Lecture 11
Euler-Lagrange Equations
and their Extension to
Multiple Functions and Multiple Derivatives
in the integrand of the Functional

ME 256, Indian Institute of Science

Variational Methods and Structural Optimization

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Outline of the lecture

Euler-Lagrange equations

Boundary conditions

Multiple functions

Multiple derivatives

What we will learn:

First variation + integration by parts + fundamental lemma = Euler-Lagrange equations

How to derive boundary conditions (essential and natural)

How to deal with multiple functions and multiple derivatives

Generality of Euler-Lagrange equations

The simplest functional, *F(y,y')*

$$\underset{y(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\left(y(x), y'(x)\right) dx$$

$$\delta_y J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right\} dx = 0$$ First variation of *J* w.r.t. *y(x)*.

The condition given above should hold good for any variation of y(x), i.e., for any δy

But there is $\delta y'$, which we will get rid of it through integration by parts.

Integration by parts…

$$\delta_y J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right\} dx = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y \right\} dx + \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y'}\delta y' \right\} dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y \right\} dx + \frac{\partial F}{\partial y'}\delta y \Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left\{ \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right)\delta y \right\} dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right\} \delta y \, dx + \frac{\partial F}{\partial y'}\delta y \Bigg|_{x_1}^{x_2} = 0$$

We can invoke fundamental lemma of calculus of variations now.

Fundamental lemma…

$$\int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right\} \delta y \, dx + \left. \frac{\partial F}{\partial y'} \delta y \right|_{x_1}^{x_2} = 0$$

$$\int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right\} \delta y \, dx = 0 \quad \text{and} \quad \left. \frac{\partial F}{\partial y'} \delta y \right|_{x_1}^{x_2} = 0$$

The two terms are equated to zero because the first term depends on the entire function whereas the second term only on the value of the function at the ends.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) = 0 \quad x \in (x_1, x_2)$$

The integral should be zero for any value of δy. So, by fundamental lemma (Lecture 10), the integrand should be zero at every point in the domain.

Boundary conditions

$$\left.\frac{\partial F}{\partial y'}\delta y\right|_{x_1}^{x_2} = 0$$

$$\Rightarrow \left.\frac{\partial F}{\partial y'}\delta y\right|_{x_2} - \left.\frac{\partial F}{\partial y'}\delta y\right|_{x_1} = 0$$

$$\frac{\partial F}{\partial y'}\delta y = 0$$

$$\Rightarrow \frac{\partial F}{\partial y'} = 0 \text{ or } \delta y = 0$$

The algebraic sum of the two terms may be zero without the two terms being equal to zero individually. We will see those cases later. For now, we will take the general case of both terms individually being equal to zero.

Thus,

$$\frac{\partial F}{\partial y'} = 0 \text{ or } \delta y = 0 \text{ at } x = x_1$$

and

$$\frac{\partial F}{\partial y'} = 0 \text{ or } \delta y = 0 \text{ at } x = x_2$$

Euler-Lagrange (EL) equation with boundary conditions

$$\underset{y(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\Big(y(x), y'(x)\Big)\, dx$$

Problem statement

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad x \in (x_1, x_2)$$

Differential equation

and

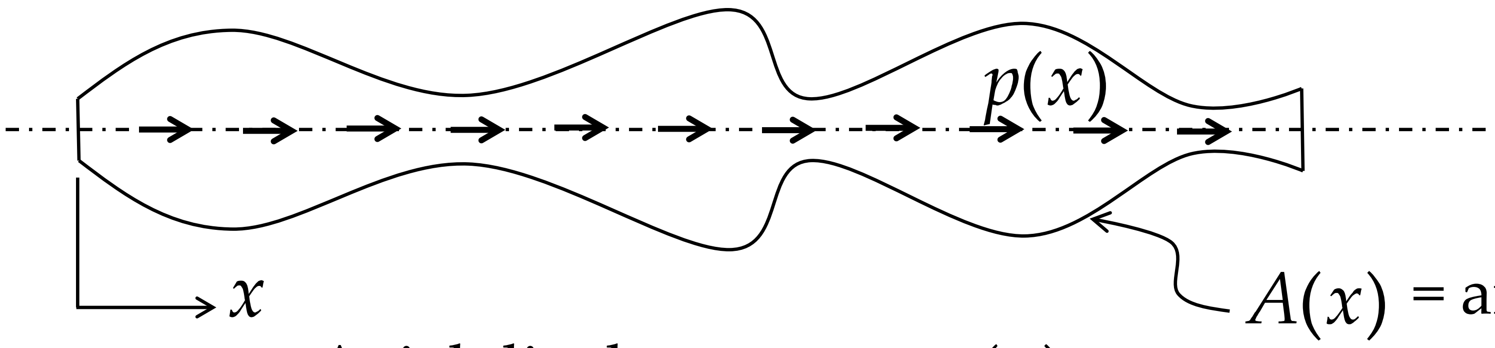
$$\frac{\partial F}{\partial y'} = 0 \;\; \text{or} \;\; \delta y = 0 \;\; \text{at } x = x_1$$

