Lecture 17

An inverse problem: from given Euler-Lagrange equation(s) to a functional to be optimized

ME 256 at the Indian Institute of Science, Bengaluru

Variational Methods and Structural Optimization

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Outline of the lecture

Simple exercises to go from the differential equation to the functional to be optimized.

A sufficient condition for the existence of a functional: self-adjointness

A method to verify self-adjointness

Two methods to find a functional for dissipative systems: (i) parallel generative system and (ii) multiplicative "generative" function

What we will learn:

How to obtain the functional for a self-adjoint differential operator

How to obtain a functional for some non-self-adjoint differential equations (when one exists)

A simple differential equation

$y'' = 0$ Which functional, when minimized will give this equation?

$$J = \int_{x_1}^{x_2} F\, dx \qquad F = ? \quad \text{such that} \quad \frac{\partial F}{\partial y} - \left(\frac{\partial F}{\partial y'}\right)' + \left(\frac{\partial F}{\partial y''}\right)'' = y'' = 0$$

Solution 1

$$F = \sqrt{1 + y'^2}$$

$$\frac{\partial F}{\partial y} - \left(\frac{\partial F}{\partial y'}\right)' = 0$$

$$0 - \left(\frac{y'}{\sqrt{1 + y'^2}}\right)' = 0$$

$$\Rightarrow y'' = 0$$

Solution 2

$$F = \left(y' y^2 - y'' y\right)$$

$$\frac{\partial F}{\partial y} - \left(\frac{\partial F}{\partial y'}\right)' + \left(\frac{\partial F}{\partial y''}\right)'' = 0$$

$$2yy' - y'' - \left(y^2\right)' - y'' = 0$$

$$\Rightarrow y'' = 0$$

There can be many solutions! Or, none! This is guesswork.

Consider this:

Given $J = \int_{x_1}^{x_2} F_1 \, dx$ and $f(x)$, what $F_2(y, y', f, f')$ can be added to F_1 so that the Euler-Lagrange equation of the new functional remains the same as that of the original functional?

$F_2 = fy' + f'y$ is an answer because…

$$\frac{\partial F_2}{\partial y} - \left(\frac{\partial F_2}{\partial y} \right)' = 0$$

This is also guesswork; not adequate.

$$F_2 = f\, y^{(2n-1)} + f'\, y^{(2n-2)}$$

for $n = 1, 2, 3, \cdots$

in general.

A sufficient condition for the existence of a functional: self-adjointness

If the differential operator of a differential equation is self-adjoint, then there exists a functional, which, when minimized, will lead to the given differential equation as the Euler-Lagrange equation.

What is a differential operator?

An operator that acts on a function to give a differential equation.

$$y'' + ky = 0$$

$$D = (\ )'' + k(\ ) = 0$$

Differential operator

What is self-adjointness?

For two given functions, *y(x)* and *z(x)*, *D* is said to be self-adjoint if…

$$\langle Dy, z \rangle = \langle y, Dz \rangle$$

$$\langle \cdots, \cdots \rangle = \text{ inner product}$$

Is this differential operator self-adjoint?

$$D = (\ )'' + k(\ ) = 0 \quad \text{with } (\ )_{x_1} = (\ )_{x_2} = 0$$

$$\langle Dy, z \rangle = \int_{x_1}^{x_2} \left(y'' + ky \right) z \, dx$$

Integrate by parts to get…

$$\langle Dy, z \rangle = zy' \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left(y'z' - kyz \right) \, dx$$

Integrate by parts again to get…

because of

So, D is self-adjoint.

$$\langle Dy, z \rangle = \cancelto{0}{zy' \Big|_{x_1}^{x_2}} - \cancelto{0}{z'y \Big|_{x_1}^{x_2}} + \int_{x_1}^{x_2} \left(z''y + kyz \right) dx = \langle y, \mathrm{D}\, z \rangle$$

How does self-adjoint operation give us the functional?

$$\langle Dy, z\rangle = \int_{x_1}^{x_2} \left(y'' + ky\right) z\, dx$$

Integrate by parts to get…

$$\langle Dy, z\rangle = \cancel{zy'\Big|_{x_1}^{x_2}}^{\,0} - \int_{x_1}^{x_2} \left(y'z' - kyz\right)\, dx$$

z is replaced by δy

Since $D = (\ \ )'' + k(\ \ ) = 0$

$$\int_{x_1}^{x_2} \left(y'\delta y' - ky\delta y\right)\, dx = 0$$

$$\Rightarrow J = \int_{x_1}^{x_2} \left(y'^2 - ky^2\right)\, dx$$

$$\underset{y(x)}{\text{Min}}\ J = \int_{x_1}^{x_2} \left(y'^2 - ky^2\right)\, dx$$

Self-adjointness is more than symmetry.

$$\langle Dy, z\rangle = \int_{x_1}^{x_2} (y'' + ky)\, z\, dx = \int_{x_1}^{x_2} (z'' + kz)\, y\, dx = \langle y, \mathrm{D}\, z\rangle$$

We notice that self-adjointness implies symmetry. But does symmetry imply self-adjointness? Let us take an example.

