# Optimal Design of Compliant Mechanisms for Uni-directional Output Displacement

## Motivation:

Compliant mechanisms are gainfully employed in MEMS applications for motion transfer. In general we want the uni-directional output displacement, otherwise it affect the performance of device. Existing mechanisms for this applications are using with symmetric geometry and symmetric loads. We want evaluate topology of compliant mechanism by eliminating the symmetry topology and load conditions.

## Optimality criterion in developing the compliant mechanism:

There are two design objectives to be meet simultaneously when a designing compliant mechanism [1],

i) Flexible enough to satisfy the kinematical requirements.
ii) Stiff enough to support external loads.

Mutual potential energy (MPE),

$$MPE = \int_{\Omega} \sigma_d^T \varepsilon d\Omega = V^T K U,$$

where $\sigma_d$ is stress field in the continuum when only the unit dummy load is applied, $\varepsilon$ is the strain field when only the input load is applied, K is the stiffness of the system, V is the displacement field when only the unit dummy load applied and U is the displacement field due to input force.

Strain energy (SE)

$$SE = \int_{\Omega} \sigma^T \varepsilon d\Omega = \frac{1}{2} U^T K U,$$

where $\sigma$ is the stress field due to the input load only.

## Statement of the optimization problem:

$$\text{Minimize} \frac{(-MPE_r + C(MPE_t)^2)}{SE}$$

where C( $\geq 0$ ) is the penalty parameter.

Subject to

$$\int_{\Omega} t d\Omega \leq V^*$$

$$\int_{\Omega} \varepsilon(u) : E : \varepsilon(w_u) d\Omega = \int_{\Gamma_{in}} F \cdot w_u d\Gamma, \ \forall w_u \in U$$

$$\int_{\Omega} \varepsilon(v_r) : E : \varepsilon(w_r) d\Omega = \int_{\Gamma_{out}} f_r \cdot w_r d\Gamma, \ \forall w_r \in U$$
$$\int_{\Omega} \varepsilon(v_t) : E : \varepsilon(w_t) d\Omega = \int_{\Gamma_{out}} f_t \cdot w_t d\Gamma, \ \forall w_t \in U$$

where E is the constitutive (material property) matrix, $v_r$ is the displacement field due to unit dummy load only along radial direction, $v_t$ is the displacement field due to unit dummy load only along tangential direction, $u$ is the displacement field due to applied loads only, $V^*$ is given volume of material and $t$ is the thickness of sheet.

$MPE_r, MPE_t$ and $SE$ are the mutual potential energy due to radial dummy load, mutual potential energy due to tangential dummy load and strain energy due to applied load only.

## Design variables:

Thickness ( $t$ ) is the design variable, it also define the topology of structure.

## State Variables:

Displacement -- $u$

## Data given:

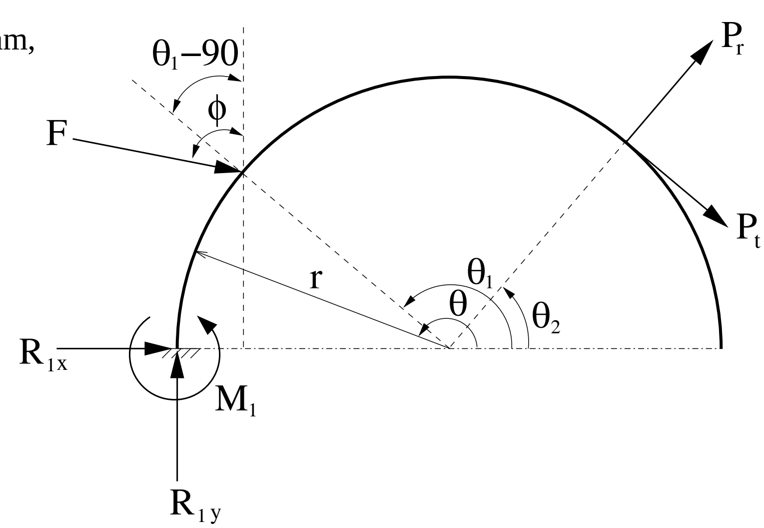
Volume of the material,
Material properties (Young's modulus, Poisson's ratio),
Applied loads,
At which point maximum displacement is required (output boundary).

## Plan for the future work:

The problem stated above has to be solved numerically with FEM to obtain the optimal topology.

## Analytical solution of the problem for one-dimensional case:

Consider the semi-circular beam, as show in figure.



Using equilibrium equations:

Reaction forces and moment is give below,

$R_{1x} = F\sin\phi$,

$R_{1y} = F\cos\phi$,

$M_1 = F\cdot r\{\cos\phi + \cos(\theta_1 - \phi)\}$,

where F is applied force at the point (r, $\theta$) at an angle of $\phi$ with vertical line.

