Midterm 2017

Problems from Midterm 2017

ME 256 at the Indian Institute of Science, Bengaluru

Variational Methods and Structural Optimization

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Outline of the lecture

Three problems and their solutions from the midterm examination of 2017.

What we will learn:

How to apply the concepts and ideas learned so far to solve problems in calculus of variations.

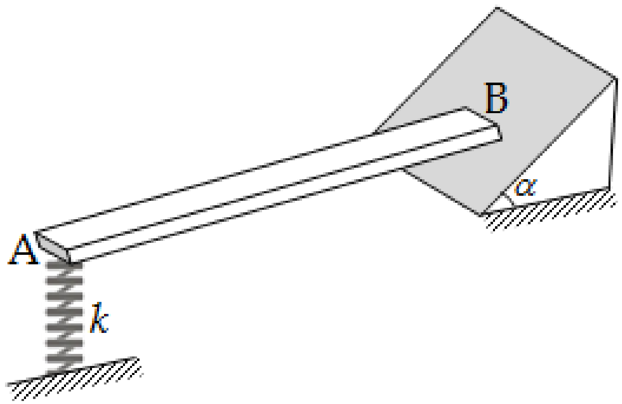
Midterm question paper

Question 1 *(7 marks)*
Find the Gâteaux variation of the following functional.

$$J = \left\{ \int_{x_1}^{x_2} \left\{ f^2(x)\ inv(f(x)) \right\} dx \right\}$$

Note: $\left\{ inv(f(x)) \right\}' = \dfrac{1}{f'\left(inv(f(x)) \right)}$

Question 2 *(8 marks)*
Write the boundary conditions at A and B for the beam shown in the figure.



Question 3 *(10 marks)*
Find the functional, which when extremized, would give the differential equation:
$\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = c\dfrac{\partial \phi}{\partial t}$.

Question 1 *(7 marks)*

Find the Gâteaux variation of the following functional.

$$J = \left\{ \int_{x_1}^{x_2} \left\{ f^2(x)\ inv(f(x)) \right\} dx \right\}$$

Solution to problem 1

$$J = \int_{x_1}^{x_2} \left\{ f(x)^2 inv\left(f(x) \right) \right\} dx$$

Question 1 *(7 marks)*
Find the Gâteaux variation of the following functional.

$$J = \left\{ \int_{x_1}^{x_2} \left\{ f^2(x)\; inv(f(x)) \right\} dx \right\}$$

Begin with the definition of Gateaux variation:

$$\delta_{f(x)} J = \frac{\partial}{\partial \varepsilon} \left(\int_{x_1}^{x_2} \left[\left\{ f(x) + \varepsilon h(x) \right\}^2 inv\left(f(x) + \varepsilon h(x) \right) \right] dx \right) \Bigg|_{\varepsilon=0}$$

Interchange the order of differentiation and integration.

$$\delta_{f(x)} J = \int_{x_1}^{x_2} \frac{\partial}{\partial \varepsilon} \left[\left\{ f(x) + \varepsilon h(x) \right\}^2 inv\left(f(x) + \varepsilon h(x) \right) \right] dx \Bigg|_{\varepsilon=0}$$

Solution to Problem 1 (contd.)

$$\delta_{f(x)} J = \int_{x_1}^{x_2} \frac{\partial}{\partial \varepsilon}\left[\left\{ f(x)+\varepsilon h(x)\right\}^2 inv\left(f(x)+\varepsilon h(x)\right)\right] dx \Bigg|_{\varepsilon=0}$$

Product rule

$$= \int_{x_1}^{x_2} \left[2\left\{ f+\varepsilon h\right\} inv\left(f+\varepsilon h\right) h + \left\{ f+\varepsilon h\right\}^2 \frac{\partial}{\partial \varepsilon}\left\{ inv\left(f+\varepsilon h\right)\right\}\right] dx \Bigg|_{\varepsilon=0}$$

$$\frac{\partial}{\partial \varepsilon}\left\{ inv\left(f+\varepsilon h\right)\right\} = ?$$

How do we get this?

