

Lecture 20a

Many problems in optimizing a beam

ME 256 at the Indian Institute of Science, Bengaluru

Variational Methods and Structural Optimization

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Outline of the lecture

Some problems in optimizing the cross-section profile of a beam under transverse loading.

What we will learn:

How to apply the concepts and ideas learned so far to solve problems in calculus of variations with particular examples of transversely loaded beams.

How is a beam different from a bar? (in the context of structural optimization)

A bar deforms axially whereas a beam displaces transversely. That is, a beam bends. The governing differential equation for a bar is of second order whereas that for a beam is fourth order. So, we should be prepared for tedious calculations in a beam.

Cross-section area is all that matters for volume and stiffness in a slender bar: its shape does not matter. That is not true for a beam. The volume of a beam depends on the value of cross-section area but the stiffness depends on the second moment of area of cross-section. So, shape of the cross-section matters.

In this set of problems, we consider only rectangular cross-section of beams. A rectangular has two dimensions, breadth, $b(x)$, and depth, $t(x)$. Both of these can be varied independently (provided that such a beam can be manufactured economically). However, most often, we vary only one of them.

If is $b(x)$ varied, $A(x) = b(x)t$; $I(x) = b(x)t^3 / 12 = b(x)t(t^2 / 12) = (t^2 / 12)A(x)$

If is $t(x)$ varied, $A(x) = b t(x)$; $I(x) = b t(x)^3 / 12 = (b t(x))^3 / (12b^2) = (1 / 12b^2)A^3(x)$

For most cross-section shapes, we can write: $I(x) = \alpha A^\beta$

Problem 1

Minimize the mean compliance of a beam for given volume of material.

$$\text{Min}_{A(x)} MC = \int_0^L q w dx$$

Subject to

$$\lambda(x): \quad (EIw'')'' - q = 0 \Rightarrow (E\alpha Aw'')'' - q = 0$$

$$\Lambda: \quad \int_0^L A dx - V^* \leq 0$$

$$\text{Data: } L, q(x), E, \alpha = \frac{t^2}{12}, V^*$$

We assume here that only $b(x)$ is variable.

$$I(x) = \alpha A^\beta$$
$$\alpha = \left(\frac{t^2}{12} \right); \beta = 1$$

Problem 2

Minimize the strain energy of a beam for given volume of material.

$$\underset{A(x)}{\text{Min}} SE = \int_0^L \frac{1}{2} E \alpha A w''^2 dx$$

Subject to

$$\lambda(x): \quad (E \alpha A w'')'' - q = 0$$

$$\Lambda: \quad \int_0^L A dx - V^* \leq 0$$

$$\text{Data: } L, q(x), E, \alpha = \frac{t^2}{12}, V^*$$

Problem 3

Minimize the volume of a beam subject to an upper bound on the strain energy.

$$\text{Min}_{A(x)} V = \int_0^L A dx$$

Subject to

$$\lambda(x): \quad (E\alpha Aw''')'' - q = 0$$

$$\Lambda: \quad \int_0^L \frac{1}{2} E\alpha Aw''^2 dx - SE^* \leq 0$$

$$\text{Data: } L, q(x), E, \alpha = \frac{t^2}{12}, SE^*$$

Problem 4

Minimize the volume of material for a given upper bound on the mean compliance.

$$\text{Min}_{A(x)} V = \int_0^L A dx$$

Subject to

$$\lambda(x): \quad (E\alpha Aw'')'' - q = 0$$

$$\Lambda: \quad \int_0^L q w dx - MC^* \leq 0$$

$$\text{Data: } L, q(x), E, \alpha = t^2/12, MC^*$$

Problem 5

$$\text{Min}_{A(x)} SE = \int_0^L \frac{M^2}{2E\alpha A} dx$$

Subject to

$$\Lambda: \int_0^L A dx - V^* \leq 0$$

$$\text{Data: } L, M(x), E, \alpha = t^2/12, V^*$$

Minimize the strain energy of a statically determinate beam for given volume of material. For a statically determinate beam, we can obtain the bending moment without knowing the area of cross-section of the beam.

Problem 6

Min-max formulation for the stiffest beam for given volume of material.

$$\underset{A(x)}{\text{Max}} \underset{u(x)}{\text{Min}} PE = \int_0^L \left(\frac{1}{2} E \alpha A w''^2 - q w \right) dx$$

Subject to

$$\Lambda : \int_0^L A dx - V^* \leq 0$$

$$\text{Data : } L, q(x), E, \alpha = \frac{t^2}{12}, V^*$$

Problem 7

$$\text{Min}_{A(x)} MC = \int_0^L q w dx$$

Subject to

$$\Gamma: \int_0^L (E\alpha A w'' v'' - qv) dx = 0$$

$$\Lambda: \int_0^L A dx - V^* \leq 0$$

$$\text{Data: } L, q(x), E, \alpha = t^2/12, V^*$$

Minimize the mean compliance of a beam for given volume of material with the governing equation in the weak form.

Problem 8

$$\text{Min}_{A(x)} MC = \int_0^L q w dx$$

Subject to

$$\Gamma: \int_0^L (E \alpha A w'' v'' - q v) dx = 0$$

$$\Lambda: \int_0^L A dx - V^* \leq 0$$

$$\mu_u(x): A - A_u \leq 0$$

$$\mu_l(x): A_l - A \leq 0$$

$$\text{Data: } L, q(x), E, \alpha = \frac{t^2}{12}, V^*, A_l, A_u$$

Minimize the mean compliance of a beam for given volume of material and upper and lower bound constraints on the area of cross-section.

