Lecture 18

Practice problems in calculus of variations

ME 256 at the Indian Institute of Science, Bengaluru

Variational Methods and Structural Optimization

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Outline of the lecture

Some problems in calculus of variations with solutions to odd-numbered problems.

What we will learn:

How to apply the concepts and ideas learned so far to solve problems in calculus of variations.

Problem 1

Write Gâteaux variation for the following functionals.

1. $\int_0^l \left(y^2 + y'^2 - 2y\sin x\right) dx$

2. $\dfrac{\int_0^l \left(w''(x)\right)^2 dx}{\int_0^l \left(w'(x)\right)^2 dx}$

3. $\frac{1}{2}\int_0^T \left[v_0 \sin\alpha(t)\left\{x_0 + w_0 t + v_0\int_0^t \cos\alpha(\tau)d\tau\right\} - \left\{v_0\cos\alpha(t) + w_0\right\}\left\{y_0 + v_0\int_0^t \sin\alpha(\tau)d\tau\right\}\right] dt$

Solution to Problem 1.1

Write Gâteaux variation for the following functionals.

1. $\int_0^1 \left(y^2 + y'^2 - 2y\sin x \right) dx$

$$F = y^2 + y'^2 - 2y\sin x$$

$$\delta\left\{ \int_0^1 \left(y^2 + y'^2 - 2y\sin x \right) dx \right\} = \int_0^1 \left\{ \left(2y - 2\sin x \right)\delta y + 2y'\delta y' \right\}\delta y dx$$

Solution for Problem 1.2

Write Gâteaux variation for the following functionals.

2. $$\frac{\int_0^l (w''(x))^2\,dx}{\int_0^l (w'(x))^2\,dx}$$

$$\delta\left\{\frac{\int_0^1 w''^2 dx}{\int_0^1 w'^2 dx}\right\} = \frac{d}{d\varepsilon}\left\{\frac{\int_0^1 \left(w''+\varepsilon h''\right)^2 dx}{\int_0^1 \left(w'+\varepsilon h'\right)^2 dx}\right\}\Bigg|_{\varepsilon=0}$$

Solution to Problem 1.2

$$\delta\left\{\frac{\int_0^1 w''^2 dx}{\int_0^1 w'^2 dx}\right\} = \frac{d}{d\varepsilon}\left\{\frac{\int_0^1 \left(w''+\varepsilon h''\right)^2 dx}{\int_0^1 \left(w'+\varepsilon h'\right)^2 dx}\right\}\Bigg|_{\varepsilon=0}$$

$$= \frac{\left\{\int_0^1 \left(w'+\varepsilon h'\right)^2 dx\right\}\int_0^1 2\left(w''+\varepsilon h''\right)h'' dx - \left\{\int_0^1 \left(w''+\varepsilon h''\right)^2 dx\right\}\int_0^1 \left(w'+\varepsilon h'\right)h' dx}{\left\{\int_0^1 \left(w'+\varepsilon h'\right)^2 dx\right\}^2}\Bigg|_0$$

$$= \frac{\left\{\int_0^1 w'^2 dx\right\}\int_0^1 2w''h'' dx - \left\{\int_0^1 w''^2 dx\right\}\int_0^1 w'h' dx}{\left\{\int_0^1 w'^2 dx\right\}^2}$$

Solution to Problem 1.3

Write Gâteaux variation for the following functionals.

3. $\frac{1}{2}\int_0^T \left[v_0 \sin\alpha(t)\left\{ x_0 + w_0 t + v_0 \int_0^t \cos\alpha(\tau)d\tau \right\} - \{ v_0 \cos\alpha(t) + w_0 \} \left\{ y_0 + v_0 \int_0^t \sin\alpha(\tau)d\tau \right\} \right] dt$

$$\delta\left(\frac{1}{2}\int_0^T \left[v_0 \sin\alpha(t)\left\{ x_0 + w_0 t + v_0 \int_0^t \cos\alpha(\tau)d\tau \right\} - \{ v_0 \cos\alpha(t) + w_0 \} \left\{ y_0 + v_0 \int_0^t \sin\alpha(\tau)d\tau \right\} \right] dt \right)$$

$$= \frac{d}{d\varepsilon}\left[\frac{1}{2}\int_0^T \left[\begin{array}{l} v_0 \sin\left(\alpha(t) + \varepsilon\,\mathrm{h(t)}\right)\left\{ x_0 + w_0 t + v_0 \int_0^t \cos\left(\alpha(\tau) + \varepsilon\,\mathrm{h}(\tau)\right) d\tau \right\} - \\ \left\{ v_0 \cos\left(\alpha(t) + \varepsilon\,\mathrm{h(t)}\right) + w_0 \right\} \left\{ y_0 + v_0 \int_0^t \sin\left(\alpha(\tau) + \varepsilon\,\mathrm{h}(\tau)\right) d\tau \right\} \end{array} \right] dt \right]\Bigg|_{\varepsilon=0}$$

Solution to Problem 1.3 (contd.)