Note that $\displaystyle \frac{\partial}{\partial x}\left\{ inv(g(x))\right\} = \frac{1}{g'(inv(g(x))}$

Why is this true?

Solution to Problem 1 (contd.)

$g(inv(g(x)) = x$ Definition of the inverse of a function.

Differentiate both sides to get

To be read as g' of $inv(g(x))$

$$g'(inv(g(x))\frac{\partial}{\partial x}\{inv(g(x))\} = 1 \Rightarrow \frac{\partial}{\partial x}\{inv(g(x))\} = \frac{1}{g'(inv(g(x))}$$

Similarly, $$\frac{\partial}{\partial x}\{inv(\mathrm{a}+g(x))\} = \frac{1}{\left(a+g(x)\right)' \text{ of } (inv(\mathrm{a}+g(x))}$$

and $$\frac{\partial}{\partial x}\{inv(\mathrm{a}+g(x)\mathrm{b})\} = \frac{b}{\left(a+g(x)\mathrm{b}\right)' \text{ of } (inv(\mathrm{a}+g(x)\mathrm{b})}$$

Solution to Problem 1 (contd.)

An example to understand.

$$\frac{\partial}{\partial x}\left\{inv(\mathrm{a}+\sin(x)\,\mathrm{b})\right\} = \frac{b}{\cos\left\{\,inv(\mathrm{a}+\sin(x)\,\mathrm{b})\right\}} = \frac{b}{\sqrt{1-\left(\dfrac{x-a}{b}\right)^2}}$$

Note that $inv\left\{\mathrm{a}+\sin(x)\,\mathrm{b}\right\} = \sin^{-1}\left(\dfrac{x-a}{b}\right)$ and verify the above calculation.

$$\frac{\partial}{\partial x}\left\{inv(\mathrm{a}+g(x)\,\mathrm{b})\right\} = \frac{b}{\left(a+g(x)\,\mathrm{b}\right)' \text{ of } (inv(\mathrm{a}+g(x)\,\mathrm{b})}$$

By following the preceding equation, we can write:

$$\frac{\partial}{\partial \varepsilon}\left\{inv\left(f+\varepsilon h\right)\right\} = \frac{h}{(f+\varepsilon h)' \text{ of } inv\left(f+\varepsilon h\right)}$$

Solution to Problem 1

Substitute $$\frac{\partial}{\partial \varepsilon}\left\{inv\left(f+\varepsilon h\right)\right\}=\frac{h}{(f+\varepsilon h)' \text{ of } inv\left(f+\varepsilon h\right)}$$

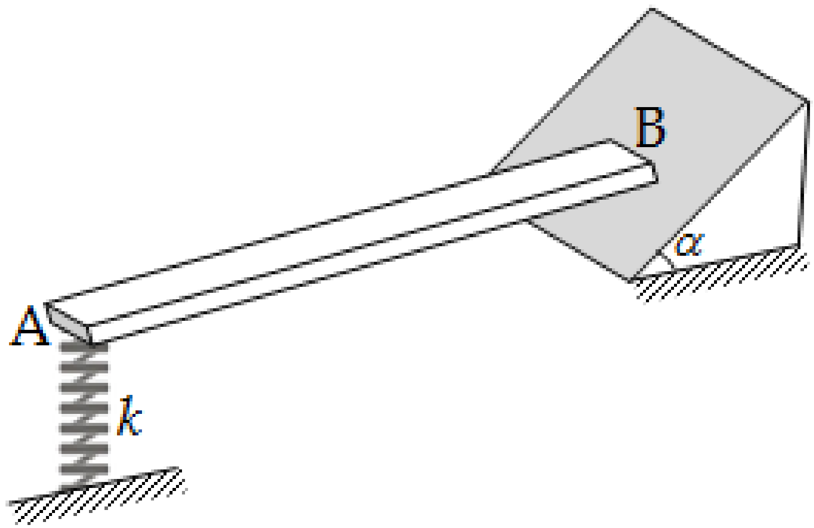
in $$\delta_{f(x)}J=\left.\int_{x_1}^{x_2}\left[2\left\{f+\varepsilon h\right\}inv\left(f+\varepsilon h\right)h+\left\{f+\varepsilon h\right\}^2\frac{\partial}{\partial \varepsilon}\left\{inv\left(f+\varepsilon h\right)\right\}\right]dx\right|_{\varepsilon=0}$$

