Lecture 16

Integrals and Invariants of Euler-Lagrange Equations

ME 256 at the Indian Institute of Science, Bengaluru

Variational Methods and Structural Optimization

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Outline of the lecture

First integrals of Euler-Lagrange equations

Noether's integral

Parametric form of E-L equations

Invariance of E-L equations

What we will learn:

How to simplify the E-L equations to easy-to-solve differential equations in some cases

How to take advantage of parametric forms and change of variables

More than formulating equations…

So far, we have learnt how to get differential equations and boundary conditions using the techniques of calculus of variations.

- Indeed it is powerful.
- We have learnt various generalizations:
  - Multiple derivatives
  - Multiple functions
  - Two and three independent variables
  - Equality and inequality constraiints
  - Variable end conditions
  - Broken extremals and corner conditions

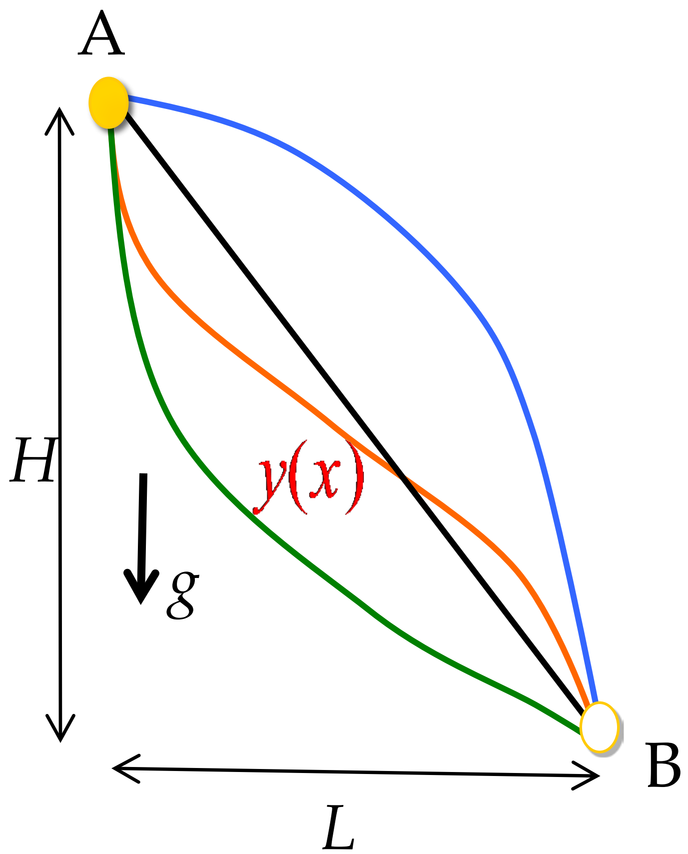
There are a few concepts that become useful when we also want to solve them using analytical (rather than numerical) techniques.

We will still not get a solution right away but we get a simpler or easily solvable form of differential equations.

In some cases, we get some insight into the problem. This is the aim of the content of this lecture.

Consider the brachistochrone problem

From Slide 14 in Lecture 11



Minimize $T = \int_0^L \frac{\sqrt{1+(y')^2}}{\sqrt{2g(H-y)}} dx$
$y(x)$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\sqrt{\frac{1+y'^2}{8g}} \frac{1}{(H-y)^{3/2}} - \left(\frac{y'}{\sqrt{2g(1+y'^2)(H-y)}}\right)' = 0$$

And we have Dirichlet (essential) boundary conditions at both the ends.

Looks formidable to solve…

At first sight, this differential equation looks to be too complicated to solve analytically…

$$\sqrt{\frac{1+y'^2}{8g}}\frac{1}{(H-y)^{3/2}}-\left(\frac{y'}{\sqrt{2g(1+y'^2)(H-y)}}\right)'=0$$

And we are far from showing that the solution of this is a cycloid.

First integral of Euler-Lagrange equations provides a way out of this.

First integrals of special forms

Solving differential equations means that we are integrating them.

- This is what we do whether we do it analytically or numerically.

So, first integrals imply that we are integrating the differential equation to some extent.

