Natural frequency

Hamilton's principle

$$\underset{u(x,t)}{\text{Extremize}} \quad A = \int_0^t \mathcal{L}\, dt$$

$\mathcal{L}$ = Lagrangian = KE − PE

Bar

$$KE = \int_0^L \frac{1}{2}(\rho A\, dx)\, \dot{u}^2 = \int_0^L \frac{1}{2} \rho A \dot{u}^2\, dx \qquad \dot{u} = \frac{du}{dt}$$

$$PE = SE + \underbrace{WP}_{=0} \text{ because of free vibration}$$

$$= \int_0^L \frac{1}{2} EA u'^2\, dx$$

$$\underset{u(x,t)}{\text{Extremize}} \quad A = \int_0^t \underbrace{\int_0^L \left[\frac{1}{2} \rho A \dot{u}^2 - \frac{1}{2} EA u'^2\right] dx}_{L}$$

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{u}}\right) = 0$$

$$0 - \left(-EAu'\right)' - \frac{\partial}{\partial t}\left(\rho A \dot{u}\right) = 0$$

$$(EAu')' - \rho A \ddot{u} = 0$$

$$u(x,t) = \phi(x)\, q(t)$$

$u' = \phi' q$ $\qquad$ $u'' = \phi'' q$

$\dot{u} = \phi \dot{q}$ $\qquad$ $\ddot{u} = \phi \ddot{q}$

Let $A(x) = A_0$

$\cancel{EA\phi'' q}$

$$E\cancel{A} u'' = \rho \cancel{A} \ddot{u}$$

$$E \phi'' q = \rho \phi \ddot{q}$$

$$\Rightarrow \quad \frac{E\phi''}{\rho\phi} = \frac{\ddot{q}}{q} = \text{Constant} = -\lambda$$

$$\ddot{q} + \lambda q = 0$$

$$\boxed{E\phi'' + \lambda \rho \phi = 0}$$

Eigenvalue problem when A is constant

$$\ddot{q} + \lambda q = 0$$

$q(t) = B \sin\sqrt{\lambda} t$ $\qquad$ $\dot{q} = +B\sqrt{\lambda} \cos\sqrt{\lambda} t$

$\ddot{q} = -B\sqrt{\lambda}^2 \sin\sqrt{\lambda} t$

@ $t=0$ $\quad \dot{q} = 0$ $\qquad$ $B \sin\sqrt{\lambda} t = 0$

Since $B \neq 0$

$\sin\sqrt{\lambda} t = 0$

Natural frequency: $\sqrt{\lambda_n} = n\pi$ $\qquad$ $n = 1, 2, \ldots \infty$

$E\phi_n'' + \lambda_n \rho \phi_n = 0$ $\qquad$ Eigenequation $\qquad$ No space b/w eigen ——

$\sqrt{\lambda} = \omega =$ Natural freq

$\phi_i =$ eigenfunction or mode shape

Properties of Eigenfunction

1) ϕ_i's are orthogonal $\Rightarrow \int_0^L \rho \phi_i \phi_j \, dx = 0$

$\phi_j (\; E\phi_i'' + \rho h_i \phi_i = 0 \;)$

$\phi_i (\; E\phi_j'' + \rho h_j \phi_j = 0 \;)$

$$E\phi_j \phi_i'' + \rho h_i \phi_i \phi_j = 0$$

$$E\phi_i \phi_j'' + \rho h_j \phi_i \phi_j = 0$$

$$\cancel{E\phi_j \phi_i'' = E\phi_i \phi_j''}$$

$$\cancel{\frac{\phi_i''}{\phi_i} = \frac{\phi_j''}{\phi_j}}$$

$$E\phi_j \phi_i'' - E\phi_i \phi_j'' + \rho(h_i - h_j)\phi_i \phi_j = 0$$

$$\int_0^L (E \underset{\mathrm{I}}{\phi_j} \underset{\mathrm{II}}{\phi_i''} - E\phi_j'' \phi_i)\, dx + \int_0^L \rho(h_i - h_j)\phi_i \phi_j \, dx = 0$$

$$\underbrace{E\phi_j \overset{=0}{\cancel{\phi_i'}}\Big|_0^L - \int_0^L \cancel{E\phi_j' \phi_i' \, dx} - E\overset{=0}{\cancel{\phi_i}}\phi_j'\Big|_0^L}_{\text{Boundary Condition lead to } 0} + \int_0^L \cancel{E\phi_j' \phi_i' \, dx} + \int_0^L \rho(h_i - h_j)\phi_i \phi_j \, dx = 0$$