$$\left.\frac{d}{d\varepsilon}\left[\frac{1}{2}\int_0^T \left[\begin{array}{l} v_0 \sin\left(\alpha(t)+\varepsilon\,\mathrm{h(t)}\right)\left\{x_0 + w_0 t + v_0\int_0^t \cos\left(\alpha(\tau)+\varepsilon\,\mathrm{h}(\tau)\right)d\tau\right\} - \\ \left\{v_0\cos\left(\alpha(t)+\varepsilon\,\mathrm{h(t)}\right)+w_0\right\}\left\{y_0 + v_0\int_0^t \sin\left(\alpha(\tau)+\varepsilon\,\mathrm{h}(\tau)\right)d\tau\right\}\end{array}\right]dt\right]\right|_{\varepsilon=0}$$

$$=\frac{1}{2}\int_0^T \left.\left[\begin{array}{l} v_0 \cos\left(\alpha(t)+\varepsilon h(t)\right)\mathrm{h(t)}\left\{x_0 + w_0 t + v_0\int_0^t \cos\left(\alpha(\tau)+\varepsilon\,\mathrm{h}(\tau)\right)d\tau\right\} + \left\{v_0\sin\left(\alpha(t)+\varepsilon\,\mathrm{h(t)}\right)\right\}v_0\int_0^t \sin\left(\alpha(\tau)+\varepsilon\,\mathrm{h}(\tau)\right)\mathrm{h}(\tau)d\tau - \\ -\left\{v_0\sin\left(\alpha(t)+\varepsilon\,\mathrm{h(t)}\right)h(t)+w_0\right\}\left\{y_0 + v_0\int_0^t \sin\left(\alpha(\tau)+\varepsilon\,\mathrm{h}(\tau)\right)d\tau\right\} + \left\{v_0\cos\left(\alpha(t)+\varepsilon\,\mathrm{h(t)}\right)+w_0\right\}v_0\int_0^t \cos\left(\alpha(\tau)+\varepsilon\,\mathrm{h}(\tau)\right)\mathrm{h}(\tau)d\tau\end{array}\right]dt\right|_{\varepsilon=0}$$

$$=\frac{1}{2}\int_0^T \left[\begin{array}{l} v_0 \cos\alpha(t)\,\mathrm{h(t)}\left\{x_0 + w_0 t + v_0\int_0^t \cos\alpha(\tau)d\tau\right\} + \left\{v_0\sin\alpha(t)\right\}v_0\int_0^t \sin\alpha(\tau)\,\mathrm{h}(\tau)d\tau - \\ -\left\{v_0\sin\alpha(t)+w_0\right\}\left\{y_0 + v_0\int_0^t \sin\alpha(\tau)d\tau\right\} + \left\{v_0\cos\alpha(t)+w_0\right\}v_0\int_0^t \cos\alpha(\tau)\,\mathrm{h}(\tau)d\tau\end{array}\right]dt$$

Problem 2

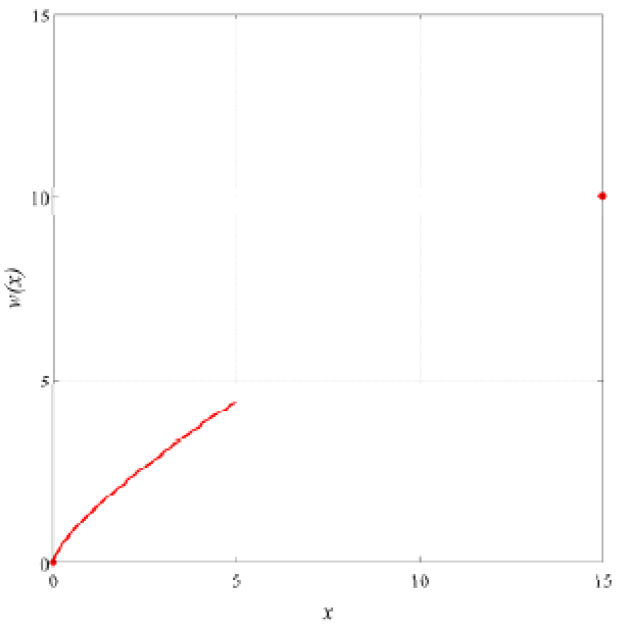
Write the Gâteaux variation of the functional, $J = \max_x (f^2 - f')$, where $f(x)$ is defined over an interval, $[x_1, x_2]$, for a particular function, $f(x) = x^2 - x$.

Problem 3

Solution over the first one third of the domain for the following problem is shown in the figure.

$$\underset{w(x)}{\text{Minimize}}\ J = \int_0^{15} w'^3 w \, dx \qquad \text{such that } w(0) = 0 \text{ and } w(15) = 10.$$

How would you proceed to plot the rest of the solution to reach the point (15,10)?