For Euler-Lagrange equations, some special forms, are amenable for writing the first integrals and thereby reduce their degree and hence their complexity.

$$J = \int_{x_1}^{x_2} F(x, y)\, dx \qquad J = \int_{x_1}^{x_2} F(x, y')\, dx \qquad J = \int_{x_1}^{x_2} F(y, y')\, dx$$

$$J = \int_{x_1}^{x_2} F(x, y, y')\, dx = \int_{x_1}^{x_2} f(x, y)\sqrt{1 + y'^2}\, dx$$

Integrand of the form $F(x,y)$

$$\underset{y(x)}{\text{Min}}\ J=\int_{x_1}^{x_2} F(x,y)\,dx$$

$$\frac{\partial F}{\partial y}=0$$ Euler-Lagrange equation has only one term, in this case.

$$\Rightarrow f(x,y)=0$$ It is simply an algebraic equation; not a differential equation. So, there is nothing to integrate here.

Notice also that it does not have a boundary condition too.

Recall that the simplest boundary condition term involves y'.

Integrand of the form $F(x, y')$

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F(x, y')\,dx$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Euler-Lagrange equation has only one term, in this case too.

$$\Rightarrow \frac{\partial F}{\partial y'} = C = \text{constant}$$

$$y' = f(x, C)$$

We can express y' in this form and now it can be directly integrated either analytically (when it is possible to do) or numerically.

See Slide 13 in Lecture 11 for an example.

Integrand of the form $F(y, y')$

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F(y, y')\, dx$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$ Euler-Lagrange equation has two terms.

$$\Rightarrow \frac{\partial F}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right) y' - \frac{\partial}{dy'}\left(\frac{\partial F}{\partial y'}\right) y'' = 0$$ Expanded.

$$\Rightarrow \frac{\partial F}{\partial y} y' - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right) y'^2 - \frac{\partial}{dy'}\left(\frac{\partial F}{\partial y'}\right) y'' y' = 0$$ Multiply by *y'* through out.

$$\Rightarrow \frac{d}{dx}\left(F - y' F_{y'}\right) = 0$$ A simple contraction of the terms.

$$\Rightarrow F - y' F_{y'} = C = \text{constant}$$ An elegant first integral.

Brachistochrone problem has the form $F(y, y')$

Minimize $T = \int_0^L \frac{\sqrt{1+(y')^2}}{\sqrt{2g(H-y)}} dx$

$y(x)$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\sqrt{\frac{1+y'^2}{8g}} \frac{1}{(H-y)^{3/2}} - \left(\frac{y'}{\sqrt{2g(1+y'^2)(H-y)}}\right)' = 0$$

Now, instead of that, we get this.

$$F - y'F_{y'} = C = \text{constant}$$

$$\sqrt{\frac{1+y'^2}{2g(H-y)}} - y'\left(\frac{y'}{\sqrt{2g(1+y'^2)(H-y)}}\right) = C$$

Simplification of the Brachistochrone differential equation

$$\sqrt{\frac{1+y'^2}{2g(H-y)}} - y'\left(\frac{y'}{\sqrt{2g(1+y'^2)(H-y)}}\right) = C$$

$$\Rightarrow 1 + y'^2 - y'^2 = C\sqrt{2g(1+y'^2)(H-y)}$$

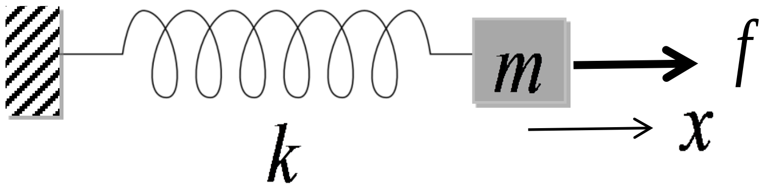
$$\Rightarrow (1+y'^2)(H-y) = \frac{1}{2gC^2} = c = \text{some other constant}$$

A much simpler form to solve.

An insight with the first integral

Consider the action integral for the dynamics of a spring-mass system:

Side 33 in Lecture 3



$$\underset{x(t)}{\text{Min}} \; J = \int_{t_1}^{t_2} F(x, \dot{x})\, dt$$

$$\underset{x(t)}{\text{Opt}} \quad A = \int_0^T \left\{ \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - \frac{1}{2} k x^2 + f x \right\} dt$$

This is of the form and hence is amenable for the elegant first integral.

$$\underset{x(t)}{\text{Opt}} \quad A = \int_0^T \{ KE - PE \}\, dt$$

$$\underset{x(t)}{\text{Opt}} \quad A = \int_0^T L\, dt$$

Hamilton's principle for dynamics.