$\therefore \int_0^L \rho(h_i - h_j)\phi_i \phi_j \, dx = 0$ when $\underline{\underline{i \neq j}}$

when $i=j$

2) $\int_0^L \rho \phi_i^2 \, dx = 1$ [Normalization Condition]

3) $\boxed{\underline{\phi(x)} = \sum_{i=1}^{\infty} c_i \phi_i(x)}$ Superposition of mode shapes

$\cancel{E\phi_i^{4} + \lambda_n \rho \phi_i = 0}$

$$\int_0^L \rho \phi \phi_i \, dx = \int_0^L \left(\sum_{j=1}^{\infty} c_j \phi_j \right) \rho \phi_i \, dx$$

$$= \sum_{j=1}^{\infty} \int_0^L \rho c_j \phi_j \phi_i \, dx = \sum_{j=1}^{\infty} \int_0^L c_j \phi_j^2 \, dx = c_i$$

$$\boxed{\int_0^L \rho \phi \phi_i \, dx = \underline{c_i}}$$

\# $\int_0^L \rho \phi^2 \, dx = 1$

$$\int_0^L \rho \phi \left[\sum_{i=1}^{\infty} c_i \phi_i(x) \right] dx = 1$$

$$\sum_{i=1}^{\infty} \left(c_i \int_0^L \rho \phi \phi_i \, dx \right) = 1$$

$$\boxed{\sum_{i=1}^{\infty} c_i^2 = 1}$$

In eigenvalue problem, always ~~$\phi(x) \int_0^L \rho \phi_i \, dx = 1$~~

$$\int_0^L (\;) \rho \phi_i \, dx = 1$$

(1) $E\phi_i'' + \lambda_i \rho \phi_i = 0$

Equation obtained after substituting $u = \phi(x) R(t)$

(2) $E\phi\phi_i'' + \lambda_i \rho\phi \rho \phi_i = 0$

$$\int_0^L (E\phi\phi_i'' + \lambda_i \rho \phi \phi_i)\, dx = 0$$

multiply by ϕ & integrate

$$\underbrace{E\phi\phi_i' \Big|_0^L}_{=0} - \int_0^L E\phi'\phi_i'\, dx + \lambda_i \int_0^L \rho\phi\phi_i\, dx = 0$$

Boundary Condition.

$$\lambda_i = \frac{\int_0^L E\phi'\phi_i'\, dx}{\int_0^L \rho\phi\phi_i\, dx}$$

$\int_0^L [w_i EI(w_i'')'' + T_i w_i'' w_i] dx$

$\overline{EI w_i''^2}$

Using orthogonal property $\phi = \sum_{i=1}^{\infty} c_i \phi_i$

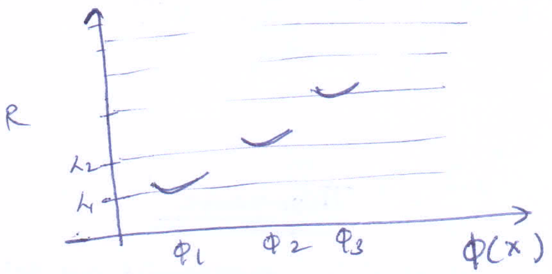
R = Rayleigh Quotient

$$\boxed{\lambda_i = \frac{\int_0^L E\phi_i'^2\, dx}{\int_0^L \rho\phi_i^2\, dx}}$$

$$\boxed{\lambda = \min(R) = \frac{\int_0^L E\phi'^2\, dx}{\int_0^L \rho\phi^2\, dx}}$$

A = Constant.