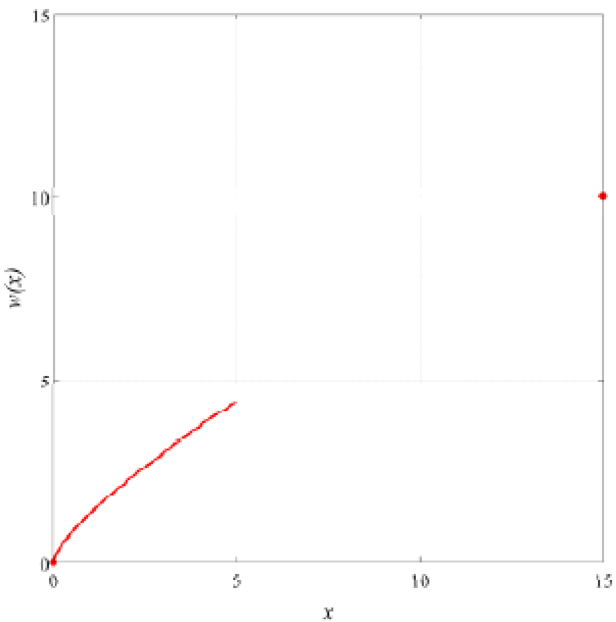


Solution to Problem 3

Solution over the first one third of the domain for the following problem is shown in the figure.

$$\underset{w(x)}{\text{Minimize}}\ J = \int_0^{15} w'^3 w\, dx \qquad \text{such that } w(0)=0 \text{ and } w(15)=10.$$

How would you proceed to plot the rest of the solution to reach the point (15,10)?



$$\underset{w(x)}{\text{Min}}\ J = \int_0^{15} w'^3 w\, dx$$

Note that $F = w'^3 w$

$$\frac{\partial F}{\partial w} - \left(\frac{\partial F}{\partial w'}\right)' = 0 \Rightarrow w'^3 - \left(3w'^2 w\right)' = 0$$

$$\Rightarrow w'^3 - 6w'w''w - 3w'^3 = 0 \Rightarrow 3ww'w'' + w'^3 = 0$$

Solution to Problem 3 (contd.)

We can write the first integral when $F = w'^3 w$

$$F - y' F_{y'} = C = \text{constant}$$

So, $w'^3 w - 3w'^2 w w' = C$

$$\Rightarrow -2w'^3 w = C$$

$$\Rightarrow w' = \sqrt[3]{-\frac{C}{2w}}$$

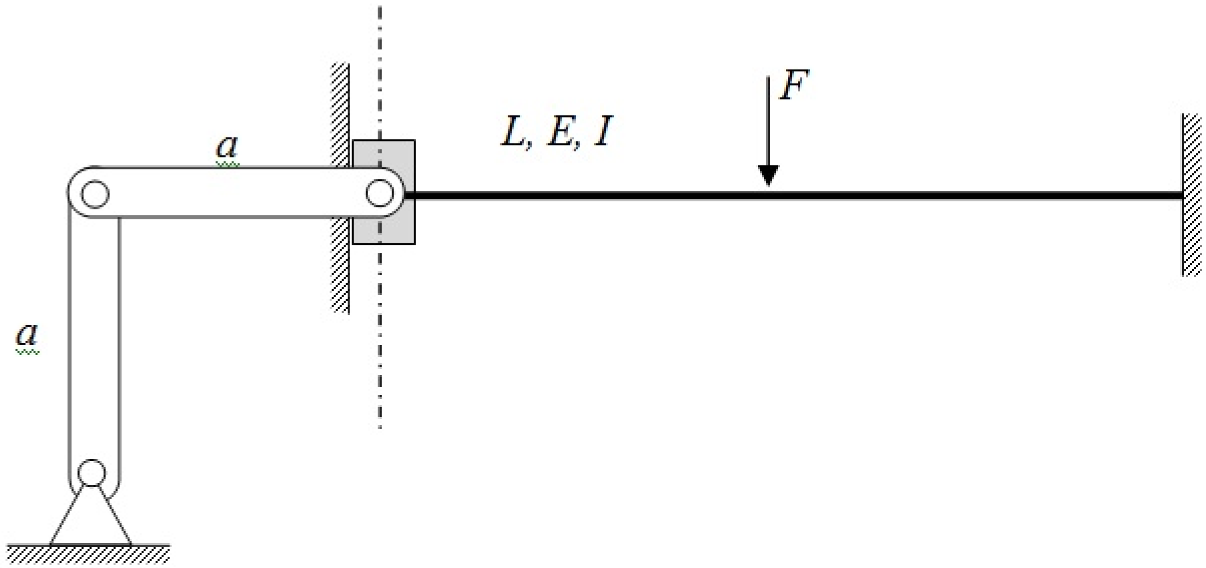
This can now be solved analytically to find $w(x)$. In that case, we do not need the first one-third of the solution. If we want to use, then we can numerically proceed from $x = 5$ because we know w' here.

$$w_{x=5+0.01} \approx w_{x=5} + w'\big|_{x=5} \times 0.01$$

Note that C can be numerically estimated from the given partial solution.