An insight with the first integral: conservation of energy

$$\underset{x(t)}{\text{Opt}} \quad A = \int_0^T \left\{ \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - \frac{1}{2} k x^2 + f x \right\} dt$$

$$F - y' F_{y'} = C = \text{constant}$$

$$\Rightarrow F - \dot{x} F_{\dot{x}} = C$$

$$\Rightarrow \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + f x \right) - \dot{x} \left(m \dot{x} \right) = C$$

$$\Rightarrow \left(\frac{1}{2} m \dot{x}^2 \right) + \left(\frac{1}{2} k x^2 - f x \right) = c$$

$$\Rightarrow KE + PE = \text{constant}$$

Thus, the first integral gave rise to the principle of conservation of energy.

An integrand of the form $f(x,y)\sqrt{1+y'^2}$

$$\underset{y(x)}{\text{Min}}\ J=\int_{x_1}^{x_2} f(x,y)\sqrt{1+y'^2}\,dx$$

$$F=f(x,y)\sqrt{1+y'^2}$$

$$F_y-\left(F_{y'}\right)'=0$$

$$\Rightarrow f_y\sqrt{1+y'^2}-\left(\frac{fy'}{\sqrt{1+y'^2}}\right)'=0$$

$$\Rightarrow f_y\sqrt{1+y'^2}-f_x\frac{y'^2}{\sqrt{1+y'^2}}-f_y\frac{y'^2}{\sqrt{1+y'^2}}-f\frac{y''}{\left(1+y'^2\right)^{3/2}}=0$$

$$\Rightarrow f_y(1+y'^2)-f_x y'^2-f_y y'^2-\frac{fy''}{1+y'^2}=0$$

$$\Rightarrow f_y-f_x y'^2-\frac{fy''}{1+y'^2}=0$$

Not integrated, but is a simpler form to deal with.

Now, try to solve this functional

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} \sqrt{y^2 + y'^2}\, dx$$

It is of the form: $F(y, y')$

Therefore, $F - y' F_{y'} = C = \text{constant}$

Thus, $$\sqrt{y^2 + y'^2} - y' \frac{y'}{\sqrt{y^2 + y'^2}} = C$$

$$\Rightarrow y^2 = C\sqrt{y^2 + y'^2}$$

No sight of solution yet!
(despite using the first integral)

Let us try change of variables:

$$x = u\cos v$$
$$y = u\sin v$$

Change of variables

$$\underset{y(x)}{\text{Min}} \ J = \int_{x_1}^{x_2} F(x, y, y')\,dx$$

$$x = x(u, v)$$
$$y = y(u, v)$$

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \begin{Bmatrix} du \\ dv \end{Bmatrix}$$

$$dx = \left(x_u + x_v \frac{dv}{du} \right) du = \left(x_u + x_v v' \right) du$$

$$dy = \left(y_u + y_v \frac{dv}{du} \right) du = \left(y_u + y_v v' \right) du$$

$$\rightarrow \underset{v(u)}{\text{Min}} \ J = \int_{x_1}^{x_2} F(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'})\,(x_u + x_v v')\,du$$

Now, this is a new functional in *u* and *v* where we need to find *v(u)*. What would be the Euler-Lagrange equations for this?

New functional satisfies the old equation!

$$\underset{v(u)}{\text{Min}} \; J = \int_{x_1}^{x_2} F\left(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'}\right)(x_u + x_v v')\,du$$

$$\Rightarrow \underset{v(u)}{\text{Min}} \; J = \int_{u_1}^{u_2} F_1(u, v, v')\,du$$

$$\frac{\partial F_1}{\partial v} - \frac{d}{du}\left(\frac{\partial F_1}{\partial v'}\right) = 0$$

is satisfied by v(u) just as y(x) satisfies

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

So, we need to get the new functional in the form shown above, when we change variables.

