Lecture 6

Formulation of Calculus of Variations Problems in Geometry and Mechanics

ME256 Indian Institute of Science

Variational Methods and Structural Optimization

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Outline of the lecture

We will discuss some geometry problems that can be cast as problems of calculus of variations.

We will also discuss the role of calculus of variations in mechanics and structural optimization.

What we will learn:

- What kinds of problems belong to calculus of variations?
- How do we formulate calculus of variations problems?
- What is the connection between mechanics and calculus of variations?
- What is the connection between structural optimization and calculus of variations?
- How does a **functional** look like?

Geometry and calculus of variations

There are many problems in geometry that relate to calculus of variations.

They pertain to minimal curves and surfaces.

Minimal curves

- Geodesics
- Maximum enclosing area for a given perimeter
- Chains hanging in a force field
- Etc.

Minimal surfaces

- Minimum surface of revolution
- Surfaces of least area enclosed by a given boundary
- Etc.

Mechanics and calculus of variations

There are three ways to write equations of statics and dynamics.

Two of these are related to calculus of variations.
- We will discuss them in this lecture and later too.

Structural optimization is essentially calculus of variations.
- What do we want to optimize in a structure?
- Stiffness, flexibility, strength, weight, cost, manufacturability, natural frequency, mode shape, stability, buckling loads, contact stress, etc.
- All of these can be posed as objective function and constraints in the framework of calculus of variations.

We will consider a few problems and formulate them in this lecture.

Three views of mechanics	Statics	Dynamics
Final result of calculus of variation!	Force balance	F = ma
An intermediate result of calculus of variations	Principle of virtual work	D'Lambert principle
Calculus of variations	Minimum potential energy principle	Hamilton's principle

Geometry and calculus of variations

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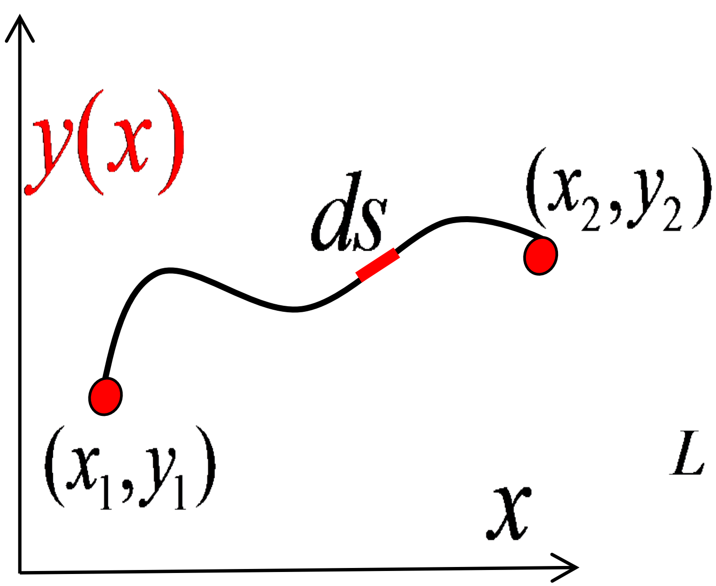
Curve of least distance between two points in a plane.

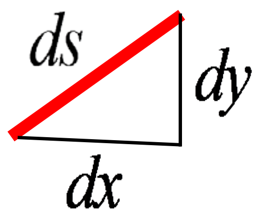
You are given two points in a flat plane. You can draw many, many curves that connect the two points. Of all those curves, which one has the least length?

The answer is obvious: it is a straight line joining the two points.

Pretend that you do not know the answer or someone is not convinced about it.

How will you pose this as a problem whose solution gives you a convincing proof?
Here is how:





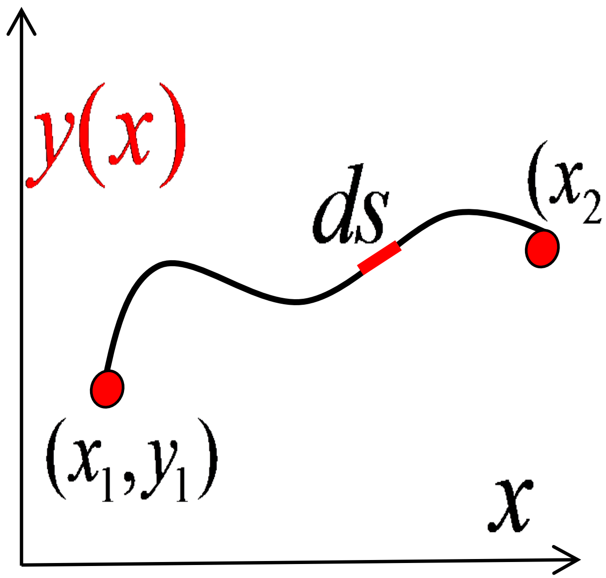
We take a small segment *ds* and integrate it to get the length of the curve *y(x)* between the two given points.

$$L = \int ds = \int \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

Geodesic in a plane

Geodesic:

- Curve of least distance between two given points.



$$\underset{y(x)}{\text{Min}}\ L = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

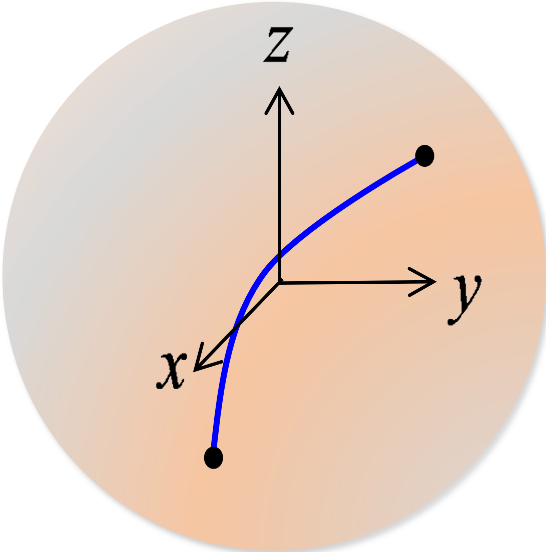
$$\text{Data}: x_1,\ x_2,\ y(x_1) = y_1,\ y(x_2) = y_2$$

L here is the **functional**. Its integrand depends on the first derivative of *y(x)*, which is denoted as *y'(x)*.

Solution in another lecture!

Observe the problem for now and understand it.

