structural optimization.

Role of calculus of variations in structural mechanics

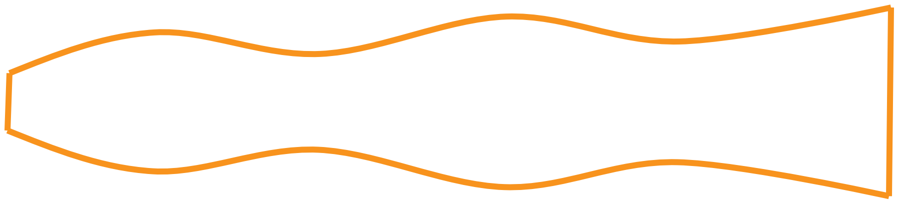


What shape does this beam take under the applied load?

The deformed profile of the beam is a function and that is the variable here.

So, it is a calculus of variations problem.

Role of calculus of variations in structural design



What should be the width profile of the beam for being stiffest under given load for a given volume of material?

The width profile along the length of the beam is a function and that is the variable here.

So, it is a calculus of variations problem.

The end note

Optimization problems

Unconstrained
Constrained

Equalities
Inequalities
Both

Single-objective
Multi-objective

Linear programming

Linear
Nonlinear

Quadratic programming
Nonlinear programming
Convex programming
Geometric programming
Semi-definite programming
More…

Smooth
Non-smooth

Finite variables
"function" variables

Continuous
Discrete

Binary
Integer
Discrete real variable set

Global
Local

Calculus of variations

We are here.

Deterministic
Stochastic

Thanks