and

$$\frac{\partial F}{\partial y'} = 0 \;\; \text{or} \;\; \delta y = 0 \;\; \text{at } x = x_2$$

Boundary conditions

Example 1: a bar under axial load



Strain energy — Work potential

$$\underset{u(x)}{\text{Min}}\ PE = \int_0^L \left(\frac{1}{2} E(x) A(x) \left(u'(x) \right)^2 - p(x)\, u(x) \right) dx$$

Data: $L, E(x), A(x), p(x)$

Principle of minimum potential energy (PE)

Among all possible axial displacement functions, the one that minimizes PE is the stable static equilibrium solution.

Bar problem: E-L equation

$$\underset{u(x)}{\text{Min}} \;\; PE = \int_0^L \left(\frac{1}{2} E(x) A(x) \left(u'(x) \right)^2 - p(x)\, u(x) \right) dx$$

$$F = \frac{1}{2} E(x) A(x) \left(u'(x) \right)^2 - p(x)\, u(x)$$ Integrand of the PE

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) = 0 \quad x \in (0, L)$$

$$\Rightarrow -p - \frac{d}{dx}\left(EAu' \right)$$

$$\Rightarrow \left(EAu' \right)' + p = 0$$ Governing differential equation

Bar problem: boundary conditions

$$\frac{\partial F}{\partial y'} = 0 \text{ or } \delta y = 0 \text{ at } x = x_1$$

and

$$\frac{\partial F}{\partial y'} = 0 \text{ or } \delta y = 0 \text{ at } x = x_2$$

$$F = \frac{1}{2} E(x) A(x) \left(u'(x) \right)^2 - p(x)\, u(x)$$

$\delta u = 0$

This means that y is specified; hence, its variation is zero. This is called the **essential** or **Dirichlet** boundary condition.

$$EAu' = 0 \text{ or } \delta u = 0 \text{ at } x = 0$$

and

$$EAu' \text{ or } \delta u = 0 \text{ at } x = L$$

$EAu' = 0$

This means that the stress is zero when the displacement is not specified. It is called the **natural** or **Neumann** boundary condition.

Weak form of the governing equation

$$\underset{u(x)}{\text{Min}} \;\; PE = \int_0^L \left(\frac{1}{2} E(x) A(x) \left(u'(x) \right)^2 - p(x)\, u(x) \right) dx$$

$$\delta_y J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right\} dx = 0$$

First variation is zero.

$$\delta_u PE = \int_0^L \left(E(x) A(x) u'(x) \delta u' - p(x)\, \delta u \right) dx = 0 \;\; \text{for any } \delta u$$

$$\int_0^L \left(E(x) A(x) u'(x) \delta u' \right) dx = \int_0^L \left(p(x)\, \delta u \right) dx$$

Internal virtual work = external virtual work

δu

Variation of u is like virtual displacement.