Geodesic on a sphere



A spherical surface can be described in parametric form by azimuthal and elevation angles and radius *R*.

$$x = R\cos\theta\cos\phi$$

$$y = R\cos\theta\sin\phi$$

$$z = R\sin\theta$$

Then, we can write the differential quantities as…

$$dx = R\left(-\sin\theta\cos\phi\, d\theta - \cos\theta\sin\phi\, d\phi\right)$$

$$dy = R\left(-\sin\theta\sin\phi\, d\theta + \cos\theta\cos\phi\, d\phi\right)$$

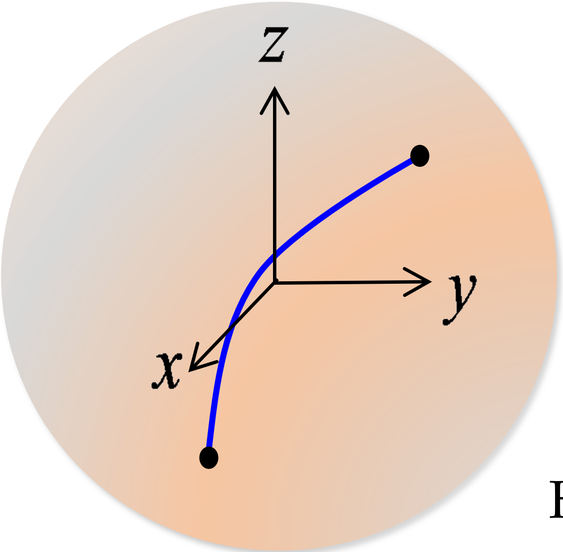
$$dz = R\cos\theta\, d\theta$$

$$ds^2 = dx^2 + dy^2 + dz^2 = R^2\left(\begin{array}{l}\sin^2\theta\cos^2\phi\, d\theta^2 + \cos^2\theta\sin^2\phi\, d\phi^2 + \sin\theta\cos\phi\cos\theta\sin\phi\, d\theta d\phi \\ +\sin^2\theta\sin^2\phi\, d\theta^2 + \cos^2\theta\cos^2\phi\, d\phi^2 - \sin\theta\cos\phi\cos\theta\sin\phi\, d\theta d\phi + \cos^2\theta\, d\theta^2\end{array}\right)$$

$$= R^2\left(d\theta^2 + \cos^2\theta\, d\phi^2\right)$$

Therefore, $ds = R\sqrt{\left(d\theta^2 + \cos^2\theta\, d\phi^2\right)}$

Geodesic on a sphere (contd.)



$$ds = R\sqrt{\left(d\theta^2 + \cos^2\theta \, d\phi^2\right)}$$

$$L = \int ds = \int R\sqrt{\left(d\theta^2 + \cos^2\theta \, d\phi^2\right)} = \int_{\theta_1}^{\theta_2} R\sqrt{\left(1 + \cos^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)}\, d\theta$$

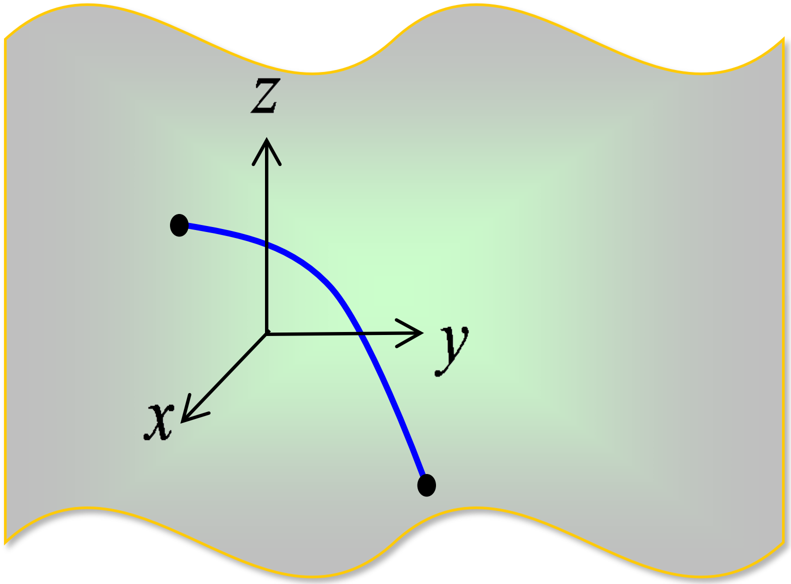
Here, we describe a curve on the sphere as $\phi(\theta)$

Thus, the geodesic problem on a sphere becomes…

$$\underset{\phi(\theta)}{\text{Min}} \;\; L = \int_{\theta_1}^{\theta_2} R\sqrt{\left(1 + \cos^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)}\, d\theta$$

$$\text{Data} : \theta_1,\ \theta_2,\ \phi(\theta_1) = \phi_1,\ \phi(\theta_2) = \phi_2$$

Geodesic on any given surface



Then, we can write the differential quantities as…

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

Now, the length of a curve on the surface, given in its parametric form, *v(u)*, is given by

Any surface can be described in parametric form using *u* and *v*

$$x = x(u,v)$$

$$y = y(u,v)$$

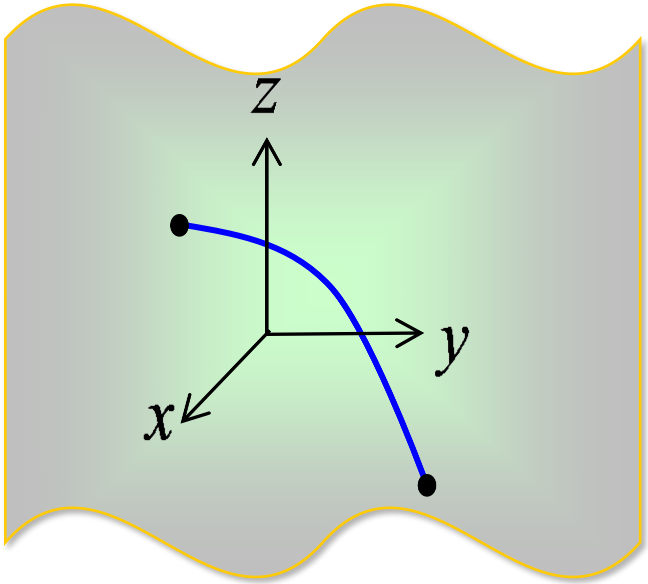
$$z = z(u,v)$$

$$L = \int ds = \int \sqrt{dx^2 + dy^2 + dz^2} = \int_{u_1}^{u_2} \sqrt{\left(P + 2Q\frac{dv}{du} + R\left(\frac{dv}{du}\right)^2\right)}\, du$$

$$P = \left(\partial x/\partial u\right)^2 + \left(\partial y/\partial u\right)^2 + \left(\partial z/\partial u\right)^2; \quad R = \left(\partial x/\partial v\right)^2 + \left(\partial y/\partial v\right)^2 + \left(\partial z/\partial v\right)^2;$$

$$Q = \left(\partial x/\partial u\right)\left(\partial x/\partial v\right) + \left(\partial y/\partial u\right)\left(\partial y/\partial v\right) + \left(\partial z/\partial u\right)\left(\partial z/\partial v\right)$$

Geodesic on any surface (contd.)



This is the general form of the geodesic problem for any surface specified in parametric form.

$$\underset{v(u)}{\text{Min}}\ L = \int_{u_1}^{u_2} \sqrt{\left(P + 2Q\frac{dv}{du} + R\left(\frac{dv}{du}\right)^2 \right)}\, du$$

$$\text{Data}: u_1, u_2, v(u_1) = v_1,\ v(u_2) = v_2$$

$$x(u,v), y(u,v), z(u,v)$$

$$P = \left(\partial x/\partial u\right)^2 + \left(\partial y/\partial u\right)^2 + \left(\partial z/\partial u\right)^2;\ R = \left(\partial x/\partial v\right)^2 + \left(\partial y/\partial v\right)^2 + \left(\partial z/\partial v\right)^2;$$

$$Q = \left(\partial x/\partial u\right)\left(\partial x/\partial v\right) + \left(\partial y/\partial u\right)\left(\partial y/\partial v\right) + \left(\partial z/\partial u\right)\left(\partial z/\partial v\right)$$

Now, with a constraint.

