Mathematical Preliminaries to Calculus of Variations (contd.)

Banach space

A complete normed vector space is called a *Banach space*.

A normed vector space X is *complete* if every *Cauchy sequence* from X has a limit in X.

A sequence $\{x_n\}$ in a normed vector space is said to be *Cauchy* (or fundamental) *sequence* if $\|x_n - x_m\| \to 0 \quad \text{as } n, m \to \infty$

In other words, given $\varepsilon > 0$ there is an integer N such that $\|x_n - x_m\| < \varepsilon$ for all $m, n > N$

$x \in X$ is called a *limit* of a convergent sequence $\{x_n\}$ in a normed vector space if the sequence $\{\|x - x_n\|\}$ converges to zero. In other words, $\lim_{n \to \infty} \|x - x_n\| = 0$.

Verifying if a given normed vector space is a Banach space requires an investigation into the limit of all Cauchy sequences. This needs tools of *real analysis*. We are not going to discuss them here. But let us try to relate to these sequences from a practical viewpoint and why we should worry about them.

In the context of structural optimization, we can interpret *sequences* as candidate designs that we obtain in a sequence in iterative numerical optimization. A design in structural optimization is represented with a function. As you may be aware, any numerical optimization technique needs an initial guess, which is improved in each iteration. Thus, we start with a function (vector) and then iteration 1 gives another function, iteration 2 yet another, and so on. Therefore, we get a sequence of “vectors” (i.e., functions). Whether such a sequence converges at all or converges to a limit within the space we are concerned with, are practically relevant questions. The abstract notion of a complete normed vector space helps us in this regard. So, it is useful to know the properties of a function space that we are dealing with. It is one way of knowing if numerical optimization would converge to a limit, which will be our optimal solution.

Hilbert space

A complete inner product space is called a *Hilbert space*.

An *inner product space* (or *pre-Hilbert space*) is a vector space X with an inner product defined on it.

An *inner product* on a vector space X is a mapping $X \times X$ into a scalar field K of X denoted as $\langle x, y \rangle, \quad x, y \in X$ and satisfies the following properties:

(i) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(iii) $\langle x, y \rangle = \langle \overline{y, x} \rangle$

The over bar denotes conjugation and is not necessary if x, y are real.

(iv) $\langle x, x \rangle \geq 0$ and

$\langle x, x \rangle = 0$ if and only if $x = \theta$

Note the following relationship between a norm and an inner product.

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Note also the relationship between a metric and an inner product.

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}$$

As an example, for $C^0[a,b]$, the norm and inner product defined as follows.

$$\|x\| = \sqrt{\int_a^b x^2(t)\,dt} = \sqrt{\langle x, x \rangle}$$

$$\langle x, y \rangle = \int_a^b x(t)\,y(t)\,dt$$

Thus, inner product spaces are normed vector spaces. Likewise, Hilbert spaces are Banach spaces.

Normed vector spaces give us the tools for algebraic operations to be performed on vector spaces because we have the notion of how close things ("vectors") are to each other by way of norm. Inner product spaces enable us to do more; they allow us to study the geometric aspects. As an example, consider that orthogonality (or perpendicularity) or lack of it is easily noticeable from the inner product.

For $x, y \in X$, if $\langle x, y \rangle = 0$, then x is said to be orthogonal to y

Banach and Hilbert spaces are classes of useful function spaces (again remember that a function space is only one type of the more general concept of a vector space). There are also some specific function spaces that we should be familiar with as they are the spaces to which the design spaces that we consider in structural optimization actually belong.

Lebesgue space

A *Lebesgue space* defined next is a Banach space.

$$L^q(\Omega) = \left\{ v : v \text{ is defined on } \Omega \text{ and } \|v\|_{L^q(\Omega)} < \infty \right\}$$

where $$\|v\|_{L^q(\Omega)} = \left(\int_\Omega |v(x)^q| \, dx \right)^{1/q} \qquad 1 \le q < \infty$$

The case of $q = 2$ gives $L^2(\Omega)$ consisting of all square-integrable functions. The integration of square of a function is important for us as it often gives the energy of some kind. Think of kinetic energy which is a scalar multiple of the square of the velocity. On many occasions, we also have other energies (usually potential energies or strain energies) that are squares of derivatives of functions. This gives us a number of energy spaces. The Sobolev space gives us exactly that.

Sobolev space

$$W^{r,q}(\Omega) = \left\{ v \in L^1(\Omega) : \|v\|_{W^{r,q}(\Omega)} < \infty \right\}, \qquad 1 \le q \le \infty$$

where

$$\left. \|v\|_{W^{r,q}(\Omega)} = \left(\sum_{|\alpha| \le r} \|D^\alpha v\|_{L^q(\Omega)}^q \right)^{1/q} \right\} \text{ is the } \textit{Sobolev norm}$$

$$L^1(\Omega) = \left\{ v : v \in L^1(K) \text{ for any compact } K \text{ inside } \Omega \right\}$$

D^α used above denoted the derivative of order α. Sobolev space is a Banach space.

<u>Note:</u> We have used the qualifying word "compact" for K above. A closed and bounded set is called a compact set. We will spare us from the definitions of closedness and boundedness of a set because we have already deviated from our main objective of knowing what a functional is. Let us return to functionals now.

We have defined a functional as a particular case of an operator whose range is a real (or complex) number set. Let us also consider another definition which says the same thing but in a different way as we have talked much about vector spaces and fields.

Functional—another definition

A *functional* J is a transformation from a vector space to its coefficient field $J : X \to K$.

Let us now look at certain types of functionals that are of main interest to us.

A *linear functional* is one for which

$J(x+y) = J(x) + J(y)$ for all $x, y \in X$ and $J(\alpha x) = \alpha J(x)$ for all $\alpha \in K,\ x \in X$

hold good. Some people write the above two linearity properties as a single property as follows.

$J(\alpha x + \beta y) = \alpha J(x) + \beta J(y)$ for all $x, y \in X;\ \alpha, \beta \in K$

A definite integral is a linear functional. We will deal with a lot of definite integrals in calculus of variations as well as variational methods and structural optimization.

A *bounded functional* is one when there exists a real number c such that $\left|J(x)\right| \le c\|x\|$ where $|\cdot|$ is the norm in K; $\|\cdot\|$ is the norm in X .

Continuous functional

Now, we have discussed in which function spaces our functions reside. In calculus of variations, our unknowns are functions. Our objective is a functional. Just as in ordinary finite-variable optimization, in calculus of variations too we need to take derivatives of functionals. What is the equivalent of a derivative for a functional? Before we define such a thing, we need to understand the concept of continuity for a functional. We do that next.

A functional J is said to be continuous at x in D (an open set in a given normed vector space X) if J has the limit $J(x)$ at x . Or symbolically, $\lim_{y\to x\in X} J(y) = J(x)$.

is said to be *continuous* on D if J is continuous at each vector in D.

J has the limit L at x if for every positive number ε there is a ball $B_r(x)$ (with radius r) contained in D such that $\left|L-J(y)\right|<\varepsilon$ for all $y\in B_r(x)$. Or symbolically, $\lim\limits_{y\to x\in X} J(y)=L$.