$$\delta_{f(x)}J=\left.\int_{x_1}^{x_2}\left[2\left\{f+\varepsilon h\right\}inv\left(f+\varepsilon h\right)h+\left\{f+\varepsilon h\right\}^2\frac{h}{(f+\varepsilon h)' \text{ of } inv\left(f+\varepsilon h\right)}\right]dx\right|_{\varepsilon=0}$$

$$=\int_{x_1}^{x_2}\left[2f\,inv\left(f\right)+\frac{f^2}{f' \text{ of } inv\left(f\right)}\right]h\,dx$$

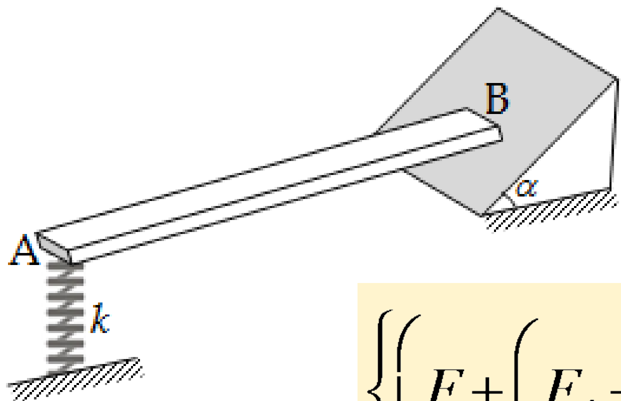
Question 2 *(8 marks)*

Write the boundary conditions at A and B for the beam shown in the figure.



Solution to Problem 2

Question 2 *(8 marks)*
Write the boundary conditions at A and B for the beam shown in the figure.



$$\left(F_{y''}\,\delta y'\right)\Big|_{x_1}^{x_2} = 0$$

$$\left\{\left(F_{y'} - \left(F_{y''}\right)'\right)\delta y\right\}\Big|_{x1}^{x_2} = 0 \quad \text{and}$$

$$\left\{\left(F - F_{y'}\,y' + \left(F_{y''}\right)' y' - F_{y''}\,y''\right)\delta x\right\}\Big|_{x1}^{x_2} = 0$$

$$\left\{\left(F + \left(F_{y'} - \left(F_{y''}\right)'\right)(\phi' - y') + F_{y''}\left(\phi'' - y''\right)\right)\delta x\right\}\Big|_{x1}^{x_2} = 0$$

$$\left(F_{y'} - \left(F_{y''}\right)'\right)\delta y\Big|_{x_1}^{x_2} = 0$$

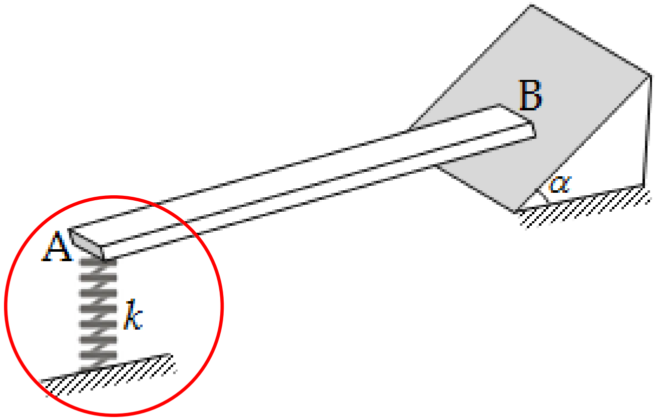
and

$$F_{y''}\,\delta y'\Big|_{x_1}^{x_2} = 0$$

In this problem, we minimize: (load is not considered)

$$\underset{w(x)}{\text{Min}}\; J = \int_0^L \left\{\frac{1}{2} EI\left(w''\right)^2\right\} dx + \frac{1}{2} k w^2_{x=0}$$