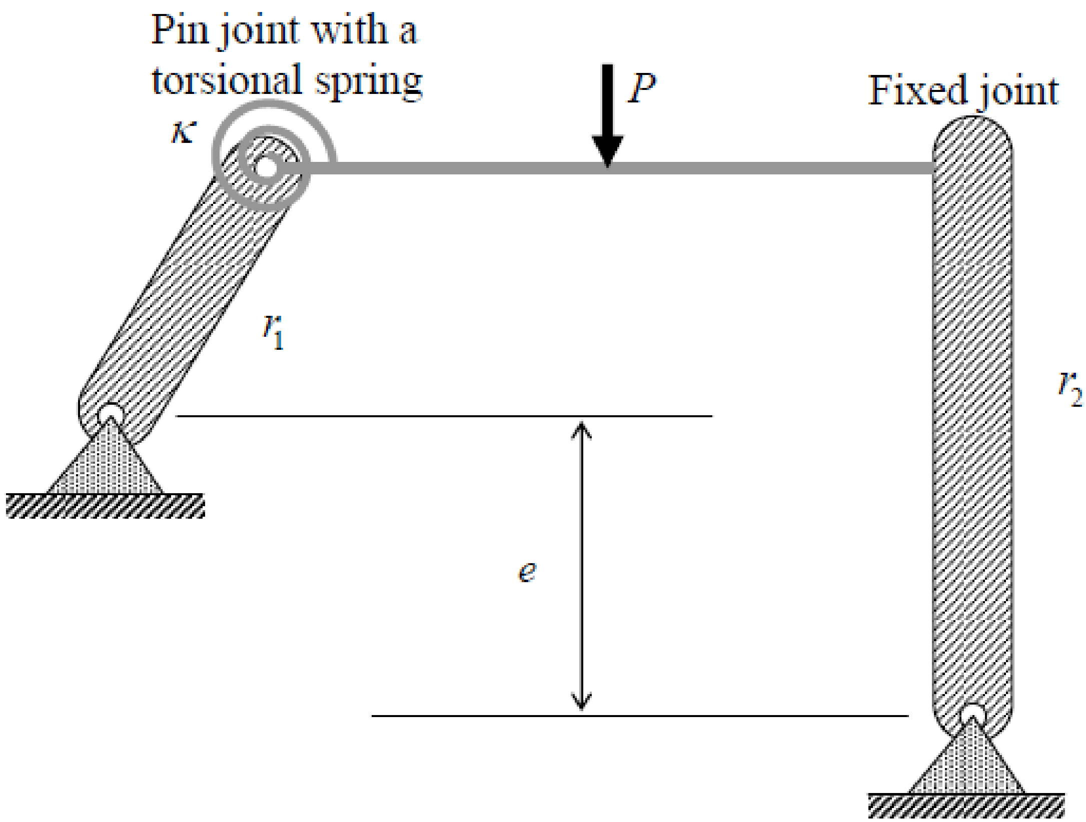
Problem 4

Write the boundary conditions for the beam shown in the figure below.



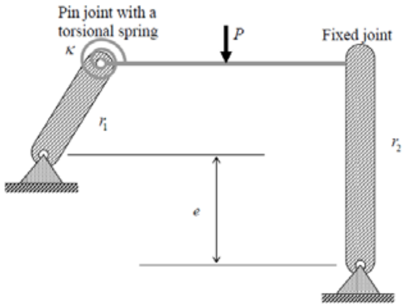
Problem 5

Write the boundary conditions for the beam shown in the figure below.



Solution to Problem 5

Let us write the potential energy for the given beam.



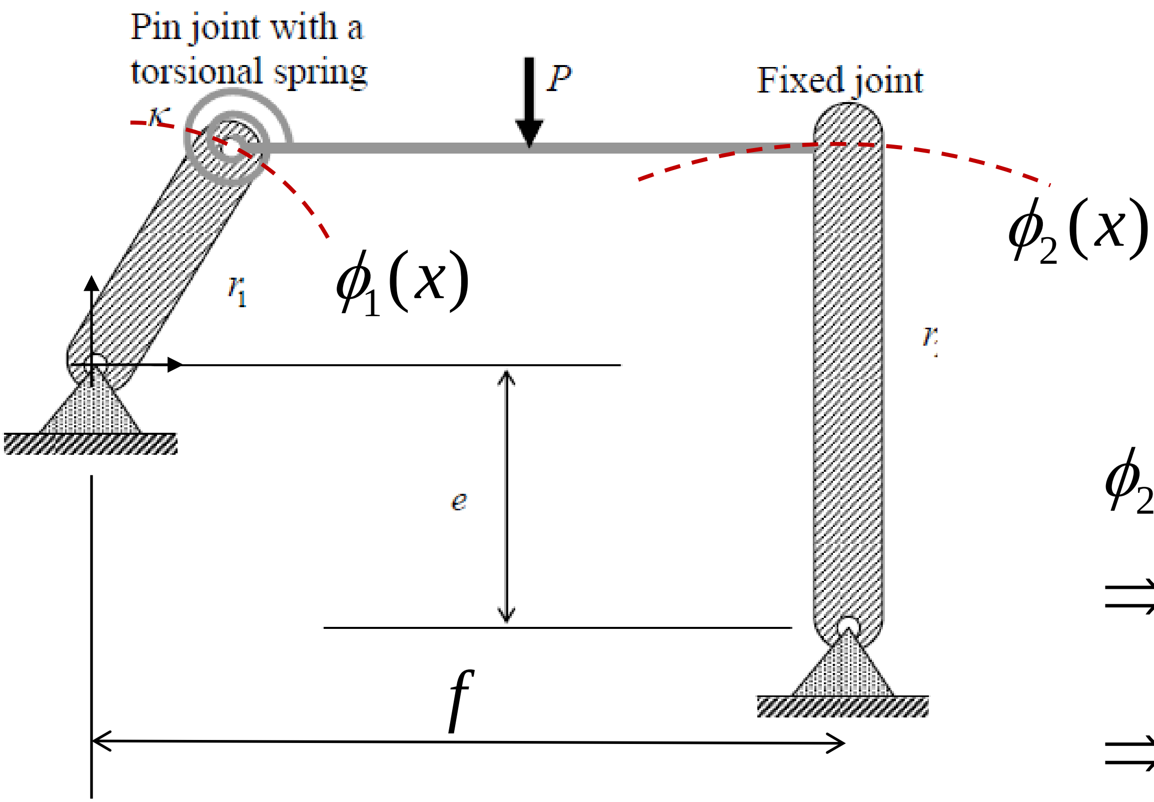
$$PE = SE + WP = \int_{x_1}^{x_2} \frac{1}{2} EIw''^2 dx + \frac{1}{2}\kappa\, w'^2\Big|_{x=x_1} + \left(-F\, w\big|_{x=0.5}\right)$$