Three ways for static equilibrium

$$\underset{u(x)}{\text{Min}}\ PE = \int_0^L \left(\frac{1}{2} E(x) A(x) \left(u'(x) \right)^2 - p(x)\, u(x) \right) dx$$

Minimum potential energy principle

$$\delta_y PE = \int_0^L \left(E(x) A(x) \delta u' - p(x)\, \delta u \right) dx = 0 \ \text{ for any } \delta u$$

Principle of virtual work;
The weak form

$$\left(EAu' \right)' + p = 0$$

$$EAu' = 0 \ \text{ or } \delta u = 0 \ \text{ at } x = 0$$

and

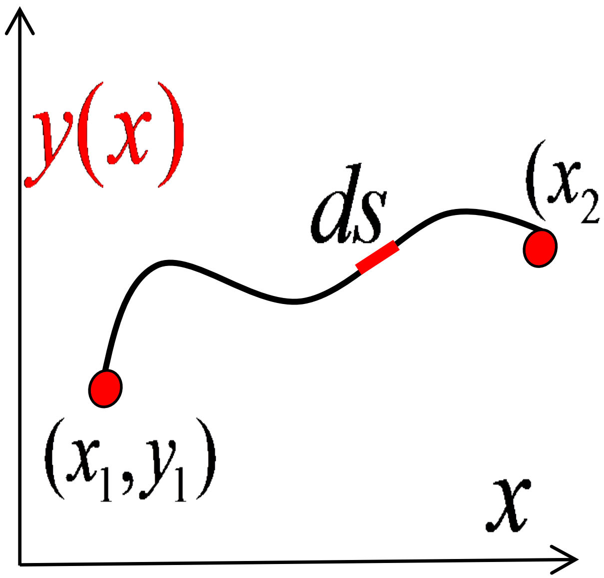
$$EAu' \ \text{ or } \delta u = 0 \ \text{ at } x = L$$

Force balance;
And boundary conditions.
The strong form.

Q: What is "weak" about the weak form?
A: It needs derivative of one less order.

Example 2: is a straight line really the least-distance curve in a plane?

From Slide 7 in Lecture 3



$$\underset{y(x)}{\text{Min}} \quad L = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

$$\text{Data}: x_1,\, x_2,\, y(x_1) = y_1,\; y(x_2) = y_2$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad x \in (x_1, x_2)$$

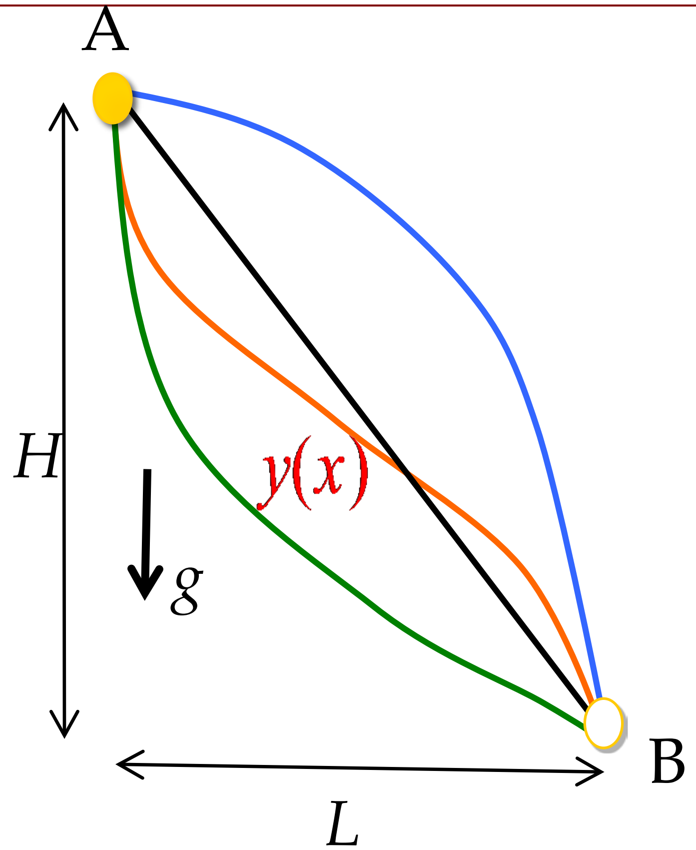
$$\Rightarrow 0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\frac{y'}{\sqrt{1+y'^2}} = C \quad \Rightarrow \quad y' = \text{constant}$$

So, straight line in indeed the geodesic in a plane.