Geodesic problems have an objective function, which is an integral. The integral depended on the derivative of the variable function.

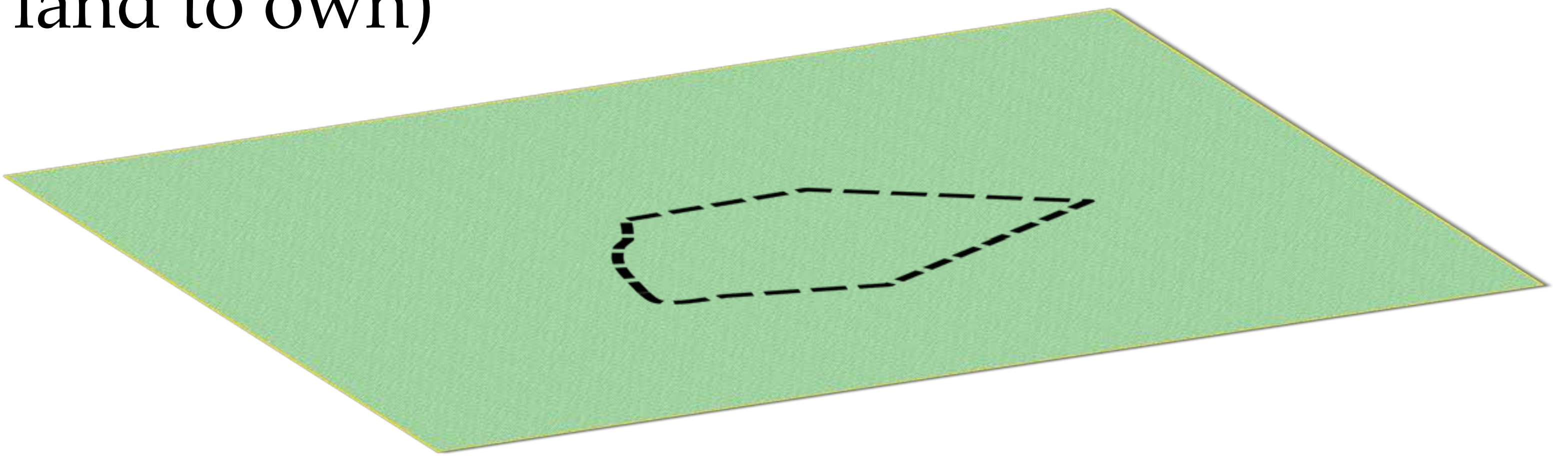
Now, we will consider a problem with a constraint that is also an integral of the variable function.

Such problems where the constraint is also an integral, we call them isoperimetric problems.

By the way, the expressions in the integral form are called **functionals**. But functionals need not be of only integral form. More later….

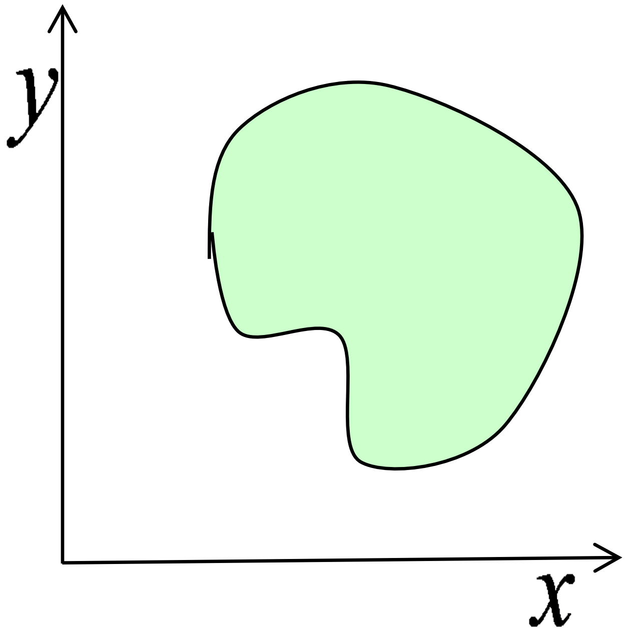
Queen Dido's "isoperimetric" problem

If someone gave you a closed loop of a chain of length L and asked you to take as much land you can enclose with it, as Dido, the Queen of Carthage (present day Tunisia) did, what shape would you put that chain on land? (provided you want to have maximum area of land to own)



Constant perimeter and hence it is called an **isoperimetric** problem.

Maximum area enclosed by a curve of given perimeter.



It is convenient to use parametric representation of a closed curve because explicit form *y(x)* may need to be multi-valued. Let $t = 0$ to L, be the parameter. Let the curve be given by *x(t)* and *y(t)*.

Perimeter

$$L = \int_0^L \left\{ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \right\} dt = \int_0^L \left(\sqrt{\dot{x}^2 + \dot{y}^2} \right) dt$$

Notation

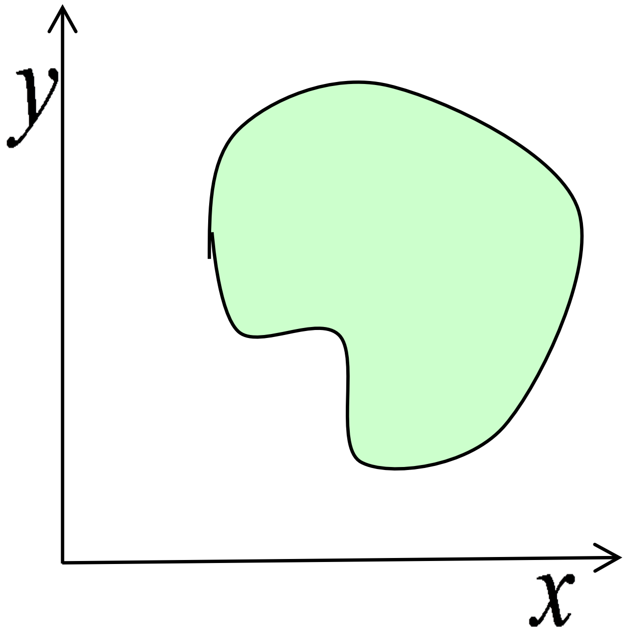
$$\dot{x} = \frac{dx}{dt}$$

$$\dot{y} = \frac{dy}{dt}$$

Enclosed area

$$= \int_0^L \frac{1}{2}\left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt} \right) dt = \int_0^L \frac{1}{2}\left(x\dot{y} - y\dot{x} \right) dt$$

Maximum enclosed area with a curve of given perimeter.



$$\underset{x(t),y(t)}{\text{Min}} \quad -A = \int_0^L \frac{1}{2}\left(y\dot{x} - x\dot{y}\right)\, dt$$

Subject to

$$\int_0^L \left(\sqrt{\dot{x}^2 + \dot{y}^2}\right) dt - L = 0$$

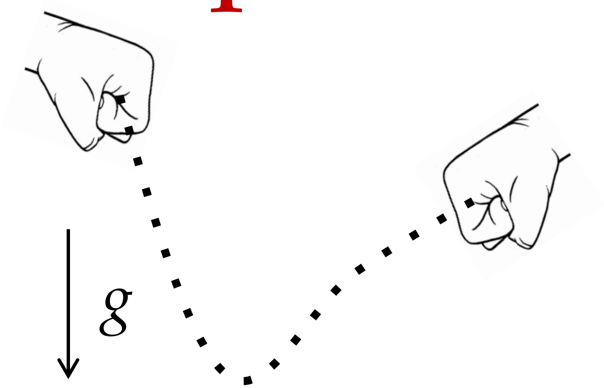
Data : L

Equality - constrained calculus of variations problem!