When ϕ become a particular mode shape then corresponding R ~~for~~ λ is ~~local~~ minimum (natural frequency)



$$\to \phi = \sum_{j=1}^{\infty} c_j \phi_j = c_i \phi_i + \sum_{j=1} \Delta c_j \phi_j$$

$$R = \frac{\int_0^L E(c_i \phi_i')^2 dx + \int_0^L E(\Delta c_j \phi_j)^2 dx}{\int_0^L \rho (c_i \phi_i)^2 dx + \int_0^L \rho (\cancel{\Delta c_j} \phi_j)^2 dx}$$

$\to 0$

Recap

free vibration of a bar
- eigenvalue problem using calculus of variations

Rayleigh quotient

$$u(x,t) = \phi(x)\, q(t)$$

$$\phi(x) = \sum_{i=1}^{\infty} c_i \phi_i(x)$$

When A is not constant

$$R = \frac{\int_0^L EA\phi'^2 dx}{\int_0^L \rho A \phi^2 dx}$$

$$\left[\left(\frac{\partial u}{\partial x}\right)^2 i + \left(\frac{\partial u}{\partial y}\right)^2 j\right]$$

Minimum characterization of eigenvalue problem

[Sturm - Liouville problems]

$$\underset{\phi(x)}{\text{Min}} \int_0^L E\phi'^2 \, dx = \lambda_k$$

Theorem

If given constraint are satisfied then min value of objfn will give eigenvalue λ_k.

subject to

$$\int_0^L \rho \phi^2 dx - 1 = 0$$

(Normalize $\phi = 1$)

$$\int_0^L \rho \phi_j \phi \, dx = 0 \qquad j = 1, 2, 3 - - (k-1)$$

(ϕ should be orthogonal to ϕ_j) ϕ_j can be state variable.

$$E\phi_i'' + \lambda_i \rho \phi_i = 0 \qquad i = 1, 2, - - - \infty$$

Governing eq^n for ϕ_i (Eigen Value)

$$\phi = \sum_{i=1}^{\infty} c_i \phi_i$$

$$\int_0^L \rho \phi^2 dx = \underline{\underline{\sum_{i=1}^{\infty} c_i^2 = 1}}$$

$$\int_0^L \rho \phi \phi_j \, dx = 0 \Rightarrow \underline{\underline{c_j = 0}}$$

$$E\phi_i'' + \lambda_i \rho \phi_i = 0 \quad \text{——} (3)$$

Minimize Quantity

$$\int_0^L E\phi'^2 dx = \int_0^L E\phi' \left(\sum_{i=k}^{\infty} c_i \phi_i' \right) dx$$

$i = 1, - - (k-1)$

$$\underline{\underline{\int \phi_i \phi = 0}}$$

$$\int_0^L E\phi'^2 dx = \sum_{i=k}^{\infty} c_i \int_0^L E \underset{2}{\phi'} \underset{1}{\phi_i'} \, dx$$

Integration by parts

0 (Coz of bc's)

Substitute from (3)

$$\sum_{i=k}^{\infty} c_i \left[\cancel{E\phi \phi_i' \Big|_0^L} - \int_0^L E\phi \phi_i'' \, dx \right]$$

$$= \sum_{i=K}^{\infty} c_i \int_0^L \lambda_i \rho \phi \phi_i \, dx$$

$$\sum_{i=K}^{\infty} \lambda_i c_i^2$$

Eigen value problem convert a continuous function into discrete function.

Now we want to prove then $\min \sum_{i=K}^{\infty} \lambda_i c_i^2 = \lambda_K$

$$= \sum_{i=K}^{\infty} \lambda_i c_i^2 + \lambda_K \Big(\underbrace{\sum_{i=K}^{\infty} c_i^2}_{\substack{=1 \\ \text{(kept as 1)}}} \Big) - \lambda_K \Big(\underbrace{\sum_{i=K}^{\infty} c_i^2}_{=1} \Big)$$

$$\int_0^L E \phi^2 \, dx = \lambda_K + \sum_{i=K}^{\infty} \underbrace{(\lambda_i - \lambda_K)}_{\geq 0} c_i^2$$