It is a problem of variable end conditions because the left and right ends of the beam are constrained to move along arc of circles. Furthermore, it is a compliant four-bar mechanism. Therefore, the rotation of the two cranks are also related to each other.

Recall the boundary conditions when the ends are variable and the integrand F contains the second derivative (see Slide 18 in Lecture 15).

$$\left\{\left(F + \left(F_{y'} - \left(F_{y''}\right)'\right)\left(\phi' - y'\right) + F_{y''}\left(\phi'' - y''\right)\right)\delta x\right\}\Bigg|_{x1}^{x_2} = 0$$

Solution to Problem 5 (contd.)



$$\phi_1(x,y) = x^2 + y^2 = r_1^2$$

$$\Rightarrow 2x\,\delta x + 2y\,\delta y = 0$$

$$\Rightarrow \delta y = -\frac{x}{y}\delta x = \phi_1'\delta x$$

$$\phi_2(x,y) = \left(x - f\right)^2 + \left(y + e\right)^2 = r_2^2$$

$$\Rightarrow 2\left(x - f\right)\delta x + 2\left(y + e\right)\delta y = 0$$

$$\Rightarrow \delta y = -\frac{x - f}{y + e}\delta x = \phi_2'\delta x$$

$$\left\{\left[\left(F + \left(F_{y'} - \left(F_{y''}\right)'\right)\right)(\phi' - y') + F_{y''}(\phi'' - y'')\right]\delta x\right\}\Bigg|_{x1}^{x_2} = 0$$

Solution to Problem 5 (contd.)

$$PE = SE + WP = \int_{x_1}^{x_2} \frac{1}{2} EIw''^2 dx + \frac{1}{2}\kappa\, w'^2\Big|_{x=x_1} + \left(-F\, w\big|_{x=0.5}\right)$$

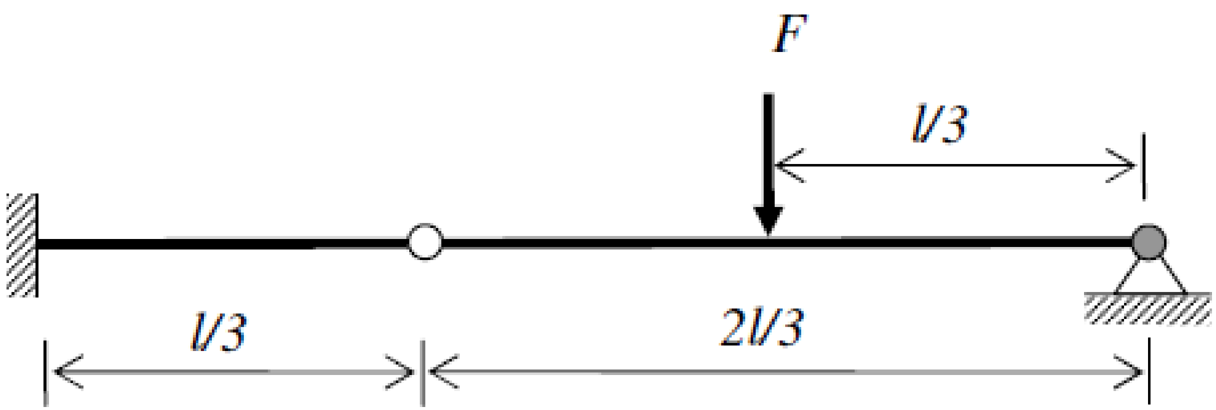
$$\left\{\left(F + \left(F_{y'} - \left(F_{y''}\right)'\right)\left(\phi' - y'\right) + F_{y''}\left(\phi'' - y''\right)\right)\delta x\right\}\Bigg|_{x1}^{x_2} = 0$$

$$\left\{\left(\frac{1}{2} EIw''^2 + \frac{1}{2}\kappa w'^2 + \left(-EIw''\right)'\left(-\frac{x}{y} - y'\right) + EIw''\left(-\frac{1}{y} - y''\right)\right)\right\}\Bigg|_{x1} = 0$$

$$\left\{\left(\frac{1}{2} EIw''^2 + \frac{1}{2}\kappa w'^2 + \left(-EIw''\right)'\left(-\frac{x-f}{y+e} - y'\right) + EIw''\left(-\frac{1}{y+e} - y''\right)\right)\right\}\Bigg|_{x2} = 0$$

Problem 6

Write the differential equation and the boundary conditions for the beam shown in the figure below.