New features in problem formulation:

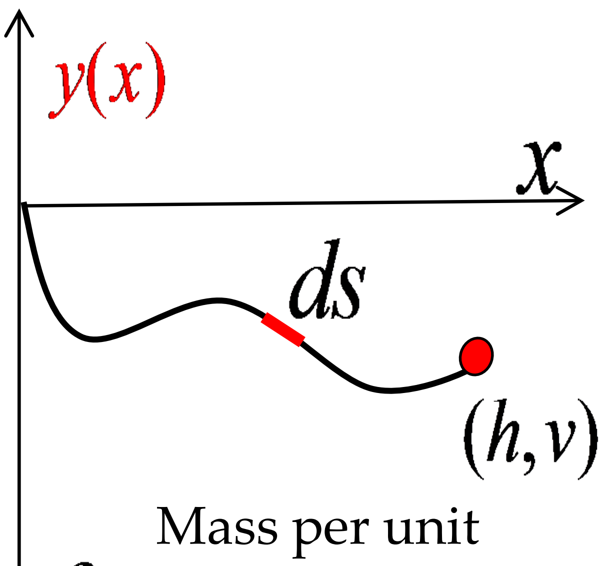
1. An integral (a form of functional) type constraint exists.
2. Two variable functions, x(t) and y(t), which need to be found.
3. Maximization problem can simply be made into a minimization problem by changing the sign.

Shape of a hanging chain



What shape does a chain held at its ends take when left freely under gravity?

It tries to minimize its potential energy by coming down as much as it could.



$\rho =$ Mass per unit length of the chain

$$\underset{y(x)}{\text{Min}} \quad PE = \int_0^h (\rho g y)\, ds = \int_0^h \rho g y \sqrt{1+y'^2}\, dx$$

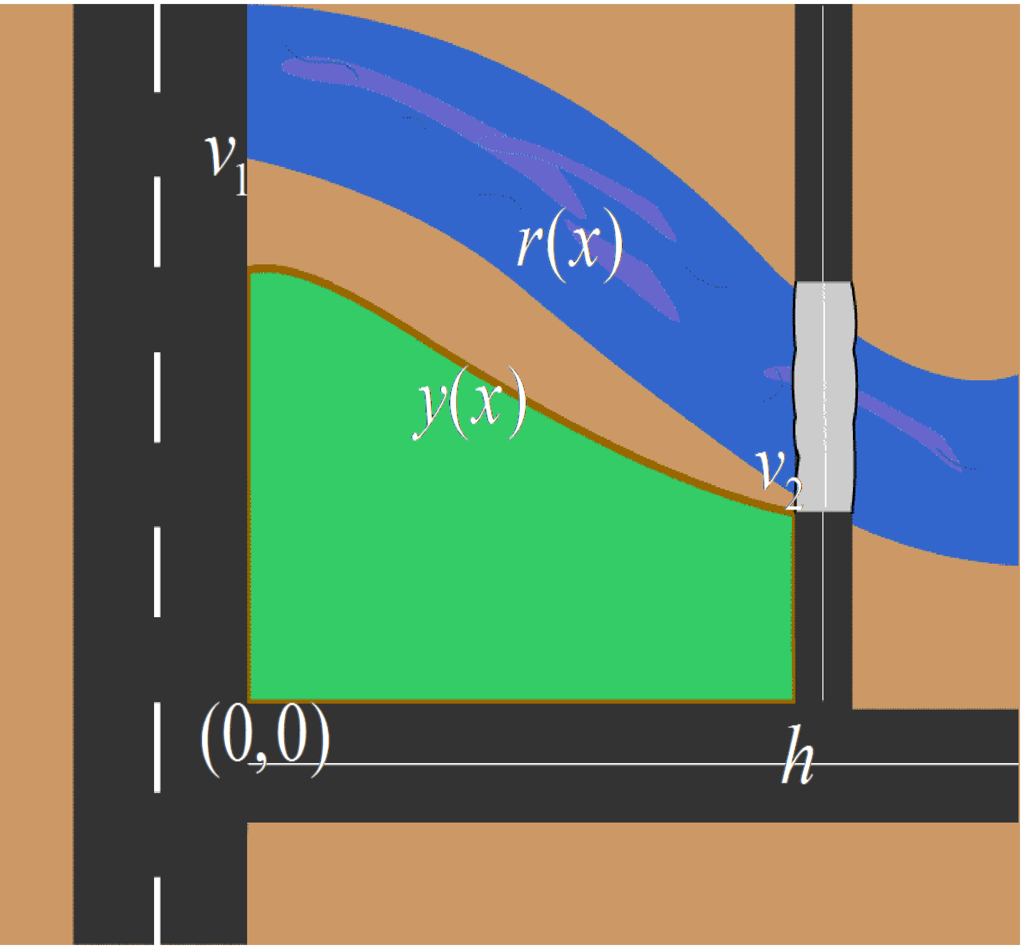
Subject to

$$\int_0^h \left(\sqrt{1+y'^2}\right) dx - L = 0$$

Data : $L, y(0)=0, h, y(h)=v, \rho, g$

Equality-constrained calculus of variations problem with one variable function.

Chatterjee problem: maximum enclosed area of a given perimeter with an inequality constraint



Posed by Prof. Anindya Chatterjee, IIT-Kanpur

A farmer is free to choose a field with a given length of fence bounded by a river and three roads as shown in the figure on the left. What should be the curve to maximize the enclosed area?