λ_i are ordered $\lambda_1 < \lambda_2 - - \lambda_K < \lambda_{K+1}$

$$\min \int_0^L E \phi^2 \, dx = \lambda_K$$

Q for free vibration of a beam

$$R = \frac{\int_0^L EI \phi''^2 \, dx}{\int_0^L \rho \phi^2 \, dx}$$

See how to find Rayleigh Quotient for diff cases

When we min R, we get λ_K

Max natural frequency of a bar

$$\max_{A(x)} \; \lambda = \min_{\phi(x)} \underbrace{\frac{\int_0^L EA\phi'^2\,dx}{\int_0^L \rho A \phi^2\,dx}}_{R}$$

subject to

$$\Lambda: \quad \int_0^L A\,dx - V^* \leq 0$$

Data: E, L, ρ, V^*

$$\mathcal{L} = \frac{\int_0^L EA\phi'^2\,dx}{\int_0^L \rho A\phi^2\,dx} + \Lambda\left(\int_0^L A\,dx - V^*\right)$$

$$\delta_\phi \mathcal{L} = 0$$

$$\Rightarrow \frac{\left(\int_0^L \rho A\phi^2\,dx\right)\left(\int_0^L 2EA\phi'\,\delta\phi'\,dx\right) - \left(\int_0^L EA\phi'^2\,dx\right)\left(\int_0^L 2\rho A\phi\,\delta\phi\,dx\right)}{\underbrace{\left(\int_0^L \rho A\phi^2\,dx\right)^2}_{>0}}$$

$$\underbrace{\left(\int_0^L EA\phi'\,\delta\phi'\,dx\right)} - \lambda\left(\int_0^L \rho A\phi\,\delta\phi\,dx\right) = 0$$

$$\cancel{EA\phi'\,\delta\phi\Big|_0^L} - \int_0^L (EA\phi')'\,\delta\phi\,dx - \lambda\int \rho A\phi\,\delta\phi\,dx = 0$$

$\delta\phi \rightarrow$ arbitrary

$$\boxed{(EA\phi')' + \lambda \rho A \phi = 0}$$

$$\delta_A L = 0$$

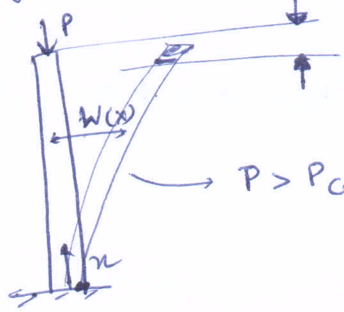
$$\frac{\left(\int_0^L \rho A \phi^2 dx\right)\left(\int_0^L E\phi'^2 \delta A\, dx\right) - \left(\int_0^L EA\phi'^2 dx\right)\left(\int_0^L \rho \phi^2 \delta A\, dx\right)}{\left(\int_0^L \rho A \phi^2 dx\right)^2} + \Lambda \int_0^L \delta A\, dx = 0$$

$$\cancel{\Lambda \int_0^L \delta A\, dx}$$

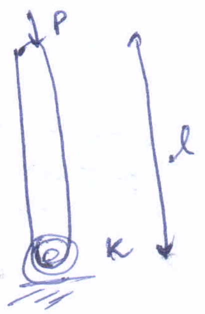
$$\left(\int_0^L E\phi'^2 \delta A\, dx\right) - \lambda\left(\int_0^L \rho\phi^2 \delta A\, dx\right) + \int_0^L \Lambda\, \delta A\, dx \left(\int_0^L \rho A \phi^2 dx\right) = 0$$

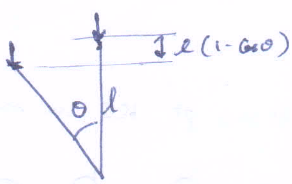
$$\boxed{E\phi'^2 - \lambda\rho\phi^2 + \Lambda\left(\int_0^L \rho A\phi^2 dx\right) = 0}$$ <u>Design Equation</u>

<u>Buckling</u>



$\longrightarrow P > P_{cr}$ (Bar Buckles)