An example

With
$$x = \sqrt{u^2 + v^2}$$
$$y = \tan^{-1}(v/u)$$

$$x_u = \frac{u}{\sqrt{u^2 + v^2}}$$

$$x_v = \frac{v}{\sqrt{u^2 + v^2}}$$

$$y_u = \frac{-v}{u^2 + v^2}$$

$$y_v = \frac{u}{u^2 + v^2}$$

$$x_u + x_v v' = \frac{u + vv'}{\sqrt{u^2 + v^2}}$$

$$y_u + y_v v' + = \frac{uv' - v}{u^2 + v^2}$$

And noting that

$$\operatorname*{Min}_{v(u)} J = \int_{x_1}^{x_2} F(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'})\,(x_u + x_v v')\,du$$

$$\operatorname*{Min}_{y(x)} J = \int_{x_1}^{x_2} \sqrt{y^2 + y'^2}\,dx$$

becomes

$$\operatorname*{Min}_{v(u)} J = \int_{u_1}^{u_2} \sqrt{1 + v'^2}\,du$$

Check the algebra by working it out in detail.

Example (contd.)

$$\underset{v(u)}{\text{Min}}\ J = \int_{u_1}^{u_2} \sqrt{1+v'^2}\, du = \int_{u_1}^{u_2} F_1\, du$$

$$\Rightarrow \frac{d}{du}\left(\frac{\partial F_1}{\partial v'}\right) = 0 \Rightarrow \frac{\partial F_1}{\partial v'} = \text{constant} = c$$

$$\Rightarrow v' = C_1 \Rightarrow v = C_1 u + C_2$$

Thus, the solution of the differential equation in slide 15 is

$$y^2 + y'^2 - yy'' = C\sqrt{y^2 + y'^2}$$

Thus,

With $x = \sqrt{u^2 + v^2}$, $y = \tan^{-1}(v/u)$ or $u = x\cos y$, $v = x\sin y$

$$x\sin y = C_1 x\cos y + C_2$$

A note about change of variables

Change of variables is a great way to solve an otherwise difficult problem. But nobody can tell us which change of variables will work for a given problem. You just have to *know* or guess.

But note that calculus of variations lets you use change of variables.

Parametric form and Euler-Lagrange equations

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

Parametric form

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned} \Rightarrow \begin{aligned} dx &= \dot{x}\; dt \\ y' &= \frac{\dot{y}}{\dot{x}} \end{aligned}$$

Then, we have

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F(x, y, y')\, dx = \int_{t_1}^{t_2} F\left(x(t), y(t), \frac{\dot{y}}{\dot{x}} \right) \dot{x}\, dt$$

$$\underset{x(t), y(t)}{\text{Min}} \; J = \int_{t_1}^{t_2} \psi(x, y, \dot{x}, \dot{y})\; dt$$

ψ should not depend on t explicitly.

Where $\psi = F\left(x(t), y(t), \frac{\dot{y}}{\dot{x}} \right) \dot{x}$ and it satisfies the following EL equations.

$$\psi_x - \frac{d}{dt}\left(\psi_{\dot{x}}\right) = 0 \quad \text{and} \quad \psi_y - \frac{d}{dt}\left(\psi_{\dot{y}}\right) = 0$$

A comment

We saw that the change of variables and the parametric form do not alter the form of Euler-Lagrange equations.

- It is very useful in a number of situations.
- Parametric form is especially useful when y(x) is to denote a closed curve.
- It is also useful in dealing with dynamics problems too.

There is a more general theorem related to invariance of Euler-Lagrange theorem. It is called Noether's theorem.

- Noether's theorem is related to the first integrals we discussed earlier in this lecture.
- It leads to conserved quantities.
- Proved by a German mathematician Emmy Noether, this theorem was praised by Einstein for its penetrating thinking.
- It is used widely in mathematical physics.

Noether's theorem next…

Invariance under transformations

Consider
$$\hat{x} = \phi(x, y, y')$$
$$\hat{y} = \psi(x, y, y')$$

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F\left(x, y, \frac{dy}{dx}\right) dx$$

$$\underset{\hat{y}(x)}{\text{Min}} \; \hat{J} = \int_{\hat{x}_1}^{\hat{x}_2} F\left(\hat{x}, \hat{y}, \frac{d\hat{y}}{d\hat{x}}\right) d\hat{x}$$

If $J = \hat{J}$, we say that the functional is invariant under the transformation shown above.