Consider the dummy forces at the point (r, $\theta_2$) along radial and tangential directions $P_r$ and $P_t$ respectively

Moment of resistance and tangential force at arbitrary $\theta$ as follows

$$M = -P_r\cdot r\{\sin(\theta-\theta_2)\} + P_t\cdot r\{1-\cos(\theta-\theta_2)\} + F\cdot r\{\sin(\theta_1-\phi)\sin(\theta-\theta_1) + \cos(\theta_1-\phi)(1-\cos(\theta-\theta_1))\}$$

$$F_t = P_r\sin(\theta-\theta_2) + P_t\cos(\theta-\theta_2) - F\sin(\theta_1-\phi)\sin(\theta-\theta_1) + F\cos(\theta_1-\phi)\cos(\theta-\theta_1)$$

Strain energy in the beam is given by

$$SE = \int_0^\pi \frac{M^2}{2EI} r\,d\theta + \int_0^\pi \frac{F^2}{2EA} r\,d\theta$$

Using the theorem of castigliano's

$$\Delta_r = \frac{\partial(SE)}{\partial P_r} = \int_0^\pi \frac{M}{EI}\frac{\partial M}{\partial P_r} r\,d\theta + \int_0^\pi \frac{F}{EA}\frac{\partial F}{\partial P_r} r\,d\theta$$

$$\Delta_t = \frac{\partial(SE)}{\partial P_t} = \int_0^\pi \frac{M}{EI}\frac{\partial M}{\partial P_t} r\,d\theta + \int_0^\pi \frac{F}{EA}\frac{\partial F}{\partial P_t} r\,d\theta$$

$$\frac{\partial M}{\partial P_r} = -r\sin(\theta-\theta_2), \frac{\partial M}{\partial P_t} = -r\{1-\cos(\theta-\theta_2)\},$$

$$\frac{\partial F}{\partial P_r} = \sin(\theta-\theta_2), \frac{\partial M}{\partial P_t} = \cos(\theta-\theta_2)$$

Radial and tangential displacements Δr and Δt are as follows,

$$\Delta_r = \int_0^\pi \frac{Mr^2}{EI}(-\sin(\theta-\theta_2)d\theta + \int_0^\pi \frac{Fr}{EA}\sin(\theta-\theta_2)d\theta$$

$$\Delta_t = \int_0^\pi \frac{Mr^2}{EI}(1-\cos(\theta-\theta_2)d\theta + \int_0^\pi \frac{Fr}{EA}\cos(\theta-\theta_2)d\theta$$

Objective function

$$\mathcal{L}^* = \frac{\Delta_r - C\Delta_t^{\,2}}{SE},$$

where penalty parameter, $C \geq 0$

Subjected to:

$$\int_0^{\pi} \mathrm{A} r d\theta - \mathrm{V}^* \le 0$$

Lagrangian as follows,

$$\mathcal{L} = \mathcal{L}^* + \Lambda\left(\int_0^{\pi} \mathrm{A} r d\theta - \mathrm{V}^*\right)$$

Moment, $M = r \cdot f(\theta)$,

Force, $F_t = g(\theta)$,

Moment of inertia, $I = \alpha \mathrm{A}$

Rewrite the deflection and strain energy equation,

$$\Delta_r = \int_0^{\pi} k_1 \frac{r^3}{\mathrm{A}} d\theta + \int_0^{\pi} k_2 \frac{r}{\mathrm{A}} d\theta$$

$$\Delta_t = \int_0^{\pi} k_3 \frac{r^3}{\mathrm{A}} d\theta + \int_0^{\pi} k_4 \frac{r}{\mathrm{A}} d\theta$$

$$SE = \int_0^{\pi} k_5 \frac{r^3}{\mathrm{A}} d\theta + \int_0^{\pi} k_6 \frac{r}{\mathrm{A}} d\theta$$

Lagrangian can rewrite as follows,

$$\mathcal{L} = \frac{\left(\int_0^{\pi} k_1 \frac{r^3}{\mathrm{A}} d\theta + \int_0^{\pi} k_2 \frac{r}{\mathrm{A}} d\theta\right) - C\left(\int_0^{\pi} k_3 \frac{r^3}{\mathrm{A}} d\theta + \int_0^{\pi} k_4 \frac{r}{\mathrm{A}} d\theta\right)^2}{\left(\int_0^{\pi} k_5 \frac{r^3}{\mathrm{A}} d\theta + \int_0^{\pi} k_6 \frac{r}{\mathrm{A}} d\theta\right)}$$

$$k_1 = -\frac{f(\theta) \cdot \sin(\theta - \theta_2)}{E\alpha}, k_2 = -\frac{g(\theta) \cdot \sin(\theta - \theta_2)}{E},$$

$$k_3 = -\frac{f(\theta) \cdot (1 - \cos(\theta - \theta_2))}{E\alpha}, k_4 = -\frac{g(\theta) \cdot \cos(\theta - \theta_2)}{E},$$

$$k_5 = -\frac{\left(f(\theta)\right)^2}{2E\alpha}, k_6 = -\frac{\left(g(\theta)\right)^2}{2E}$$

To get the solution we have take below variations,

$\delta_{\mathrm{A}} \mathcal{L} = 0,$

$\delta_r \mathcal{L} = 0,$

$\delta_\theta \mathcal{L} = 0$

Here equations are coming implicit we can’t solve by hand. We have to go for numerical methods to handle the equations.