Problem 7

Four boundary conditions are obtained when we minimize the potential energy of a beam under transverse load. Think of a physical arrangement of a beam where two or more boundary conditions are coupled to one another so that the four boundary conditions are not all independently zero. Sketch your physical arrangement clearly and mathematically interpret how the boundary conditions are coupled.

Solution to Problem 7

Four boundary conditions are obtained when we minimize the potential energy of a beam under transverse load. Think of a physical arrangement of a beam where two or more boundary conditions are coupled to one another so that the four boundary conditions are not all independently zero. Sketch your physical arrangement clearly and mathematically interpret how the boundary conditions are coupled.

$$\left(EIw''\right)' \delta w\Big|_0^L + \left(EIw''\right) \delta w'\Big|_0^L = 0$$

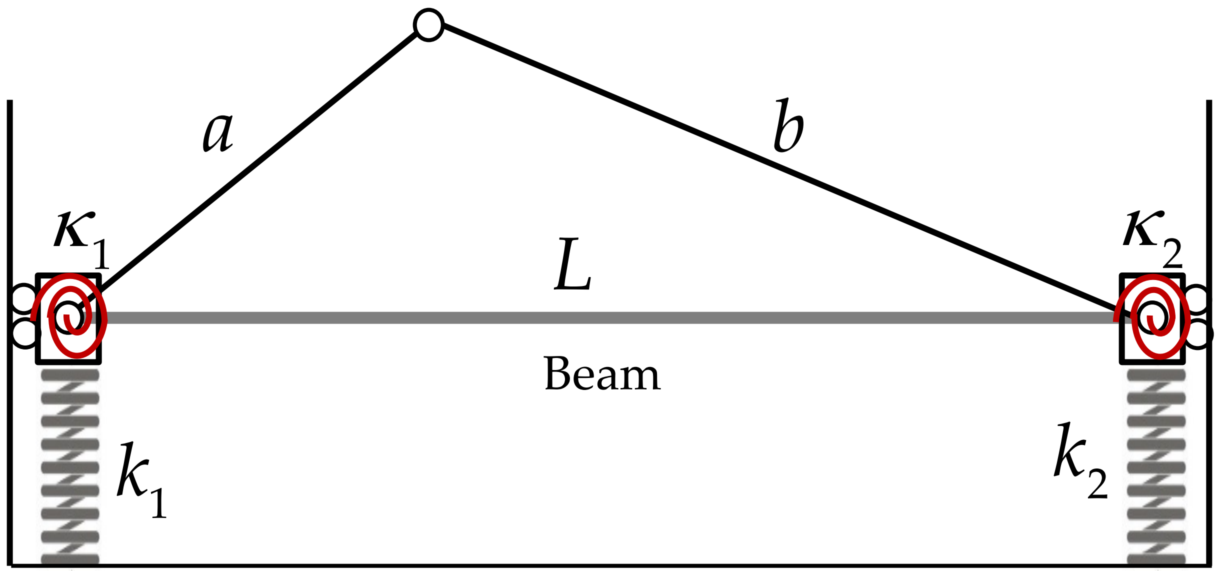
This is the sum of the four boundary conditions. If each of the four terms is individually equated to zero, we get four boundary conditions.

We need to find a physical arrangement where the four terms are not individually zero. Then, the sum must be zero.

Go to the next slide to see such an arrangement.

Solution to Problem 7 (contd.)

$$\left(EIw''\right)'\delta w\Big|_0+\left(EIw''\right)'\delta w\Big|_L+\left(EIw''\right)\delta w'\Big|_0+\left(EIw''\right)\delta w\Big|_L=0$$



Here, the transverse displacements and rotations at the two ends are coupled to one another because of the asymmetric linkage that connects them. We need to write the strain energy and then take Gâteaux variation.