Let $D = i(\ )'$ with $(\ )_{x_1} = (\ )_{x_2} = 0$

$$\langle Dy, z\rangle = \int_{x_1}^{x_2} iy'\, z\, dx \qquad \langle Dy, z\rangle = \int_{x_1}^{x_2} iz'\, y\, dx$$

Since it involves a complex number, symmetry necessitates taking the complex conjugate. Let us verify (see the next slide…).

Check for symmetry and self-adjointness

$$\langle Dy, z\rangle = \int_{x_1}^{x_2} iy'\,z\,dx$$

Integrate by parts to get…

$$\langle Dy, z\rangle = \cancel{izy}\Big|_{x_1}^{x}{}^{0} - \int_{x_1}^{x_2} iy\,z'\,dx$$

$\boxed{\text{Note: } y(x_1) = y(x_2) = 0}$

$$\langle y, Dz\rangle = \cancel{iyz}\Big|_{x_1}^{x}{}^{0} - \int_{x_1}^{x_2} iz\,y'\,dx$$

$\boxed{\text{Note: } z(x_1) = z(x_2) = 0}$

$$\overline{\langle Dz, y\rangle} = -\int_{x_1}^{x_2} i^2 y\,z'\,dx = \langle y, Dz\rangle \Rightarrow$$ Symmetric

$$\langle Dy, z\rangle \neq \langle y, Dz\rangle \Rightarrow$$ Not self-adjoint

So, symmetry does not imply self-adjointness.

Verifying self-adjointness and obtaining a functional.

$$D=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \quad \text{for the differential equation,} \quad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

Is this true? $$\int_S\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)\psi\,dS=\int_S\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\phi\,dS$$

$$\int_S\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)\psi\,dS=-\int_S\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}\right)dS+\cancelto{0}{\int_S\left\{\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\psi\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\psi\right)\right\}dS}$$

(Green's theorem and boundary condition)

because
$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\psi\right)=\frac{\partial^2\phi}{\partial x^2}\psi+\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\psi\right)=\frac{\partial^2\phi}{\partial y^2}\psi+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}$$

(contd.)

$$\int_S \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \psi \, dS = -\int_S \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} \right) dS$$

Similarly,

$$\int_S \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \phi \, dS = -\int_S \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} \right) dS$$

Therefore,

$$\int_S \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \psi \, dS = \int_S \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \phi \, dS$$

Self-adjointness is verified; so, there exists a functional.

Obtaining a functional…

$$\int_S \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \psi \, dS = 0$$

$$\Rightarrow \int_S \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} \right) dS = 0$$

$$\Rightarrow \int_S \left(\frac{\partial \phi}{\partial x}\frac{\partial \delta\phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \delta\phi}{\partial y} \right) dS = 0$$

This implies:

$$\underset{\phi(x,y)}{\mathrm{Min}} \quad J = \int_S \left\{ \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right\} dS$$

because

$$\delta J = \int_S \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} \right) dS = 0$$

Self-adjointness is a sufficient condition; not a necessary condition.

What does this mean?

It means that a functional that gives the given differential equation might exist even if the differential operator is not self-adjoint. This is because self-adjointness is not a *necessary* condition.

Let us take an example to be convinced about it.

