ME 256: VARIATIONAL METHODS AND STRUCTURAL OPTIMIZATION

TERM PROJECT

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1.0 BACKGROUND TO THE PROBLEM

Any aircraft wing is made of the structural members namely Spars, Ribs and Skin mainly to take different loads coming on the wing structure. The spars and skin resist the bending load while the ribs take the shear load. The skin and the spar box resist the torque load. For the initial structural design of the wing structure, the spars are assumed and designed to take the bending load and are idealized as the beam structure which is fixed at the fuselage (Cantilever Beam Structure) end.

For this term project, the wing spar of a flying Unmanned Aerial Vehicle (UAV) is considered. The spar was designed for strength and stiffness and the cross section is fixed as channel (C) section. The dimensions of the channel section (namely the flange width and flange thickness) were kept uniform throughout the span. The depth of the channel section is fixed and is constant across the span as that parameter is controlled by the thickness of the airfoil that is in turn governed by the aerodynamic design. Also the cross section of the spar is fixed as channel section only because it is most ideally suited for taking torque load as a box and also best suited in terms of the manufacturing considerations.

Here, an attempt is made to optimize the cross section of the spar (flange width and flange thickness) for minimum volume for the given air load distribution, keeping the strength and stiffness as the design constraints. The allowable stress for the given material is 32 Kg/mm^2. Also, from the aviation design standards, the maximum deflection allowed is only 1% of the span, which puts a constraint on the stiffness in terms of the maximum deflection at the tip. The optimally designed spar will be compared with the existing spar in terms of the weight.

2.0 OPTIMIZATION PROBLEM

$$Min \int_0^L A\,dx \quad Subject\ to \quad \left(EIw''\right)'' - q = 0$$

$$w\left(\hat{x}\right) - \Delta^* \leq 0$$

$$\sigma - S \leq 0$$

$$-\sigma - S \leq 0$$

Objective Function: - Minimize the Volume. The volume can be written as A dx and is minimized. The cross sectional area can further be written as follows,

$$A = 2\left(FT \times FW\right) + d \times FT$$

Similarly the Moment of Inertia is approximated as follows

$$I = 2\left(FT \times FW\right)\left(\frac{d}{2}\right)^2$$

Therefore the objective function becomes

$$Min \int \left\{2\left(FT \times FW\right) + d \times FT\right\} dx$$

Constraints: - The deflection at the tip should not be more than deltastar and the stress at any point should not exceed the allowable stress value S.

Design Variables: - The flange width (FW) and flange thickness (FT)

State Variable: - The transverse displacement of the beam - w

Data: - (i) Air load distribution (loading) – q (x) – Given in Table 1

(ii) $\Delta^* = 247.5\ mm$

(iii) $S = 32\ Kg/mm^2$

(iv) E = 7000 Kg/mm^2

(v) d = depth of the cross section = 87.6 mm

(vi) Half Span of the wing = L = 3288.5 mm

3.0 ANALYTICAL SOLUTION

$$Min \quad \int_0^L \{2(FT \times FW) + d \times FT\}\, dx \quad Subject\ to \quad (EIw'')'' - q \quad = 0$$

$$w\left(\hat{x}\right) - \Delta^* \leq 0$$

$$\sigma - S \leq 0$$

$$-\sigma - S \leq 0$$

From the expression for the mutual strain energy we get,

$$w\left(\hat{x}\right) = \int_0^L EIw''w_1''dx$$

Where w – deformation under q(x)

w_1 – deformation under unit load at xhat.

Recalling

$$M = EIw''$$

$$M_1 = EIw_1''$$

$$\sigma = \frac{Md}{2I}$$

Where M – Bending moment under q(x) and M_1 – bending moment under unit load at xhat. For statically determinate beams, which is our case here, M and M_1 are independent of the cross sectional area.

Hence the problem gets simplified as follows,

$$\underset{FT(x),FW(x)}{Min} \quad \int_0^L \left\{2\left(FT \times FW\right) + d \times FT\right\} dx \quad Subject\ to$$

$$\Lambda : \int_0^L \frac{MM_1}{EI} dx - \Delta^* \leq 0$$

$$\mu_1 : \frac{Md}{2I} - S \leq 0$$

$$\mu_2 : -\frac{Md}{2I} - S \leq 0$$

The Lagrangian can be written as follows,

$$L = \int_0^L \left\{2\left(FT \times FW\right) + d \times FT\right\} dx + \Lambda \left\{ \int_0^L \frac{MM_1}{EI} dx - \Delta^* \right\} + \int_0^L \mu_1 \left(\frac{Md}{2I} - S \right) dx + \int_0^L \mu_2 \left(-\frac{Md}{2I} - S \right) dx$$

Substituting for moment of inertia in terms of the design variables, we get the Lagrangian as follows,

$$L = \int_0^L \left\{2\left(FT \times FW\right) + d \times FT\right\} dx + \Lambda \left\{ \int_0^L \frac{2MM_1}{Ed^2\left(FT \times FW\right)} dx - \Delta^* \right\} + \int_0^L \mu_1 \left(\frac{M}{d \times FT \times FW} - S \right) dx + \int_0^L \mu_2 \left(-\frac{M}{d \times FT \times FW} - S \right) dx$$

The necessary conditions can be obtained by writing Euler – Lagrangian equations,

$$\delta_{FT} L = 0 \quad \Rightarrow \frac{dF}{d(FT)} = 0$$

$$\Rightarrow 2 \times FW + d - \Lambda \frac{2MM_1}{Ed^2 FW \times FT^2} - \mu_1 \frac{M}{d \times FW \times FT^2} + \mu_2 \frac{M}{d \times FW \times FT^2} = 0$$

$$\delta_{FW} L = 0 \quad \Rightarrow \frac{dF}{d(FW)} = 0$$

$$\Rightarrow 2 \times FT - \Lambda \frac{2MM_1}{Ed^2 FT \times FW^2} - \mu_1 \frac{M}{d \times FT \times FW^2} + \mu_2 \frac{M}{d \times FT \times FW^2} = 0$$

Now, for the given loading and unit load, the variation of bending moment have to be plotted and the equations have to be solved for getting the design variables.

4.0 FURTHER WORK

- Analytically solve the problem for the given loading (converting the given concentrated loading into a uniformly distributed loading)
- Numerical Optimization using Matlab Tool box
- Fine Tune the moment of inertia by replacing the approximations
- Introduce upper and lower bounds on the flange thickness
- Introduce the manufacturing constraints and try to improve optimum solution by optimal remodeling

Table 1: Load on the Spar – Distribution along the Span

S.No	Station	Load
	(mm)	(Kg)
1	0	29.4
2	138.5	29.33
3	288.5	29.33
4	438.5	29.19
5	588.5	28.91
6	738.5	28.63
7	888.5	28.35
8	1038.5	27.93
9	1188.5	27.44
10	1338.5	26.88
11	1488.5	26.25
12	1638.5	25.48
13	1788.5	24.71
14	1938.5	23.73
15	2088.5	22.75
16	2238.5	21.56
17	2388.5	20.23
18	2538.5	18.69
19	2688.5	16.94
20	2838.5	14.84
21	2988.5	12.25
22	3138.5	8.82
23	3288.5	0