Example 3: Brachistochrone problem

From Slide 11 in Lecture 2



Minimize $\quad T = \int_0^L \frac{\sqrt{1+(y')^2}}{\sqrt{2g(H-y)}}dx$

$y(x)$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\sqrt{\frac{1+y'^2}{8g}}\frac{1}{(H-y)^{3/2}} - \left(\frac{y'}{\sqrt{2g(1+y'^2)(H-y)}}\right)' = 0$$

And we have Dirichlet (essential) boundary conditions at both the ends.

A functional with two derivatives: *F(y,y',y'')*

$$\underset{y(x)}{\text{Min}} \;\; J = \int_{x_1}^{x_2} F\Big(y(x), y'(x), y''(x)\Big)\,dx$$

$$\delta_y J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' \right\} dx = 0$$

First variation of J w.r.t. $y(x)$.

We now need to integrate by parts twice to get rid of the second derivative of y.

Integration by parts… twice!

$$\delta_y J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' \right\} dx = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y \right\} dx + \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y'}\delta y' \right\} dx + \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y''}\delta y'' \right\} dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y \right\} dx + \frac{\partial F}{\partial y'}\delta y \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left\{ \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y \right\} dx + \frac{\partial F}{\partial y''}\delta y' \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left\{ \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y' \right\} dx = 0$$

$$\Rightarrow \underbrace{\int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) \right\} \delta y \, dx}_{} + \underbrace{\left(\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) \right) \delta y \Big|_{x_1}^{x_2} + \frac{\partial F}{\partial y''}\delta y' \Big|_{x_1}^{x_2}}_{} = 0$$

= 0 gives differential equation by using the fundamental lemma.

Two sets of boundary conditions

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \text{ for } x \in (x_1, x_2)$$

E-L equation and BCs for *F(y,y',y'')*

$$\underset{y(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\left(y(x), y'(x), y''(x)\right) dx$$

Things are getting lengthy;
Let us use short-hand notation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \ \text{ for } x \in (x_1, x_2)$$

$$\frac{\partial F}{\partial y} = F_y; \ \frac{\partial F}{\partial y'} = F_{y'}; \ \frac{\partial F}{\partial y''} = F_{y''}$$

$$\left(\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right)\delta y\bigg|_{x_1}^{x_2} = 0$$

and

$$\frac{\partial F}{\partial y''}\delta y'\bigg|_{x_1}^{x_2} = 0$$

$$F_y - (F_{y'})' + (F_{y''})'' = 0$$

$$\left(F_{y'} - \left(F_{y''}\right)'\right)\delta y\bigg|_{x_1}^{x_2} = 0$$

and

$$F_{y''}\delta y'\bigg|_{x_1}^{x_2} = 0$$

Example 4: beam deformation

From Slide 27 in Lecture 3

$$\underset{w(x)}{\text{Min}} \;\; PE = \int_0^L \left\{ \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw \right\} dx$$

$$\text{Data} : q(x), E, I$$

$$F = \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw$$

$$w(x)$$

$$F_y - (F_{y'})' + (F_{y''})'' = 0$$

$$-q - 0 + \left(EIw'' \right)'' = 0$$

$$\Rightarrow \left(EIw'' \right)'' = q$$

When E and I are uniform, we get the familiar: $EIw^{iv} = q$

Boundary conditions for the beam

$$F = \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw$$

$w(x)$

$$\left(F_{y'} - \left(F_{y''} \right)' \right) \delta y \Big|_{x_1}^{x_2} = 0$$

and

$$F_{y''} \delta y' \Big|_{x_1}^{x_2} = 0$$

Physical interpretation

$$\left(EIw'' \right)' \delta w \Big|_0^L = 0$$

Either shear stress is zero or the transverse displacement is specified.

$$\left(EIw'' \right) \delta w' \Big|_0^L = 0$$

Either bending moment is zero or the slope is specified.

Do we see a trend for multiple derivatives in the functional?