A differential operator of a dissipative system

Consider this differential equation: $y'' + by' + ky = 0$

with $\left(\ \right)_{x_1} = \left(\ \right)_{x_2} = \left(\ \right)'_{x_1} = \left(\ \right)'_{x_2} = 0$

This is the differential operator: $D = \left(\ \right)'' + b\left(\ \right)' + k\left(\ \right)$ Is this self-adjoint?

$$\langle Dy, z\rangle = \int_{x_1}^{x_2} \left(y'' + by' + ky\right) z\, dx$$

$$\Rightarrow \langle Dy, z\rangle = \cancelto{0}{\left(zy' + bzy\right)\Big|_{x_1}^{x_2}} - \int_{x_1}^{x_2} \left(y'z' + byz' - kyz\right) dx$$

$$\Rightarrow \langle Dy, z\rangle = \cancelto{0}{\left(z'y\right)\Big|_{x_1}^{x_2}} + \int_{x_1}^{x_2} \left(yz'' - byz' + kyz\right) dx$$

$$\langle Dz, y\rangle = \int_{x_1}^{x_2} \left(z'' + bz' + kz\right) y\, dx$$

$$\Rightarrow \langle Dz, y\rangle = \cancelto{0}{\left(yz' + byz\right)\Big|_{x_1}^{x_2}} - \int_{x_1}^{x_2} \left(y'z' + bzy' - kzy\right) dx$$

$$\Rightarrow \langle Dz, y\rangle = \cancelto{0}{\left(y'z\right)\Big|_{x_1}^{x_2}} + \int_{x_1}^{x_2} \left(y''z - bzy' + kzy\right) dx$$

These two are not equal.

Not self-adjoint.

Minimized functional may exist even if the operator is not self-adjoint.

$$D = (\ )'' + b(\ )' + k(\ )$$

We saw in the previous slide that this operator is not self-adjoint.

Consider this and write E-L equations

$$\underset{y(x)}{\text{Min}}\ J = \int_{x_1}^{x_2} \left(y'^2 - ky^2 \right) e^{bx}\ dx$$

$$F = \left(y'^2 - ky^2 \right) e^{bx}$$

$$\frac{\partial F}{\partial y} - \left(\frac{\partial F}{\partial y'} \right)' = 0$$

$$\Rightarrow -2kye^{bx} - \left(2y'e^{bx} \right)' = 0$$

$$\Rightarrow -2kye^{bx} - 2y''e^{bx} - 2by'e^{bx} = 0$$

$$\Rightarrow y'' + by' + ky = 0$$

We got it!

If a minimizable functional exists, we need to find a suitable multiplicative factor like e^{bx}

There is another way too…

$$D = (\ )'' + b(\ )' + k(\ )$$

We saw in the previous slide that this operator is not self-adjoint.

Consider this and write E-L equations

$$\underset{y(x),z(x)}{\text{Min}}\ J = \int_{x_1}^{x_2} \left(y'z' + \tfrac{1}{2} byz' - \tfrac{1}{2} bzy' - kyz \right)\ dx$$

$$F = \left(y'z' + \tfrac{1}{2} byz' - \tfrac{1}{2} bzy' - kyz \right)$$

$$\frac{\partial F}{\partial y} - \left(\frac{\partial F}{\partial y'} \right)' = 0$$

$$\Rightarrow \tfrac{1}{2} bz' - kz - \left(z' - \tfrac{1}{2} bz \right)' = 0$$

$$\Rightarrow bz' - kz - z'' + \tfrac{1}{2} bz' = 0$$

$$\Rightarrow z'' - bz' + ky = 0$$

$$\frac{\partial F}{\partial z} - \left(\frac{\partial F}{\partial z'} \right)' = 0$$

$$\Rightarrow -\tfrac{1}{2} by' - ky - \left(y' + \tfrac{1}{2} by \right)' = 0$$

$$\Rightarrow -\tfrac{1}{2} by' - ky - y'' - \tfrac{1}{2} by' = 0$$

$$\Rightarrow y'' + by' + ky = 0$$

Look at what we got.

Non-self-adjoint dissipative systems too can have functionals to be minimized.

$$\underset{y(x)}{\text{Min}}\ J = \int_{x_1}^{x_2} \left(y'^2 - k y^2 \right) e^{bx} \, dx$$

Multiplicative factor considered.

$$\underset{y(x),\mathrm{z(x)}}{\text{Min}}\ J = \int_{x_1}^{x_2} \left(y'z' + \tfrac{1}{2} byz' - \tfrac{1}{2} bzy' - kyz \right) \, dx$$

Parallel generative system appended.

$$y'' + by' + ky = 0$$

We learned two methods for non-self-adjoint systems too. But we need to think creatively to find the multiplicative factor or a parallel generative system.

The end note

The inverse problem of obtaining the functional from EL-equations

- Simple guess-work (does not work most of the time)
- A systematic method (works only with self-adjoint operators)
- What is self-adjointness? How to verify it?
- Method 1 for dissipative (non-self-adjoint) system, a "compenatory" multiplicative factor
- Method 2 for dissipative (non-self-adjoint) system, a parallel generative system

Thanks