Problem 9

Minimize the mean compliance of a beam for given volume of material where the depth of the beam is the design variable.

$$\text{Min}_{A(x)} MC = \int_0^L q w dx$$

Subject to

$$\Gamma: \int_0^L (E\alpha A^3 w'' v'' - qv) dx = 0$$

$$\Lambda: \int_0^L A dx - V^* \leq 0$$

$$\mu_u(x): A - A_u \leq 0$$

$$\mu_l(x): A_l - A \leq 0$$

$$\text{Data: } L, q(x), E, \alpha = \frac{1}{12b^2}, V^*, A_l, A_u$$

We assume here that only $t(x)$ is variable.

$$I(x) = \alpha A^\beta$$

$$\alpha = \left(\frac{1}{12b^2} \right); \beta = 3$$

Problem 10

Minimize the volume of a statically determinate beam with a deflection constraint in its span.

$$\text{Min}_{A(x)} V = \int_0^L A dx$$

Subject to

$$\Lambda: \int_0^L \frac{MM_d}{E\alpha A} dx - \Delta^* = 0$$

$$\text{Data: } L, M(x), M_d(x), E, \alpha = t^2/12, \Delta^*$$

Problem 11

Minimize the volume of material of a beam (statically determinate or indeterminate) for a deflection constraint in its span.

$$\text{Min}_{A(x)} V = \int_0^L A dx$$

Subject to

$$\lambda(x): \quad (E\alpha Aw'')'' - q = 0$$

$$\lambda_d(x): \quad (E\alpha Av'')'' - q_d = 0$$

$$\Lambda: \quad \int_0^L E\alpha Aw''v'' dx - \Delta^* = 0$$

$$\text{Data: } L, q(x), q_d(x), E, \alpha = t^2/12, \Delta^*$$

Problem 12

$$\underset{A(x)}{\text{Min}} V = \int_0^L A dx$$

Subject to

$$\lambda(x): \quad (E\alpha Aw'')'' - q = 0$$

$$\lambda_d(x): \quad (E\alpha Av'')'' - q_d = 0$$

$$\Lambda: \quad \int_0^L E\alpha Aw''v'' dx - \Delta^* = 0$$

$$\Gamma: \quad \int_0^L \frac{1}{2} E\alpha Aw''^2 dx - SE^* = 0$$

$$\text{Data: } L, q(x), q_d(x), \alpha = t^2/12, E, \Delta^*, SE^*$$

Minimize the volume of material of a beam (statically determinate or indeterminate) for a deflection constraint in its span with an upper bound on the strain energy.

Problem 13

$$\text{Min}_{A(x)} MC = \int_0^L q w dx$$

Subject to

$$\Gamma: \int_0^L (E \alpha A w'' v'' - q v) dx = 0$$

$$\Lambda: \int_0^L A dx - V^* \leq 0$$

$$\mu_u(x): w - w_u \leq 0$$

$$\mu_l(x): w_l - w \leq 0$$

$$\text{Data: } L, q(x), E, \alpha = t^2/12, V^*, w_l, w_u$$

Minimize the mean compliance of a beam for given volume of material and upper and lower bound constraints on the transverse displacement.

Problem 14

$$\text{Min}_{A(x)} MC = \int_0^L q w dx$$

Subject to

$$\Gamma: \int_0^L (E\alpha A w'' v'' - qv) dx = 0$$

$$\Lambda: \int_0^L A dx - V^* \leq 0$$

$$\mu_u(x): E\alpha w'' - \sigma_t \leq 0$$

$$\mu_l(x): \sigma_c - E\alpha w'' \leq 0$$

$$\text{Data: } L, q(x), E, \alpha = t^2/12, V^*, \sigma_t, \sigma_c$$

Minimize the mean compliance of a beam for given volume of material and upper and lower bound constraints on the stress.

Problem 15

$$\underset{q(x)}{\text{Max}} MC = \int_0^L q w dx$$

Subject to

$$\lambda(x): \quad (E\alpha A w'')'' - q = 0$$

$$\Lambda: \quad \int_0^L p dx - W^* \leq 0$$

$$\text{Data: } L, A(x), E, \alpha = t^2/12, W^*$$

Determine the world load distribution of a beam. Note that the upper bound on the overall load is specified.

Problem 16

A general objective function for a beam problem.

$$\text{Min}_{A(x)} MC = \int_0^L q w^2 dx$$

Subject to

$$\lambda(x): \quad (E\alpha A w'')'' - q = 0$$

$$\Lambda: \quad \int_0^L A dx - V^* \leq 0$$

$$\text{Data: } L, q(x), E, \alpha = t^2/12, V^*$$

Problem 17

$$\underset{b(x), t(x)}{\text{Min}} \quad MC = \int_0^L q w dx$$

Subject to

$$\lambda(x) : \quad \left(\frac{1}{12} E b t^3 A w'' \right)'' - q = 0$$

$$\Lambda : \quad \int_0^L A dx - V^* \leq 0$$

Data : $L, q(x), E, V^*$

Minimize the mean compliance of a beam for given volume of material by varying both breadth and depth of the rectangular cross-section of the beam.

The end note

Practice problems in calculus of variations

Transversely deforming beam is the second simplest structural optimization problem. Either breadth or depth can be varied; both can be varied too.

Mean compliance and strain energy are measures of stiffness.

Volume of material used is a cost-measure.

Objective function and functional constraint can be interchanged without affecting the nature of the solution.

Equilibrium equation can be posed in strong or weak form without changing the nature of the solution.

Constraints can also be imposed on profile of the area of cross-section.

Constraints on displacements and strains (stresses) can be imposed.

Thanks