Lecture 13

Global Constraints
in calculus of Variations

ME 256 at the Indian Institute of Science, Bangalore

Variational Methods and Structural Optimization

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Outline of the lecture

Global and local constraints

Dealing with global constraints

Euler-Lagrange equations with constraints; Lagrange multipliers

Inequality constraints

What we will learn:

How to identify a constraint as global as local

When is Lagrange multiplier a scalar

How to write Euler-Lagrange equations and boundary conditions for a problem with global constraints

Interpreting the Lagrange multipliers and understanding the complementarity conditions

Global vs. local constraints

Global vs. local here pertains to whether a constraint is imposed at each point in the domain or it is imposed on a quantity that pertains to the entire domain.

- Global constraints pertain to the entire domain.
- Local constraints are imposed at every point in the domain, individually.

Mathematically, it tells whether a constraint is a functional or a function.

- Global constraint is a functional
- Local constraint is a function. It can also be a differential equation.

It also has implications when we discretize.

- Upon discretization, a global constraint gives rise to only one constraint.
- A local constraint, on the other hand, gives as many constraints as the number of discretization points.

Examples of global and local constraints

Global constraints

Length of a curve

Area of a surface

Time of travel

Weight of a structure

Deflection at a particular point

Maximum stress

Buckling load

Natural frequency

Local constraints

Upper or lower bound on a curve

Bounds on the deflection of a structure

Bounds on stress

Governing differential equation

Bounds on the mode shape

It is important to understand this difference.

Global constraint: isoperimetric problem

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F\big(y(x), y'(x)\big)\, dx$$

Subject to

$$K = \int_{x_1}^{x_2} G\big(y(x), y'(x)\big)\, dx - K^* = 0$$

This problem statement means that we need to find *y(x)* that minimizes *J* and satisfies the equality constraint, *K*.

It is a global constraint because *K* here depends on the entire domain. It is a functional. It is a single value.

A problem with a global constraint is also called isoperimetric problem. This is because the perimeter constraint is the historic global constraint.

How do we solve this?

$$\underset{y(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\Big(y(x), y'(x)\Big)\, dx$$

Subject to

$$K = \int_{x_1}^{x_2} G\Big(y(x), y'(x)\Big)\, dx - K^* = 0$$

Recall how we handled equality constraints in finite-variable optimization. (Lecture 5)

You may recall from lecture 5 that…
We linearized the constraint and used the first-order term to eliminate a variable and made the problem unconstrained.
We also came up with the concept of Lagrange multiplier.
Here too, we will follow the same idea.