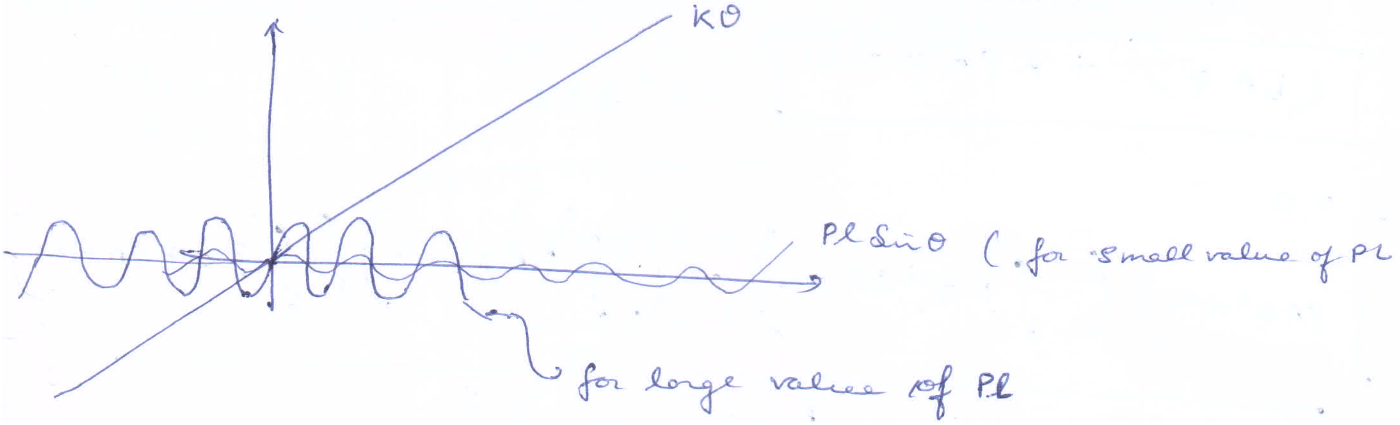


$$PE = \frac{1}{2}K\theta^2 + \left\{-Pl(1-\cos\theta)\right\}$$

for equilibrium

$$\frac{\partial PE}{\partial \theta} = 0 \Rightarrow K\theta - Pl\sin\theta = 0 \quad \text{—①}$$

$$\frac{\partial^2 PE}{\partial \theta^2} = K - Pl\cos\theta \quad \text{—②}$$

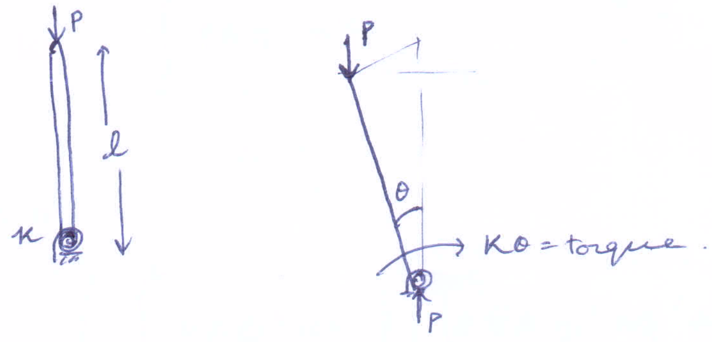


$$\frac{\partial^2 PE}{\partial \theta^2} > 0 \Rightarrow \text{min PE} \quad \text{So all Solution.}$$