Solution to Problem 2



$$F = \frac{1}{2} EI \left(w'' \right)^2$$

$$\frac{1}{2} k w^2_{x=0}$$

This term in J leads to

$$\left(F_{y'} - \left(F_{y''} \right)' \right) \delta y \Big|_{x_1}^{x_2} = 0$$

and

$$F_{y''} \delta y' \Big|_{x_1}^{x_2} = 0$$

At A…

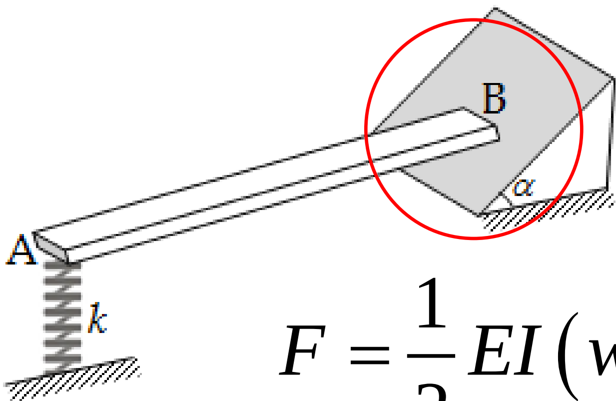
1 $$-EIw'''_{x=0} + kw_{x=0} = 0$$ Shear force is equal to the spring force.

and

2 $$EIw''\,\delta w'\Big|_{x=0} = 0 \Rightarrow EIw''\Big|_{x=0} = 0$$

Since the slope is not specified.

Solution to Problem 2



$$F = \frac{1}{2}EI\left(w''\right)^2 + \frac{1}{2}kw^2_{x=0}$$

$$\left\{\left(F + \left(F_{y'} - \left(F_{y''}\right)'\right)\right)(\phi' - y') + F_{y''}(\phi'' - y'')\right\}\delta x\Bigg\}\Bigg|_{x1}^{x_2} = 0$$

At B…

$$\left\{\left(\frac{1}{2}EI\left(w''\right)^2 - \left(EIw''\right)'\left(\tan\alpha - w'\right) + EIw''\left(0 - w'''\right)\right)\right\}\Bigg|_L = 0$$ 3

This assumes that the slope at B is not permitted to change.

So, $w'_{x=L} = 0$ (This must be specified.) 4a

If slope at B is permitted to change…

$$\delta y' = \delta w' = \phi''\delta x$$

But here, $\phi''_{x=L} = 0$

So, we should consider:

$$\left(F_{y''}\,\delta y'\right)\Big|_{x_1}^{x_2} = 0$$

At B…

$$\left(EIw''\,\delta w'\right)\Big|_{x=L} = 0$$

If we assume that the slope at B is allowed to change,

$$\left(EIw''\right)\Big|_{x=L} = 0$$ 4b

i.e., moment is zero.

Question 3 *(10 marks)*

Find the functional, which when extremized, would give the differential equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = c\,\frac{\partial \phi}{\partial t}.$$

Solution to Problem 3

Question 3 *(10 marks)*
Find the functional, which when extremized, would give the differential equation:
$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=c\frac{\partial \phi}{\partial t}$.

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=c\frac{\partial \phi}{\partial t}$$

Given differential equation.

$$\int_\Omega \left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}-c\frac{\partial \phi}{\partial t}\right)\psi \, d\Omega = 0$$

Multiply by a trial function and integrate.