Problem 8

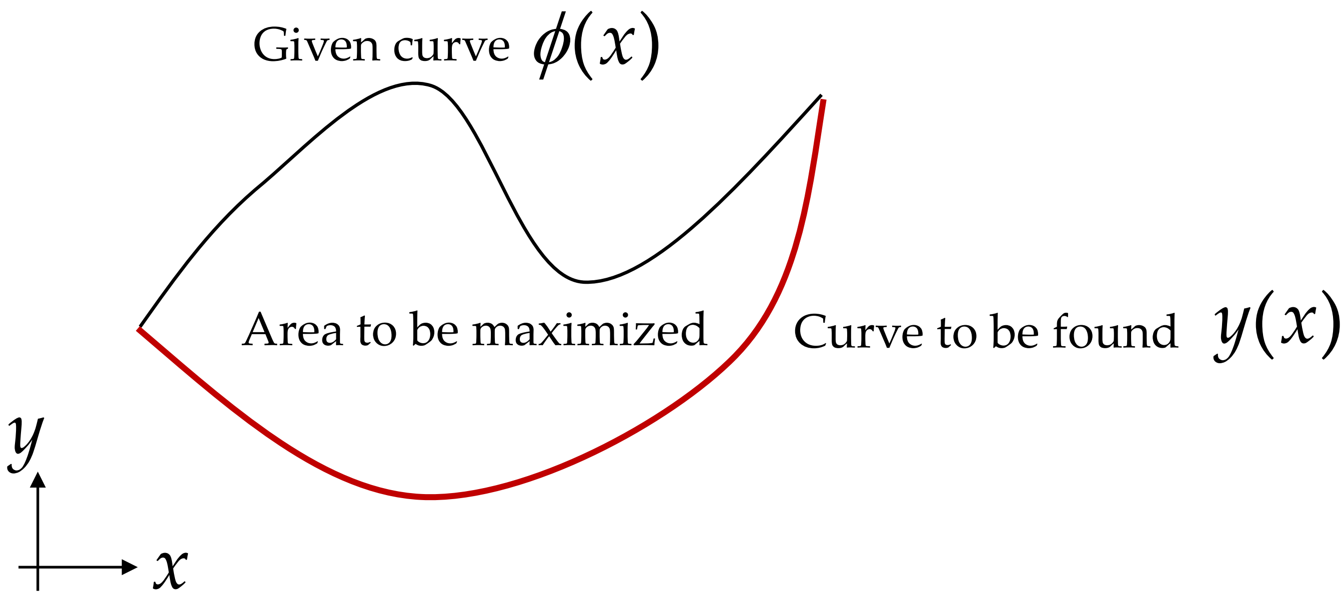
Solve the problem of hanging-chain problem using constrained calculus of variations problem. The length of the chain is 1 m and the horizontal span between the points from which it hangs is 0.5 m. The two points are at the same height. The mass per unit length of the chain is 1 kg/m. Acceleration dues to gravity is 9.81 m/s^2. Compute the Lagrange multiplier and interpet it physically in the context of this problem

Problem 9

Given two points in the xy-plane, namely (x1,y1) and (x2,y2), and a curve given by f(x,y) = 0 passing through those points, find a second curve of length L passing through the same two points and enclosing maximum area between this curve and the first curve. Pose the problem first and the write the differential equation and boundary conditions. Solve it numerically by taking a particular f(x,y) = 0 and some values for x1, y1, x2, and y2.

Solution to Problem 9

Given two points in the xy-plane, namely (x1,y1) and (x2,y2), and a curve given by f(x,y) = 0 passing through those points, find a second curve of length L passing through the same two points and enclosing maximum area between this curve and the first cirve. Pose the problem first and the write the differential equation and boundary conditions. Solve it numerically by taking a particular f(x,y) = 0 and some values for x1, y1, x2, and y2.



Solution to Problem 9 (contd.)

$$\underset{y(x)}{\text{Min}}\ A = -\int_{x_1}^{x_2} (\phi - y)\,dx$$

Area to be maximized (hence negative sign for minimization).

Subject to

$$\Lambda: \qquad \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx - L = 0$$

Constraint on the length of the curve.

$$L = \int_{x_1}^{x_2} \left\{ -(\phi - y) + \Lambda\sqrt{1+y'^2} \right\} dx - \Lambda L$$

Lagrangian

$$F = -(\phi - y) + \Lambda\sqrt{1+y'^2}$$

Integrand to be used to write Euler-Lagrange equation and boundary conditions.

Solution to Problem 9 (contd.)

$$F = \left(\phi - y\right) + \Lambda\sqrt{1 + y'^2}$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad x \in (x_1, x_2)$$

$$\Rightarrow -1 - \frac{d}{dx}\left(\frac{\Lambda y'}{\sqrt{1 + y'^2}}\right) = 0$$

This shows that the curvature is constant for $y(x)$.

Problem 10

Write the necessary conditions, including the boundary conditions, for the following problem.

$$\underset{A(x)}{\text{Max}}\ \underset{u(x)}{\text{Min}}\ \ PE = \int_0^L \left\{ \frac{EA(x)u'^2(x)}{2} - p(x)u(x) \right\} dx$$

Subject to

$$\Lambda: \qquad \int_0^L A(x)\,dx - V^* \le 0$$

$$\text{Data}: L, E, p(x), V^*$$

Problem 11

Write the necessary conditions, including the boundary conditions, for the following problem.

$$\underset{x(s,t),y(s,t)}{Min\ I} \ : \int_0^L \int_0^T \sqrt{\left(\frac{dT_x}{dt} + \frac{\partial x(s,t)}{\partial t}\right)^2 + \left(\frac{dT_y}{dt} + \frac{\partial y(s,t)}{\partial t}\right)^2} dtds$$

$$Subject\ to$$

$$\Lambda(t) : A = \int_0^L \left(\sqrt{\left(\frac{\partial x(s,t)}{\partial s}\right)^2 + \left(\frac{\partial y(s,t)}{\partial s}\right)^2} - 1 \right) ds = 0$$

$$Data : x(s,0), y(s,0), T_x(t), T_y(t), x(0,t) = 0, y(0,t) = 0$$

Solution to Problem 11

$$\underset{x(s,t),y(s,t)}{\text{Min}} \quad I = \int_0^L \int_0^T \sqrt{\left(\frac{dT_x}{dt} + \frac{\partial x}{\partial t}\right)^2 + \left(\frac{dT_y}{dt} + \frac{\partial y}{\partial t}\right)^2} \; dt\, ds$$

Subject to

$$\Lambda : \qquad \int_0^L \int_0^T \frac{1}{T} \left\{ \sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2} - 1 \right\} ds\, dt = 0$$

$$\text{Data}: L, T, T_x(t), T_y(t), x(s,0), y(s,0), x(0,t) = y(0,t) = 0$$

Notice that we have included time-integration in the constraint equation without affecting it. This is just for convenience so that we can write the integrand together for the objective function and constraint.