$$K - Pl\cos\theta > 0$$

for $P > P_{cr}$ equilibrium straight position is no longer stable

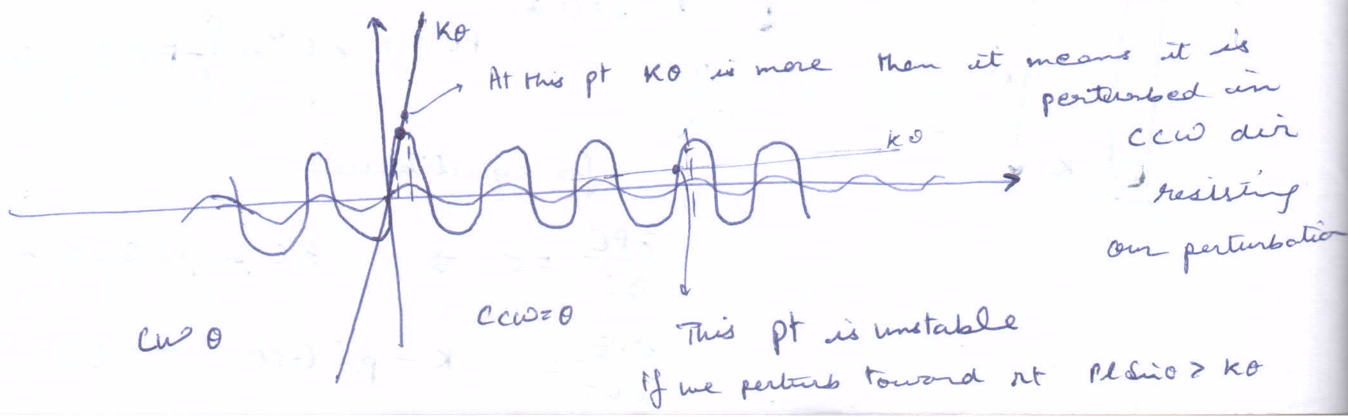
Buckling Stability



$Pl\sin\theta = K\theta$ moment balance

$$PE = \frac{1}{2}K\theta^2 + \{-P(l - l\cos\theta)\}$$

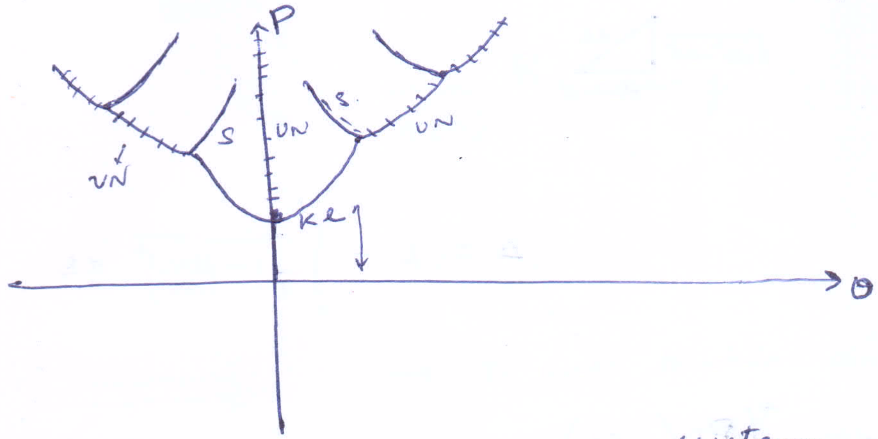
$$\frac{\partial PE}{\partial \theta} = K\theta - Pl\sin\theta = 0 \quad (\text{Moment balance})$$



$\frac{\partial^2 PE}{\partial \theta^2} > 0$ $\qquad$ $K - Pl\cos\theta > 0$

@ $\theta = 0$ $\qquad$ $K - Pl > 0$

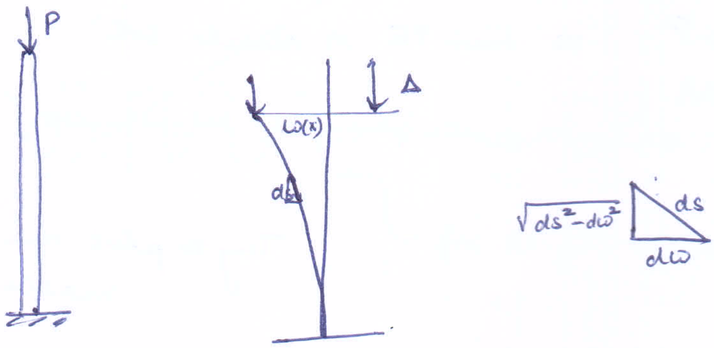
$\frac{K}{l} > P$ $\Rightarrow$ min PE $\Rightarrow$ stable solⁿ



Try to plot this in matlab.

Bifurcation in dynamic system

Buckling



$$\Delta = L - \int_0^L \sqrt{1-\left(\frac{dw}{ds}\right)^2}\, ds$$

PE = SE + WP

$$PE = \int_0^L \frac{1}{2} EI\, w''^2\, dx + (-P\Delta)$$

$$\Delta = L - \int_0^L \left[1 - \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] dx$$

$$\Delta = \int_0^L \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$$

$$PE = \int_0^L \frac{1}{2} EI\, w''^2\, dx - P\int_0^L \frac{1}{2} w'^2\, dx$$

$$PE = \int_0^L \left[\frac{EI\,w''^2}{2}\, dx - \frac{P\,w'^2}{2}\right] dx$$

$$\delta_w PE = -(-Pw')' + (EI\,w'')'' = 0$$

$$\boxed{(EI\,w'')'' + P\,w'' = 0}$$

This is twice differentiated moment eq.