$$\underset{y(x)}{\text{Min}} \; -A = -\int_0^h y \, dx$$

Subject to

$$\int_0^h \left(\sqrt{1+y'^2} \right) dx - L = 0$$

$$y(x) - r(x) \le 0$$

$$\text{Data}: L, y(0) = v_1, y(h) = v_2$$

New feature:
An inequality constraint

Geometry and calculus of variations

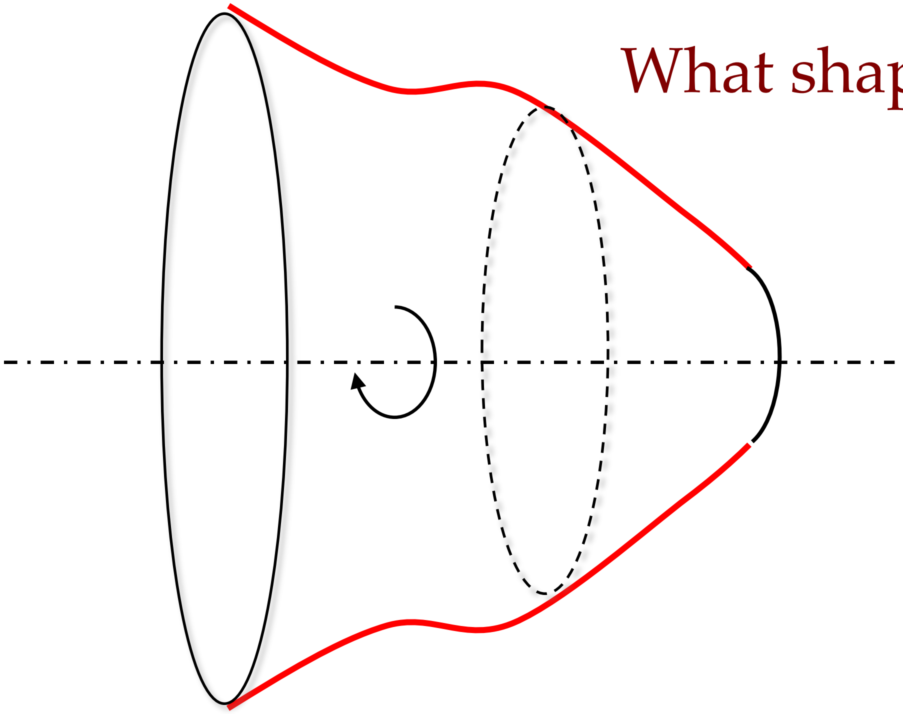
There are many problems in geometry that relate to calculus of variations.

They pertain to minimal curves and surfaces.

Minimal curves
- Geodesics
- Maximum enclosing area for a givenWe will consider a few of them
- perimeter length
- Chains hanging in a force field
- Etc.

Minimal surfaces
- Minimum surface of revolution
- Surfaces of least area enclosed by a given boundary
- Etc.

Minimum surface of revolution of a curve



Here is a problem that looks exactly like the hanging chain problem as far as mathematical formulation is concerned.

So, don't you expect the solution to be the same as well?

$$\underset{y(x)}{\text{Min}} \quad S = \int_0^L 2\pi y \, ds = \int_{x_1}^{x_2} 2\pi y \sqrt{1+y'^2} \, dx$$

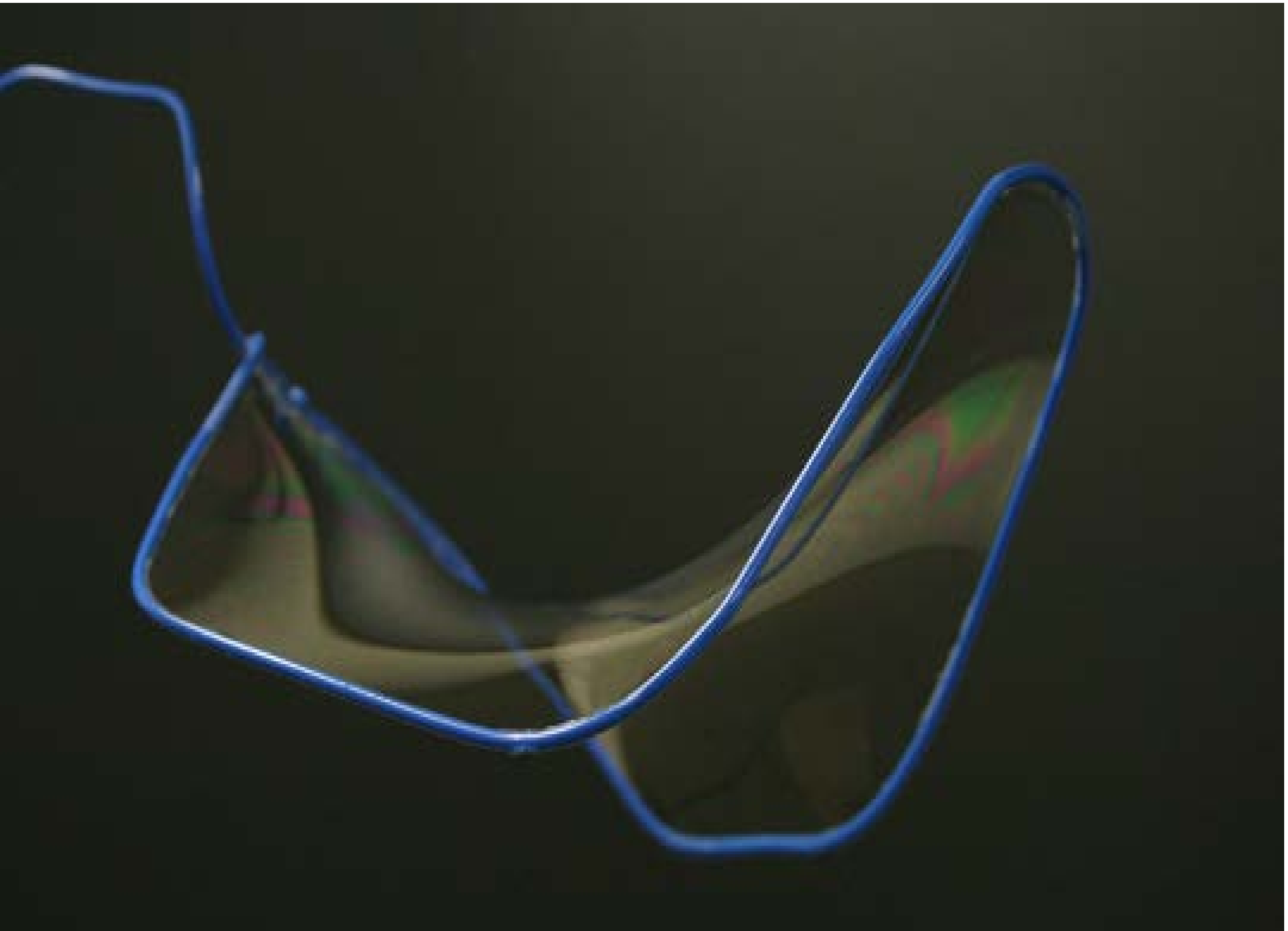
Subject to

$$\int_{x_1}^{x_2} \left(\sqrt{1+y'^2} \right) dx - L = 0$$

$$\text{Data}: L, x_1, y(x_1) = y_1, x_2, y(x_2) = y_2$$

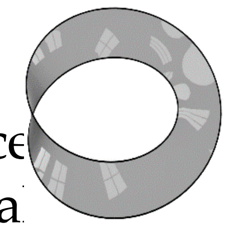
Given end points (x_1,y_1) and (x_2,y_2), find the curve which when rotated about the *x*-axis will have least surface of revolution.

Soap films solve a calculus of variations problem!