Equivalent of first-order term of a functional

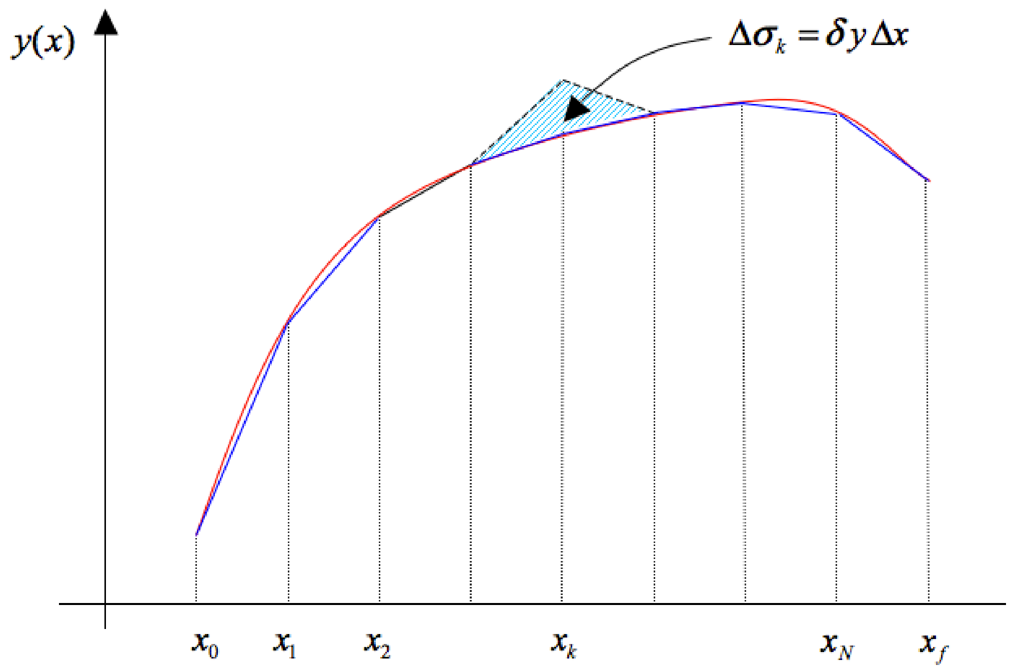
From Eq. (6) in Slide 26 of Lecture 9

$$\Delta J = J(y+h) - J(y) = \left\{ \left. \frac{\delta J}{\delta y} \right|_{x=\hat{x}} + \varepsilon \right\} \Delta\sigma$$

$$\frac{\delta J}{\delta y} = F_y - \frac{d}{dx}\left(F_{y'}\right)$$

Variational derivative, which is the expression in the Euler-Lagrange equation.

From Eq. (1) in Slide 22 of Lecture 9



First-order term of the global constraint

$$K = \int_{x_1}^{x_2} G\Big(y(x), y'(x)\Big)\, dx - K^* = 0$$

$$\Delta K = K(y+h) - K(y) = \left\{ \left.\frac{\delta K}{\delta y}\right|_{x=\hat{x}} + \varepsilon \right\} \Delta\sigma$$

$$\frac{\delta K}{\delta y} = G_y - \frac{d}{dx}\left(G_{y'}\right)$$

The first-order term shows that the constraint has non-zero value whenever we perturb the function at a point. So, it won't satisfy the equality constraint anymore.

So, we will perturb *y(x)* at two points…

Two perturbations of the global constraint

$$\Delta K_a = K(y+h) - K(y) = \left\{ \left. \frac{\delta K}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a \qquad \Delta\sigma_a = \delta y_a \, \Delta x_a$$

$$\Delta K_b = K(y+h) - K(y) = \left\{ \left. \frac{\delta K}{\delta y} \right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b \qquad \Delta\sigma_b = \delta y_b \, \Delta x_b$$

We choose x_a and x_b such that the first-order changes due to the two perturbations cancel each other and we retain the feasibility of the constraint.

$$\Delta K_a + \Delta K_b = 0$$

$$\Rightarrow \left\{ \left. \frac{\delta K}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ \left. \frac{\delta K}{\delta y} \right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b = 0$$

One perturbation of the function in terms of the other

$$\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b = 0$$

$$\Rightarrow \Delta\sigma_b = -\frac{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\}}{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\}} \Delta\sigma_a$$

In order to divide like this, we require that there should be at least one point x where the variational derivative is not zero. This is the equivalent of constraint qualification of finite-variable optimization. See Slide 13 of Lecture 5.

Perturbation of the objective functional at the same two points by the same amounts

$$\Delta J_a = J(y+h) - J(y) = \left\{ \left. \frac{\delta J}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a \qquad \Delta\sigma_a = \delta y_a \, \Delta x_a$$

$$\Delta J_b = J(y+h) - J(y) = \left\{ \left. \frac{\delta J}{\delta y} \right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b \qquad \Delta\sigma_b = \delta y_b \, \Delta x_b$$

$$\Delta J_a + \Delta J_b = \Delta J_{a+b}$$

$$\Rightarrow \left\{ \left. \frac{\delta J}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ \left. \frac{\delta J}{\delta y} \right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b = \Delta J_{a+b}$$

Eliminating one perturbation…

$$\Delta\sigma_b = -\frac{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\}}{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\}} \Delta\sigma_a$$

$$\Delta J_{a+b} = \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b$$

$$\Delta J_{a+b} = \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a - \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\} \frac{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\}}{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\}} \Delta\sigma_a$$

Defining a multiplier…

$$\Delta J_{a+b} = \left[\left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \underbrace{- \frac{\left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\}}{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\}}}_{\Lambda} \left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \right] \Delta\sigma_a$$

$$\Delta J_{a+b} = \left[\left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} + \Lambda \left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \right] \Delta\sigma_a$$