Solution to Problem 11 (contd.)

Begin with the Lagrangian.

$$L = \int_0^L \int_0^T \left[\sqrt{\left(\frac{dT_x}{dt} + \frac{\partial x}{\partial t} \right)^2 + \left(\frac{dT_y}{dt} + \frac{\partial y}{\partial t} \right)^2} + \frac{\Lambda}{T} \left\{ \sqrt{\left(\frac{\partial x}{\partial s} \right)^2 + \left(\frac{\partial y}{\partial s} \right)^2} - 1 \right\} \right] dt\, ds$$

$$F = \sqrt{\left(T_{xt} + x_t \right)^2 + \left(T_{yt} + y_t \right)^2} + \frac{\Lambda}{T} \left\{ \sqrt{x_s^2 + y_s^2} - 1 \right\}$$

With shorthand notation for partial derivatives.

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial x_s} \right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x_t} \right) = 0$$

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial y_s} \right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y_t} \right) = 0$$

Euler-Lagrange equations

$$\int_S \left(-\frac{\partial F}{\partial x_t} ds + \frac{\partial F}{\partial x_s} dt \right) \delta x = 0$$

$$\int_S \left(-\frac{\partial F}{\partial y_t} ds + \frac{\partial F}{\partial y_s} dt \right) \delta y = 0$$

Boundary conditions

Solution to Problem 11 (contd.)

$$F = \sqrt{\left(T_{xt} + x_t\right)^2 + \left(T_{yt} + y_t\right)^2} + \frac{\Lambda}{T}\left\{\sqrt{x_s^2 + y_s^2} - 1\right\}$$

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial x_s}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x_t}\right) = 0 \text{ and } \frac{\partial F}{\partial y} - \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial y_s}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y_t}\right) = 0$$

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial x_s}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x_t}\right) = 0$$

Similarly, for $y(s,t)$ too.

$$\Rightarrow 0 - \frac{\partial}{\partial s}\left(\frac{\Lambda x_s}{T\sqrt{x_s^2 + y_s^2}}\right) - \frac{\partial}{\partial t}\left(\frac{T_{xt} + x_t}{\sqrt{\left(T_{xt} + x_t\right)^2 + \left(T_{yt} + y_t\right)^2}}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial s}\left(\frac{\Lambda x_s}{T\sqrt{x_s^2 + y_s^2}}\right) + \frac{\partial}{\partial t}\left(\frac{T_{xt} + x_t}{\sqrt{\left(T_{xt} + x_t\right)^2 + \left(T_{yt} + y_t\right)^2}}\right) = 0$$

Solution to Problem 11 (contd.)

$$F = \sqrt{\left(T_{xt} + x_t\right)^2 + \left(T_{yt} + y_t\right)^2} + \frac{\Lambda}{T}\left\{\sqrt{x_s^2 + y_s^2} - 1\right\}$$

$$\int_S \left(-\frac{\partial F}{\partial x_t} ds + \frac{\partial F}{\partial x_s} dt \right) \delta x = 0 \quad \text{and} \quad \int_S \left(-\frac{\partial F}{\partial y_t} ds + \frac{\partial F}{\partial y_s} dt \right) \delta y = 0$$

Similarly, for $y(s,t)$ too.

$$\int_S \left(-\frac{\partial F}{\partial x_t} ds + \frac{\partial F}{\partial x_s} dt \right) \delta x = 0$$

$$\Rightarrow \int_S \left(\frac{\Lambda x_s}{T\sqrt{x_s^2 + y_s^2}} ds - \frac{T_{xt} + x_t}{\sqrt{\left(T_{xt} + x_t\right)^2 + \left(T_{yt} + y_t\right)^2}} dt \right) \delta x = 0$$

Problem 12

Write the necessary conditions, including the boundary conditions, for the following problem.

$$\underset{b(x)}{\text{Minimize}} \; \frac{\displaystyle\int_0^L \frac{(L-x)^2}{b}\,dx}{\alpha\,\tilde{E}\,\Delta}$$

Subject to

$$\Lambda_1 : \quad \Delta - \int_0^L \frac{M_l m_u}{\alpha\, b\, \tilde{E}}\,dx \le 0$$

$$\Lambda_2 : \quad \int_0^L t\, b\, dx - V_{max} \le 0$$

Data: $\alpha = t^3/12, t, \Delta,\ \tilde{E} = E/(1-\nu)^2,\ L, V_{max}, M_l(x), m_u(x)$

The end note

Practice problems in calculus of variations

- Problems in taking Gâteaux variation
- Writing the Euler-Lagrange equations for unconstrained and constrained problems
- Boundary conditions for beams to apply the concepts related to coupled boundary conditions, variable end conditions, and corner conditions.
- Formulating calculus of variations.
- Writing the necessary conditions for calculus of variations problem with constraints.

Thanks