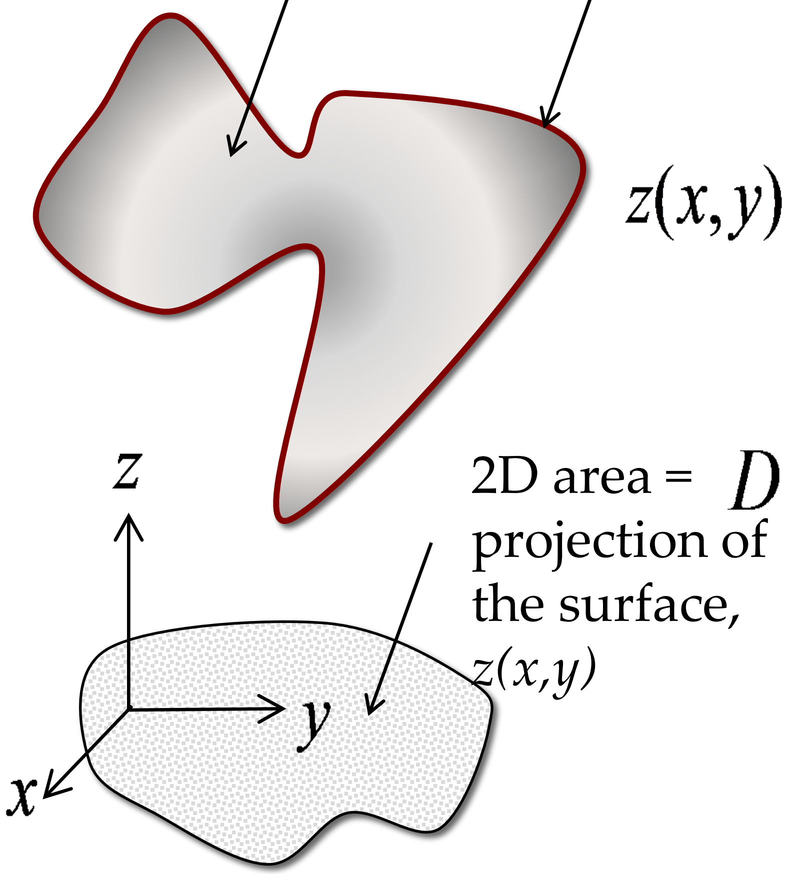
http://www.math.hmc.edu/~jacobsen/demolab/soapfilm.html

Take an easily bendable wire and make a loop or even multiple loops with it. Dip it in soap water and watch the shape of the soap film that forms.



Soap films want to minimize the surface tension and hence ta the surface of least area as they attach to the boundary of the wire.

Plateau's problem of least surface area for a given boundary curve in 3D (simpler version)



$z(x,y)$ = surface (single-valued)

$$\underset{z(x,y)}{\text{Min}} \quad S = \iint_D \left(\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \right) dx\, dy$$

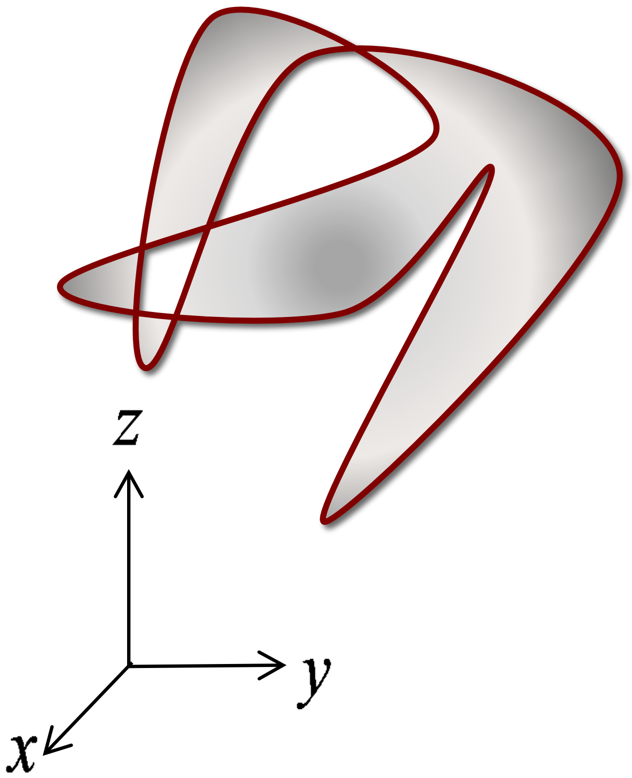
Data : D

New features:

The functional can be a double-integral.

The variable function can depend on two independent variables.

Plateau's problem of least surface area for a given boundary curve in 3D (more complex version)



What if the contour is irregular and it is multi-valued within the projected 2D domain D?

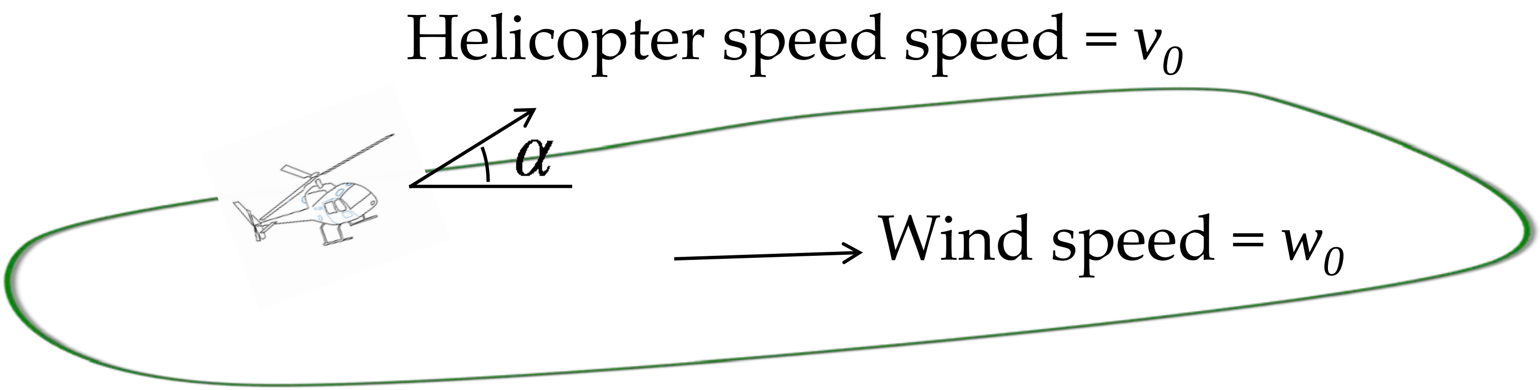
Posing and solving the problem become difficult.

Field's medals have

http://fathom-the-universe.tumblr.com/post/55740943330/the-beauty-of-minimal-surfaces-there-are-many

Douglas, Jesse (1931). "Solution of the problem of Plateau". *Trans. Amer. Math. Soc.* (Transactions of the American Mathematical Society, Vol. 33, No. 1) **33** (1): 263–321.

An optimal control problem: area maximization problem with optimal steering



A surveillance helicopter travelling at constant speed (v_o) under the constant wind speed of (w_0) needs to enclose maximum area by taking a closed path in a given time T. The optimization variable is the steering angle, $\alpha(t)$. The starting point is (x_0,y_0).

$$\underset{\alpha(t)}{\text{Min}} \; -A = -\frac{1}{2}\int_0^T \left[v_0 \sin\alpha(t) \left\{ x_0 + w_0 t + v_0 \int_0^t \cos\alpha(\tau)d\tau \right\} - \left\{ v_0 \cos(t) + w_0 \right\} \left\{ y_0 + v_0 \int_0^t \sin\alpha(\tau)d\tau \right\} \right] dt$$

$$\text{Data}: w_0, v_0, x_0, y_0, T$$

Study this functional…

$$\underset{\alpha(t)}{\text{Min}} \; -A = -\frac{1}{2}\int_0^T \left[v_0 \sin\alpha(t) \left\{ x_0 + w_0 t + v_0 \int_0^t \cos\alpha(\tau)\, d\tau \right\} - \left\{ v_0 \cos(t) + w_0 \right\} \left\{ y_0 + v_0 \int_0^t \sin\alpha(\tau)\, d\tau \right\} \right] dt$$

Data : w_0, v_0, x_0, y_0, T

The objective functional in this problem is interesting. Its **new feature** is that it is an integral but it has integrals to be evaluated within it and those integrals have the unknown variable function in their integrands.
The purpose of these examples is to let us appreciate the variety of functionals. We will study the formal notion of a functional in a later lecture.