First order change in the objective functional

$$\Delta J_{a+b} = \left[\left\{ \left. \frac{\delta J}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\} + \Lambda \left\{ \left. \frac{\delta K}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\} \right] \Delta \sigma_a$$

$$\Rightarrow \Delta J_{a+b} = \left[\left. \frac{\delta J}{\delta y} \right|_{x=x_a} + \Lambda \left. \frac{\delta K}{\delta y} \right|_{x=x_a} + \varepsilon \right] \Delta \sigma_a = 0$$

This is zero because now it is the first-order term due to one arbitrary feasible perturbation because the other one is eliminated.

$$\left. \frac{\delta J}{\delta y} \right|_{x=x_a} + \Lambda \left. \frac{\delta K}{\delta y} \right|_{x=x_a} = 0$$

because $\Delta \sigma_a \neq 0$

and $\varepsilon \Delta \sigma_a = 0$ (the second order term)

Putting things together…

From Slide 13…

$$-\frac{\left\{\left.\frac{\delta J}{\delta y}\right|_{x=x_b}+\varepsilon_b\right\}}{\left\{\left.\frac{\delta K}{\delta y}\right|_{x=x_b}+\varepsilon_b\right\}}=\Lambda \Rightarrow \left\{\left.\frac{\delta J}{\delta y}\right|_{x=x_b}+\varepsilon_b\right\}+\Lambda\left\{\left.\frac{\delta K}{\delta y}\right|_{x=x_b}+\varepsilon_b\right\}=0$$

$$\Rightarrow \left.\frac{\delta J}{\delta y}\right|_{x=x_b}+\Lambda\left.\frac{\delta K}{\delta y}\right|_{x=x_b}=0$$

From Slide 14…

$$\left.\frac{\delta J}{\delta y}\right|_{x=x_a}+\Lambda\left.\frac{\delta K}{\delta y}\right|_{x=x_b}=0$$

Since x_a and x_b are arbitrary, the following should be true for any x. And Λ must be a constant.

$$\boxed{\frac{\delta J}{\delta y}+\Lambda\frac{\delta K}{\delta y}=0}$$

Lagrangian can now be defined.

$$\underset{y(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\big(y(x), y'(x)\big)\,dx$$

Subject to

$$K = \int_{x_1}^{x_2} G\big(y(x), y'(x)\big)\,dx - K^* = 0$$

$$\boxed{\frac{\delta J}{\delta y} + \Lambda \frac{\delta K}{\delta y} = 0}$$

Necessary coniditon

$$\boxed{\int_{x_1}^{x_2} G\big(y(x), y'(x)\big)\,dx - K^* = 0}$$

Feasibility coniditon

$$\underset{y(x)}{\text{Min}} \quad L = \left\{ \int_{x_1}^{x_2} F\big(y(x), y'(x)\big)\,dx \right\} + \Lambda \left\{ \int_{x_1}^{x_2} G\big(y(x), y'(x)\big)\,dx \right\}$$

Necessary conditions

Equations

$$\frac{\delta J}{\delta y} + \Lambda \frac{\delta K}{\delta y} = 0$$

$$\int_{x_1}^{x_2} G\Big(y(x), y'(x)\Big) dx - K^* = 0$$

Differential equation

Scalar equation

$y(x)$

Function

Λ

Scalar variable

Unknowns

What if we have an inequality constraint?

$$\underset{y(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\big(y(x), y'(x)\big)\, dx$$

Subject to

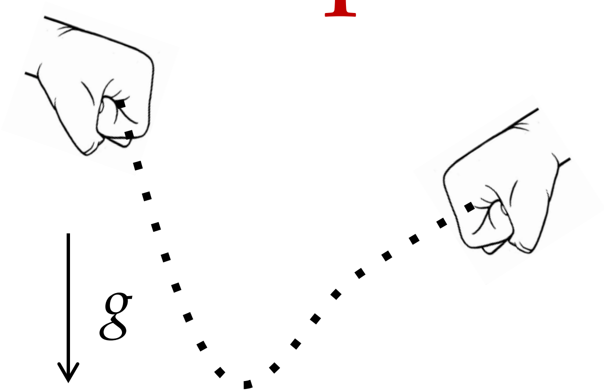
$$K = \int_{x_1}^{x_2} G\big(y(x), y'(x)\big)\, dx - K^* \leq 0$$