Noether's theorem

Consider
$$\hat{x} = \phi(x, y, y', \alpha)$$
$$\hat{y} = \psi(x, y, y', \alpha)$$

A one-parameter transformation.

If
$$J = \int_{x_1}^{x_2} F\left(x, y, \frac{dy}{dx}\right) dx = \hat{J} = \int_{\hat{x}_1}^{\hat{x}_2} F\left(\hat{x}, \hat{y}, \frac{d\hat{y}}{d\hat{x}}\right) d\hat{x}$$

we say that the functional is invariant under the transformation shown above. Then,

$$\left(F_{y'}\right)\left(\left.\frac{\partial \psi}{\partial \alpha}\right|_{\alpha=0}\right) - \left(F - y'F_{y'}\right)\left(\left.\frac{\partial \phi}{\partial \alpha}\right|_{\alpha=0}\right) = \text{constant}$$

Noether's theorem (case of many functions)

Consider

$$\hat{x} = \phi(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n', \alpha)$$

$$\hat{y}_i = \psi_i(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n', \alpha), \quad \mathrm{i} = 1, 2, \cdots, \mathrm{n}$$

If $J = \int_{x_1}^{x_2} F\left(x, \bar{y}, \frac{d\bar{y}}{dx}\right) dx = \hat{J} = \int_{\hat{x}_1}^{\hat{x}_2} F\left(\hat{x}, \bar{\hat{y}}, \frac{d\bar{\hat{y}}}{d\hat{x}}\right) d\hat{x}$

we say that the functional is invariant under the transformation shown above. Then,

$$\sum_{i=1}^{n} \left\{ \left(F_{y_i'}\right)\left(\left.\frac{\partial \psi_i}{\partial \alpha}\right|_{\alpha=0}\right) - \left(F - y_i' F_{y_i'}\right)\left(\left.\frac{\partial \phi}{\partial \alpha}\right|_{\alpha=0}\right) \right\} = \text{constant}$$

An application of Noether's theorem

Consider a system of n particles with position coordinates:

$x_i(\mathrm{t}), y_i(\mathrm{t}), z_i(t)\ (i=1,2,\cdots,n)$

The kinetic energy of such a system = $KE = \frac{1}{2}\sum_{i=1}^{n} m_i\left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)$

Let the potential energy be $= PE = U(x_1, y_1, z_1, \cdots, x_n, y_n, z_n)$

Consider the action integral = $A = \int_{t_0}^{t_1}\left(KE - PE\right)dt$

Consider

$$x_i^* = x_i\cos\theta + y_i\sin\theta$$
$$y_i^* = -x_i\sin\theta + y_i\cos\theta$$
$$z_i^* = z_i$$

A one-parameter family of transformations for the rotation of the system of particles about the z-axis.

Suppose that A is invariant under the above transformation.

Compare with the generic transformation.

$$\hat{x} = \phi(x, \overline{y}, \overline{y}', \alpha)$$

$$\hat{y}_i = \psi_i(x, \overline{y}, \overline{y}', \alpha)$$

$t = t \Rightarrow \phi$ No transformation in the independent variable.

$$t^* = \phi(\mathrm{t}, \overline{\mathrm{x}}, \overline{\mathrm{y}}, \overline{\mathrm{z}}, \dot{\overline{\mathrm{x}}}, \dot{\overline{\mathrm{y}}}, \dot{\overline{\mathrm{z}}}_i, \theta)$$

$$x^*_i = \psi_1(\mathrm{t}, \overline{\mathrm{x}}, \overline{\mathrm{y}}, \overline{\mathrm{z}}, \dot{\overline{\mathrm{x}}}, \dot{\overline{\mathrm{y}}}, \dot{\overline{\mathrm{z}}}, \theta)$$

$$y^*_i = \psi_2(\mathrm{t}, \overline{\mathrm{x}}, \overline{\mathrm{y}}, \overline{\mathrm{z}}, \dot{\overline{\mathrm{x}}}, \dot{\overline{\mathrm{y}}}, \dot{\overline{\mathrm{z}}}, \theta)$$

$$z^*_i = \psi_3(\mathrm{t}, \overline{\mathrm{x}}, \overline{\mathrm{y}}, \overline{\mathrm{z}}, \dot{\overline{\mathrm{x}}}, \dot{\overline{\mathrm{y}}}, \dot{\overline{\mathrm{z}}}, \theta)$$

$$\left.\begin{aligned} x_j^* &= x_j \cos\theta + y_j \sin\theta \\ y_j^* &= -x_j \sin\theta + y_j \cos\theta \\ z_j^* &= z_j \qquad j = 1, 2, \cdots N \end{aligned}\right\} \Rightarrow \psi_{i=1,2,\cdots 3N}$$