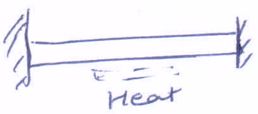
$(EI\,w'')'' \longrightarrow w''''$

Buckling is a eigen value problem

Has infinite no of solution.

$$R_{buckling} = P = \frac{\int_0^L EI\,\phi''^2\,dx}{\int_0^L \phi'^2\,dx}$$

$$P_{cr} = \min_{\phi(x)} R_{buckling}$$



→ It will buckle due to residual comp stresses.

<u>Most stable coloumn</u>

$$\max_{A(x)} \; \min_{\phi(x)} \; \underbrace{\frac{\int_0^L E\alpha A\,\phi''^2\,dx}{\int_0^L \phi'^2\,dx}}_{P_{cr}}$$

<u>Design of stable coloumn</u>

subject to $\int_0^L A\,dx - V^* \leq 0$

<u>Bower</u> (Flat plate)

Recap

Buckling of columns in the framework of calculus of variations

Most stable column for given volume ($\max_{A(x)} \min_{\phi(x)} R_{buckling}$)

Code

Bar

$$Eu'^2 = \Delta$$

$$A^{(k+1)} = A^{(k)} + (Eu'^2 - \Delta)$$

or

$$A^{(k+1)} = A^{(k)}\left(\frac{Eu'^2}{\Delta^{(k)}}\right)^{\eta}$$

$$\sum_{i=1}^{n} A_i^{(k+1)} l_i = V^*$$

Δ: $\int A(x)\,dx - V^* \leq 0$

$\sum A_i l_i = V^*$

$$\sum_{i=1} A_i^{(k)} l_i \left(\frac{Eu'^2}{\Delta}\right)^{\eta} = V^*$$

$$A^{(k+1)} = A^{(k)}\left(\frac{C_i}{\Delta}\right)^{\eta}$$

$$\sum A_i^k C_i^{\eta} l_i = V^*$$

$$\Delta^{\eta} V^* = \sum_{i=1}^{n} A_i^{(k)} l_i (Eu'^2)^{\eta}$$

$$\Delta = \left[\frac{\sum_{i=1}^{n} A_i^{(k)} l_i (Eu'^2)^{\eta}}{V^*}\right]^{1/\eta}$$

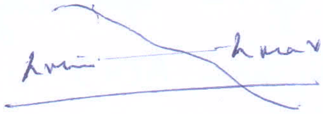
$$A^{(k+1)} = A^{(k)}\left(\frac{Eu'^2}{\Delta}\right)^{\eta}$$

$$\sum A_i l_i = A_{max}\, l_i\,(\text{upcount}) + A_{min}\, l_i\,(\text{lowcount})$$

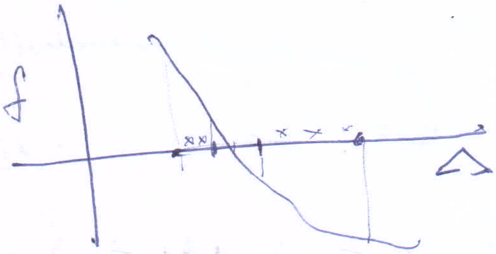
$$+ \underbrace{\sum_{i \in R} A_i l_i}_{\text{Those which have not reached upper or lower bound}} \longrightarrow \text{These area do not req } \mu_u \,\&\, \mu_L. \text{ Th}$$

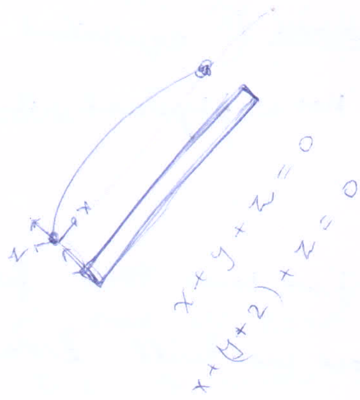
Now for the remaining we are calculating Lambda.

for non linear variables, use bisection and other methods (iteratively)



In case of non linear fn.





zero for A_{max} & A_{min}, μ_u & $\mu_L \neq 0$

But