Mechanics and calculus of variations

There are three ways to write equations of statics and dynamics.

Two of these are related to calculus of variations.

- We will discuss them in this lecture and later too.

Structural optimization is essentially calculus of variations.

- What do we want to optimize in a structure?
- Stiffness, flexibility, strength, weight, cost, manufacturability, natural frequency, mode shape, stability, buckling loads, contact stress, etc.
- All of these can be posed as objective function and constraints in the framework of calculus of variations.

We will consider a few problems and formulate them in this lecture.

Three views of mechanics	Statics	Dynamics
A result of calculus of variation!	Force balance	F = ma
Calculus of variations	Principle of virtual work	D'Lambert principle
	Minimum potential energy principle	Hamilton's principle

Static equilibrium of a beam

Method 1: Force and moment balance approach

$$EI\frac{d^4w}{dx^4} = q(x)$$

This differential equation for the small transverse displacement *w(x)* of a beam under transverse load, *q(x)* is derived based on moment balance at a cross-section and the bending moment itself is computed based on force and moment balance.

Static equilibrium of a beam

Method 2: Minimum potential energy principle

$$\underset{w(x)}{\text{Min}} \; PE = \int_0^L \left\{ \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw \right\} dx$$

$$\text{Data}: q(x), E, I$$

As an alternative to force/moment balance, we can simply minimize the potential energy (*PE*) with respect to the unknown variable function, *w(x)*.

The solution to this calculus of variations problem is the differential equation shown in the pervious slide.

Static equilibrium of a beam

Method 3: Principle of virtual work

$$\int_0^L EI\left(\frac{d^2 w}{dx^2}\right)\left(\frac{d^2 \delta w}{dx^2}\right) dx = \int_0^L q\, \delta w\, dx$$

For all kinematically admissible $\delta w(x)$.

Internal virtual work = external virtual work

As the second alternative to force/moment balance, we can simply solve this equation that is valid for any kinematically admissible function,

. $\delta w(x)$

This statement is a consequence of the minimization of the potential energy functional of the previous slide.

But this is an independent way of stating static equilibrium!

Static equilibrium of a beam

Now, we know three **independent** ways of writing conditions for static equilibrium.

Method 1: Force/moment balance approach

- The differential equation with boundary conditions
- Called the strong form

Method 2: Principle of minimum potential energy (calculus of variations)

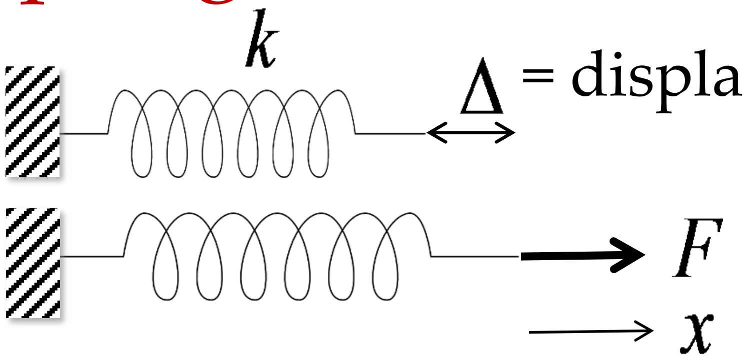
- All we need to know is an expression for the potential energy.
- The boundary conditions will emerge out of this statement.

Method 3: Principle of virtual work

- An intermediate result of calculus of variations
- Called also the weak form
- Notice that the highest order derivative of the unknown function is lower here as compared to the one in the strong form.

We will discuss details of Methods 2 and 3 in later lectures.

Understand the three methods with a simple spring.



Δ = displacement (stretch) of the spring at equilibrium

Since there is just one scalar variable *x*, it is a finite-variable optimization here and NOT calculus of variations.

Method 1
Force equilibrium

$$kx = F$$

Internal force = external force

Method 2
Minimum potential energy

$$\underset{x}{\text{Min}} \quad PE = \frac{1}{2}kx^2 - Fx$$

$$\frac{\partial PE}{\partial x} = 0 \Rightarrow kx = F$$

Method 3
Principle of virtual work

$$kx\,\delta x = F\,\delta x$$

Internal virtual work = external virtual work

Static equilibrium of a general elastic body

Method 1
Force equilibrium

$$\nabla \cdot (\mathbf{D} : \varepsilon) + \mathbf{b} = 0 \quad \text{where} \quad \varepsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

Method 2
Minimum potential energy

$$\underset{\mathbf{u}}{\text{Min}} \;\; PE = \int_{\Omega} \left(\frac{1}{2} \boldsymbol{\varepsilon} : \mathbf{D} : \boldsymbol{\varepsilon} - \mathbf{b} \cdot \mathbf{u} \right) d\Omega$$

$$\text{Data} : \mathbf{D}, \mathbf{b}, \Omega$$

Method 3
Principle of virtual work

$$\int_{\Omega} (\varepsilon : \mathbf{D} : \delta\varepsilon) \, d\Omega = \int_{\Omega} (\mathbf{b} \cdot \delta\mathbf{u}) \, d\Omega =$$

We will discuss the notation and derivations in later lectures.

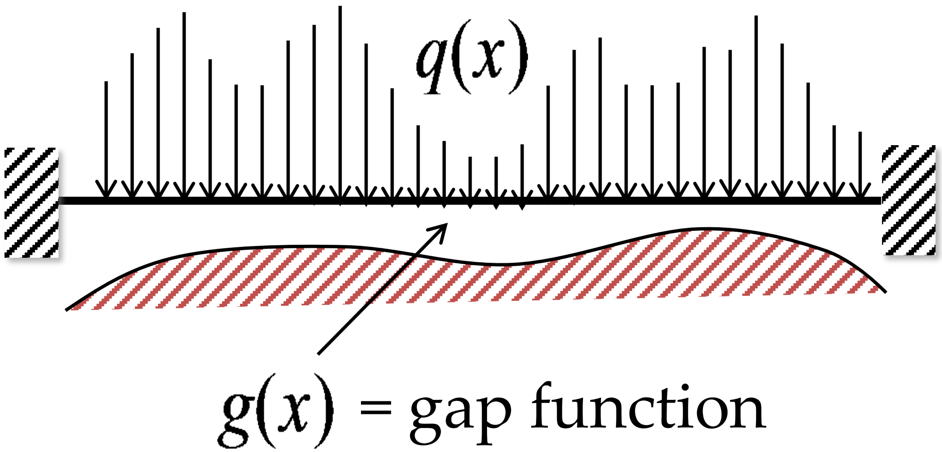
Contact problems in elasticity: beam

$$\underset{w(x)}{\text{Min}}\ PE = \int_0^L \left\{ \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw \right\} dx$$

Subject to

$$w(x) - g(x) \le 0$$

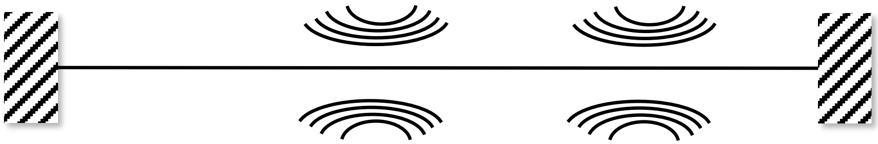
Data : $q(x), E, I$



Calculus of variations problem, in the framework of minimum potential energy principle, can easily account for contact conditions, as shown here.