Now, as per Noether's theorem, we have

$$\sum_{i=1}^{n} \left\{ \left(F_{y_i'}\right) \left(\left. \frac{\partial \psi_i}{\partial \alpha} \right|_{\alpha=0} \right) - \left(F - y_i' F_{y_i'}\right) \left(\left. \frac{\partial \phi}{\partial \alpha} \right|_{\alpha=0} \right) \right\} = \text{constant}$$

$$\sum_{i=1}^{N} \left\{ \left(\frac{\partial KE}{\partial \dot{x}_i} \right) \left(\left. \frac{\partial \psi_1}{\partial \theta} \right|_{\theta=0} \right) + \left(\frac{\partial KE}{\partial \dot{y}_i} \right) \left(\left. \frac{\partial \psi_2}{\partial \theta} \right|_{\theta=0} \right) + \left(\frac{\partial KE}{\partial \dot{z}_i} \right) \left(\left. \frac{\partial \psi_3}{\partial \theta} \right|_{\theta=0} \right) \right\} = \text{constant}$$

Note that

$$\left(\left. \frac{\partial \phi}{\partial \theta} \right|_{\theta=0} \right) = 0$$

N = # particles

(contd.)

Noether's theorem application (contd.)

Note that

$$\left.\frac{\partial x_i^*}{\partial \theta}\right|_{\theta=0} = y_i \qquad \left.\frac{\partial y_i^*}{\partial \theta}\right|_{\theta=0} = -x_i \qquad \left.\frac{\partial z_i^*}{\partial \theta}\right|_{\theta=0} = 0$$

and

$$\left(\left.\frac{\partial \phi}{\partial \theta}\right|_{\theta=0}\right) = 0$$

(contd.)

$$\sum_{i=1}^{n}\left\{\left(\frac{\partial KE}{\partial \dot{x}_i}\right)\left(\left.\frac{\partial \psi_1}{\partial \theta}\right|_{\theta=0}\right)+\left(\frac{\partial KE}{\partial \dot{y}_i}\right)\left(\left.\frac{\partial \psi_2}{\partial \theta}\right|_{\theta=0}\right)+\left(\frac{\partial KE}{\partial \dot{z}_i}\right)\left(\left.\frac{\partial \psi_3}{\partial \theta}\right|_{\theta=0}\right)\right\} = \text{constant}$$

$$\Rightarrow \sum_{i=1}^{N}\left\{(m\dot{x}_i)(y_i)-(m\dot{y}_i)(x_i)\right\} = \text{constant} \Rightarrow \sum_{i=1}^{N}\left\{\mathbf{r}_i \times \mathbf{p}_i\right\} = \text{constant}$$

Conservation of angular momentum!

where $\mathbf{p}_i = \left(m\dot{x}_i, \mathrm{m}\,\dot{y}_i, \mathrm{m}\,\dot{z}_i\right)$ and $\mathbf{r}_i = \left(x_i, y_i, z_i\right)$

Linear momentum vector

Position vector

Why is Noether's theorem important?

Because it lets us find conserved quantities for any calculus of variations problems leading to first integrals.

It can be extended to multiple functions.

It can be extended to multiple derivatives.

In mechanics, conservation of energy, conservation of linear momentum, and conservation of angular momentum, etc., follow from Noether's theorem.

The previous example illustrated it for the conservation of angular momentum.

The end note

First integrals and invariance of Euler-Lagrange equations

- First integrals for various forms of functionals
- Ways to simplify Euler-Lagrange equations and thereby solve them analytically.
- Change of variables does not alter the form of Euler-Lagrange equations.
- Parametric form too does not alter the form of the El equations.
- **Invariant transformations and conserved quantities using Noether's theorem**

Thanks