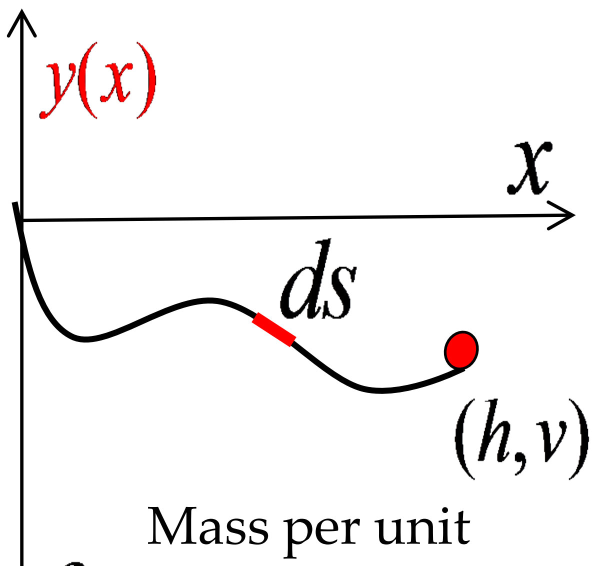
$$\Lambda \left(\int_{x_1}^{x_2} G\big(y(x), y'(x)\big)\, dx - K^* \right) = 0$$

$$\Lambda \geq 0$$

We introduce complementarity condition and require non-negativity of the Lagrange multiplier…
just as we did in finite-variable optimization; see Slide 23 in Lecture 5. The same argument applies here too.

Example 1: hanging chain problem





$\rho =$ Mass per unit length of the chain

$$\underset{y(x)}{\text{Min}} \quad PE = \int_0^h (\rho g y)\, ds = \int_0^h \rho g y \sqrt{1+y'^2}\, dx$$

Subject to

$$\int_0^h \left(\sqrt{1+y'^2} \right) dx - L = 0$$

$$\text{Data}: L, y(0)=0, h, y(h)=v, \rho, g$$

$$\underset{y(x)}{\text{Min}} \quad L = \int_0^h \rho g y \sqrt{1+y'^2}\, dx + \Lambda \left(\int_0^h \left(\sqrt{1+y'^2} \right) dx - L \right)$$

$$\text{Data}: L, y(0)=0, h, y(h)=v, \rho, g$$

Necessary conditions for the hanging chain problem

$$\underset{y(x)}{\text{Min}} \quad L = \int_0^h \rho g y \sqrt{1+y'^2}\, dx + \Lambda \left(\int_0^h \left(\sqrt{1+y'^2} \right) dx - L \right)$$

$$\text{Data} : L, y(0) = 0, h, y(h) = v, \rho, g$$

$$\delta_y L = 0$$

$$\int_0^h \left(\sqrt{1+y'^2} \right) dx - L = 0$$

$$\delta_y L = \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) = 0$$

Differential equation

Example 2: Stiffest beam of given volume

$$\underset{b(x)}{\text{Min}} \quad SE = \int_0^L \left\{ \frac{1}{2} \frac{Ebd^3}{12} \left(\frac{d^2 w}{dx^2} \right)^2 \right\} dx$$

Subject to

$$\frac{d^2}{dx^2}\left(Ebd^3 \frac{d^2 w}{dx^2} \right) + q = 0$$

This is a local constraint; we discuss this in Lecture 14

$$\int_0^L bd \, dx - V^* \le 0$$

We now know how to deal with this global constraint

$$\text{Data}: L, q(x), d, V^*, E$$

The end note

Global constraints in calculus of variations

Distinguishing between global and local constraints

First-order perturbation of a functional using the concept of
Variational derivative
Two perturbations to cancel the effects of each other to retain feasibility of
The equality constraint.

Concept of Lagrange multiplier and Lagrangian

Necessary conditions
Extension to inequality constraints

Necessary constraints for global constraints in calculus of variations

Thanks