Combine and split.

$$\Rightarrow \int_\Omega \left(\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\psi\right)+\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\psi\right)-\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x}-\frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y}-c\frac{\partial \phi}{\partial t}\psi\right)d\Omega = 0$$

Separate the terms.

$$\Rightarrow \int_\Omega \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y}+c\frac{\partial \phi}{\partial t}\psi\right)d\Omega - \int_\Omega \left(\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\psi\right)+\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\psi\right)\right)d\Omega = 0$$

Solution to Problem 3

$$\int_{\Omega}\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}+c\frac{\partial\phi}{\partial t}\psi\right)d\Omega-\int_{\Omega}\left(\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\psi\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\psi\right)\right)d\Omega=0$$

Using Divergence theorem on the second term, and equating the domain and boundary terms to zero.

$$\Rightarrow\int_{\Omega}\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}+c\frac{\partial\phi}{\partial t}\psi\right)d\Omega=0\ \text{and}\ \int_{\partial\Omega}\left(\frac{\partial\phi}{\partial x}\psi\,\hat{i}+\frac{\partial\phi}{\partial y}\psi\,\hat{j}\right)\cdot\hat{\mathbf{n}}\ d\partial\Omega=0$$

The term in red is asymmetric, making this differential operator non-self-adjoint.

The asymmetric term is a dissipative term. So, let us add a generative term.

$$\int_{\Omega}\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}+c\frac{\partial\phi}{\partial t}\psi-c\frac{\partial\psi}{\partial t}\phi\right)d\Omega=0$$

Solution to Problem 3

Consider this problem (written on the basis of the last equation in the preceding slide.

$$\underset{\phi(x,y,t),\,\psi(x,y,t)}{\text{Min}} \quad J = \int_{\Omega} \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} + \frac{c}{2}\frac{\partial \phi}{\partial t}\psi - \frac{c}{2}\frac{\partial \psi}{\partial t}\phi \right) d\Omega$$

$$\text{Data}: c$$

Noting that $F = \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} + \frac{c}{2}\frac{\partial \phi}{\partial t}\psi - \frac{c}{2}\frac{\partial \psi}{\partial t}\phi \right)$

we write two Euler-Lagrange equations.

$$\frac{\partial F}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \phi_y}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \phi_t}\right) = 0$$

$$\frac{\partial F}{\partial \psi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \psi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \psi_y}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \psi_t}\right) = 0$$

Solution to Problem 3 (contd.)

$$F = \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} + \frac{c}{2}\frac{\partial \phi}{\partial t}\psi - \frac{c}{2}\frac{\partial \psi}{\partial t}\phi \right)$$

$$\frac{\partial F}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \phi_y}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \phi_t}\right) = 0$$

$$\Rightarrow -\frac{c}{2}\frac{\partial \psi}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial t}\left(\frac{c}{2}\psi\right) = 0 \Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -c\frac{\partial \psi}{\partial t}$$

$$\frac{\partial F}{\partial \psi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \psi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \psi_y}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \psi_t}\right) = 0$$

This is what we need:

$$\Rightarrow \frac{c}{2}\frac{\partial \phi}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial t}\left(-\frac{c}{2}\phi\right) = 0 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = c\frac{\partial \phi}{\partial t}$$

The end note

Practice problems in calculus of variations

- For Gateaux variation, simply apply the operationally useful definition and follow the rules of differentiation. There is absolutely no trick involved.
- For beams, we should write all four boundary conditions as you we still have a fourth-degree differential equation.
- For "inverse E-L problem", first check if the differential operator is self-adjoint (i.e., there are no asymmetric terms).
  If the differential operator is not self-adjoint, it then means that there is "dissipation" in the systems. Then, try to use either an integrating factor (there is guesswork here) or a parallel generative system.
- Calculus of variations is straightforward to use if you are clear about all concepts. No other tricks are needed.

Thanks