Just an inequality constraint!

Vibrating string: Hamilton's principle



A taut vibration string with tension, T.
Length = L; mass per unit length = ρ

$$T\frac{\partial^2 w}{\partial x^2} = \rho\frac{\partial^2 w}{\partial t^2}$$

Equation of motion obtained using force-balance.

$$\underset{w(x,t)}{\text{Extremize}} \ H = \frac{1}{2}\int_{t_1}^{t_2}\int_0^L \left(\rho\dot{w}^2 - Tw'^2\right) dx\, dt$$

Calculus of variations statement: **Hamilton's principle**

Notice that it is not minimization or maximization; it is simply **extremization of a functional**; also notice that the variable function depends on space variable x and time variable t.

Equation of motion of a beam

$$\rho \frac{d^2 w}{dt^2} + EI \frac{d^4 w}{dx^4} = q(x)$$

Equation of motion obtained using force-balance.

$$\underset{w(x,t)}{\text{Extremize}} \; H = \frac{1}{2} \int_{t_1}^{t_2} \int_0^L \left\{ \frac{1}{2} \rho \left(\frac{\partial w}{\partial t} \right)^2 - \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 + q w \right\} dx \, dt$$

Calculus of variations statement: **Hamilton's principle**

Which function *w(x,t)* will extremize *H*, the Hamiltonian?

Mechanics and calculus of variations

There are three ways to write equations of statics and dynamics.

Two of these are related to calculus of variations.
- We will discuss them in this lecture and later too.

Structural optimization is essentially calculus of variations.
- What do we want to optimize in a structure?
- Stiffness, flexibility, strength, weight, cost, manufacturability, natural frequency, mode shape, stability, buckling loads, contact stress, etc.
- All of these can be posed as objective function and constraints in the framework of calculus of variations.

We will consider a few problems and formulate them in this lecture.

Three views of mechanics	Statics	Dynamics
A result of calculus of variation!	Force balance	F = ma
Calculus of variations	Principle of virtual work	D'Lambert principle
	Minimum potential energy principle	Hamilton's principle

Objectives and constraints in structural optimization

Weight

Stiffness

Strength

Flexibility

Cost

Stability

Buckling load

Natural frequency

Mode shape

Dynamic response

Contact stress

Etc.

Any of these can be the objective function or be part of a constraint.

Variable functions, the design variables, will be related shape and size; and topology (how many holes are there?)

Structural optimization of a beam

Minimize the strain energy of the beam for an upper bound on the volume of material.

$$\underset{b(x)}{\text{Min}}\ SE = \int_0^L \left\{ \frac{1}{2} \frac{Ebd^3}{12} \left(\frac{d^2 w}{dx^2} \right)^2 \right\} dx$$

The less the strain energy, the stiffer the beam.

The breadth of the beam is the design variable.

The displacement of the beam (w(x)) is the state variable.

Subject to

$$\frac{d^2}{dx^2}\left(Ebd^3 \frac{d^2 w}{dx^2} \right) + q = 0$$

The **governing equation** (the equilibrium equation) for the state variable.

$$\int_0^L bd\, dx - V^* \le 0$$

The **volume constraint** is an inequality.

$$\text{Data}: L, q(x), d, V^*, E$$

Data constitutes the known quantities.

This will be the typical structure of any structural optimization problem.

Min-max of stress: design for a strong beam

Minimize the maximum stress for an upper bound on the volume of material.

$$\underset{b(x)}{\text{Min}}\ \underset{x}{\text{Max}}\left(\sigma = \frac{1}{2}Edw''\right)$$

Subject to

$$\frac{d^2}{dx^2}\left(Ebd^3\frac{d^2w}{dx^2}\right)+q=0$$

$$\int_0^L bd\,dx - V^* \le 0$$

$$\text{Data}: L, q(x), d, V^*, E$$

New feature in the formulation:
The functional has another maximization problem in it.
This is a min-max problem.
Note that minimization and maximization of the same quantity is with respect to two different variables.
They are not uncommon in structural optimization.

Electro-thermal-compliant actuator design

$$\underset{\rho(x,y)}{\text{Min}} \left(-u_{out}\right)$$

Subject to

$$\int_{\Omega} t\, \nabla^T V\, k_e \nabla^T V_v\, d\Omega = 0$$

$$\int_{\Omega} t\, \nabla^T T\, k_t \nabla^T T_v\, d\Omega - \int_{\Omega} t\, \nabla^T V\, k_e \nabla^T V\, d\Omega = 0$$

$$\int_{\Omega} t \left(\varepsilon^T \mathbf{E} \varepsilon_v - \left\{ 1 \quad 1 \quad \alpha T \right\} \mathbf{E} \varepsilon_v \right) d\Omega$$

$$\int_{\Omega} t\, d\Omega - V^* \leq 0$$

$$\text{Data}: \Omega, V^*, k_e = k_{e0}\rho^\eta, k_t = k_{t0}\rho^\eta, \alpha = \alpha_0 \rho^\eta, \mathbf{E} = \mathbf{E}\ \rho^\eta$$

New features in the formulation:

The functional is simply one variable, the displacement at a point.

There are three governing equations pertaining to electrical, thermal, and elastic problems.

There are six state variables, V, V_v, T, T_v, $\boldsymbol{u}$, $\boldsymbol{u}_v$.

Features of calculus of variations problems

There can be constraints which are functionals or functions.

Constraints can be equalities are inequalities.

Objective functions are always functionals.

A functional can be of many forms.

- Just an integral
- Ratio of integrals
- Integral with another integral inside it
- Maximum or a minimum of a function
- Etc.

You have now seen what a functional is, in many of its forms. We will learn about them formally after a brief de tour of theory of finite-variable optimization.

The end note

Calculus of variations in geometry and mechanics

Many problems in geometry can be posed as calculus of variations problems.
Curves of least length and surfaces of least area are popular.

Mechanics problems can be posed in three different ways;
Two of them are directly under the purview of calculus of variations.

Structural optimization problems are essentially calculus of variations problems.

Constraints can be equalities and inequalities in calculus of variations too.

Functionals can be…

- **Integrals**
- **Integrals within an integral**
- **Ratio of two integrals**
- **Min or max of a function**
- **Can depend on more than one variable function**
- **Can involve more than one independent variable**
- **Can depend on space and time variables**

Thanks