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F\big(y(x), y'(x)\big)\,dx$$

$$F_y - \left(F_{y'}\right)' = 0$$

$$\left(F_{y'}\right)\delta y\Big|_{x_1}^{x_2} = 0$$

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F\big(y(x), y'(x), y''(x)\big)\,dx$$

$$F_y - (F_{y'})' + (F_{y''})'' = 0$$

$$\left(F_{y'} - \left(F_{y''}\right)'\right)\delta y\Big|_{x_1}^{x_2} = 0$$

and

$$F_{y''}\delta y'\Big|_{x_1}^{x_2} = 0$$

Three derivatives… *F(y,y',y'',y''')*

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F\left(y(x), y'(x), y''(x), y'''(x)\right) dx$$

$$F_y - (F_{y'})' + (F_{y''})'' - (F_{y'''})''' = 0$$

$$\left(F_{y'} - \left(F_{y''}\right)' + \left(F_{y'''}\right)'' \right) \delta y \Big|_{x_1}^{x_2} = 0 \;, \qquad \left(F_{y''} - \left(F_{y'''}\right)' \right) \delta y' \Big|_{x_1}^{x_2} = 0 \quad \text{and}$$

$$F_{y'''} \delta y'' \Big|_{x_1}^{x_2} = 0$$

Many derivatives… $F(y, y', y'', \ldots, y^{(n)})$

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F\left(y(x), y'(x), y''(x), \cdots, y^{(n)}(x)\right) dx$$

$$F_y - (F_{y'})' + (F_{y''})'' - (F_{y'''})''' + \ldots = \sum_{i=0}^{n} \left(-1\right)^i \left(F_{y^{(i)}}\right)^{(i)} = 0$$

$$\left(\sum_{i=j}^{n} \left(-1\right)^{i-j} \left(F_{y^{(i)}}\right)^{(i-1)} \right) \delta y^{(j-1)} \quad \text{for } \; j = 1, 2, \cdots n$$

Most general form with one function and many derivatives

What if we have two functions?

$$\underset{y_1(x),y_2(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\left(y_1(x), y_1'(x), y_2(x), y_2'(x)\right) dx$$

$$\delta_{y_1} J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y_1} \delta y_1 + \frac{\partial F}{\partial y_1'} \delta y_1' \right\} dx = 0$$

$$\delta_{y_2} J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y_2} \delta y_2 + \frac{\partial F}{\partial y_2'} \delta y_2' \right\} dx = 0$$

Now, we need to take the first variation with respect to both the functions, **separately**.

What if we have two functions? (contd.)

$$\underset{y_1(x),y_2(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\Big(y_1(x), y_1'(x), y_2(x), y_2'(x)\Big)\,dx$$

$$F_{y_1} - \left(F_{y_1'}\right)' = 0 \quad \text{and} \quad \left(F_{y_1'}\right)\delta y_1\Big|_{x_1}^{x_2} = 0$$

$$F_{y_2} - \left(F_{y_2'}\right)' = 0 \quad \text{and} \quad \left(F_{y_2'}\right)\delta y_2\Big|_{x_1}^{x_2} = 0$$

And, we will have two differential equations and two sets of boundary conditions.

Two unknown functions need two differential equations and two sets of BCs. That is all!

Most general form: *m* functions with *n* derivatives.

The most general form when we have one independent variable x.

$$\underset{y(x)}{\text{Min}}\ J = \int_{x_1}^{x_2} F\left(y_1, y_1', \cdots, y_1^{(n)}, y_2, y_2', \cdots, y_2^{(n)}, \cdots, y_m, y_m', \cdots, y_m^{(n)}\right) dx$$

$$\left.\begin{array}{l} F_{y_k} - (F_{y_k'})' + (F_{y_k''})'' - (F_{y_k'''})''' + \ldots = \displaystyle\sum_{i=0}^{n} \left(-1\right)^{i} \left(F_{y_k^{(i)}} \right)^{(i)} = 0 \\ \left(\displaystyle\sum_{i=j}^{n} \left(-1\right)^{i-j} \left(F_{y_k^{(i)}} \right)^{(i-1)} \right) \delta y_k^{(j-1)} \quad \text{for} \quad j = 1, 2, \cdots n \end{array}\right\} k = 1, 2, \cdots, m$$

The end note

Euler-Lagrange equations and their extension to multiple functions and multiple derivatives

- Euler-Lagrange equations = first variation + integration by parts + fundamental lemma
- Boundary conditions
  Essential (Dirichlet)
  Natural (Neumann)
- Dealing with multiple derivatives along with boundary conditions (need to do integration by parts as many times as the order of the highest derivative)
- Dealing with multiple functions (rather easy)
- **General form of Euler-Lagrange equations in one independent variable**

Thanks