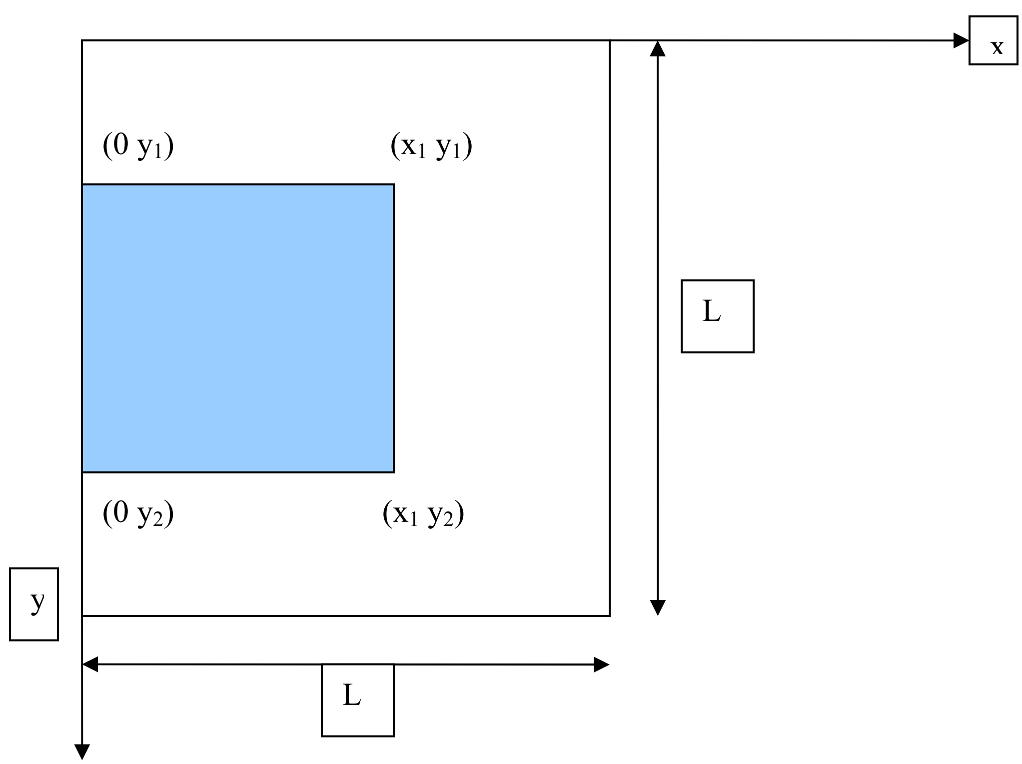
Background to the problem

Normal topology optimization method requires division of the domain into a fine number of grids and then based on the objective function and constraints successive elimination of the members of this grid is done to arrive at an optimal topology defining the shape of the structure. So the geometry of the final structure consists of a large number of elements. I was interested in what way the topology and shape of the final structure can be arrived at using a small number of elements. This project aims at solving a problem using the wide curve theory in which a minimum number of grid has been considered.

Statement of the problem

For a given load and a given amount of material design the stiffest hook (the hook has to be sufficiently large to support the member (a rope or a chain) through which the load is applied.

Minimize: (SE) with respect to design variables.
Data: a given mass of a specified material.



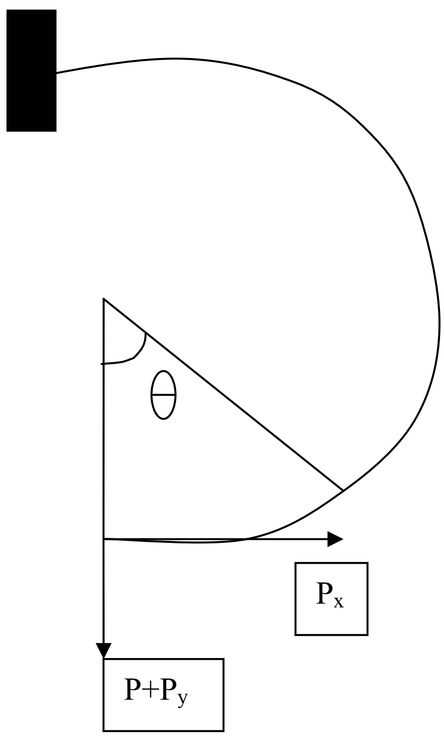
The domain of the problem

Constraints :

No part of the hook can be within the blue shaded region which is kept for the device thru which load is applied and it has to be within the square box of side L which is the domain of the problem.

Theoretical solution:

I don't think that the topology of the hook can be solved analytically. However if the topology is selected (for e.g. a semicircle) then the shape of the beam may be tried to be derived. I am taking a semi-circular beam



The above figure shows a semi-circular beam (never mind the drawing , that's all I could make)

$$M = (P + P_y)R\sin\theta + P_x R(1-\cos\theta)$$
$$F = (P + P_y)\sin\theta - P_x\cos\theta$$
$$SE = \int_0^{\pi} M^2/(2EI)Rd\theta + \int_0^{\pi} F^2/(2EA)Rd\theta$$
$$= \int_0^{\pi} \{(P+P_y)R\sin\theta + P_x R(1-\cos\theta)\}^2 Rd\theta/(2EI) + \int_0^{\pi} \{(P+P_y)\sin\theta - P_x\cos\theta\}^2 Rd\theta/(2EA)$$
$$\partial SE/\partial P_y = \int_0^{\pi} \{(P+P_y)R\sin\theta + P_x R(1-\cos\theta)\} Rd\theta/(EI) + \int_0^{\pi} \{(P+P_y)\sin\theta - P_x\cos\theta\} Rd\theta/(EA)$$

Now substituting $P_y = 0$ & $P_x = 0$

$$\delta_y = \int_0^{\pi} PR^3\sin^2\theta d\theta/(EI) + \int_0^{\pi} PR\sin^2\theta d\theta/(EA)$$
$$= \frac{\pi}{2}\left(\frac{PR^3}{EI} + \frac{PR}{AE}\right)$$

Let

$$I = bh^3/12$$
$$A = bh$$
$$b = 1$$
$$=> I = h^3/12$$
$$A = h$$
$$=> I = A^3/12$$
$$\delta_y = \frac{\pi}{2}\left(\frac{12PR^3}{EA^3} + \frac{PR}{AE}\right)$$

Now the mean compliance can be written as

$$MC = \frac{\pi}{2}\left(\frac{12P^2R^3}{EA^3} + \frac{P^2R}{AE}\right)$$

Now the Lagrangian may be written as

$$L = \frac{P^2 R\pi}{2E}\left(\frac{12R^2}{A^3} + \frac{1}{A}\right) + \Lambda\left(\int A dx - V^*\right)$$

$$\delta L_A = 0$$

$$=> \frac{P^2 R\pi}{2E}\left(-\frac{36R^2}{A^4} - \frac{1}{A^2}\right) + \Lambda = 0$$

$$=> \frac{36R^2}{A^4} + \frac{1}{A^2} - \frac{2E\Lambda}{P^2 R\pi} = 0$$

let

$$\alpha = \frac{1}{A}$$

$$K = \frac{2E\Lambda}{P^2 R\pi}$$

$$36R^2\alpha^4 + \alpha^2 - K = 0$$

$$=> \alpha^2 = \frac{-1 \pm \sqrt{1+C}}{72R^2}$$

where

$$C = \frac{288\Lambda ER}{P^2\pi}$$

$$\alpha^2 > 0$$

$$=> \alpha^2 = \frac{\sqrt{1+C} - 1}{72R^2}$$

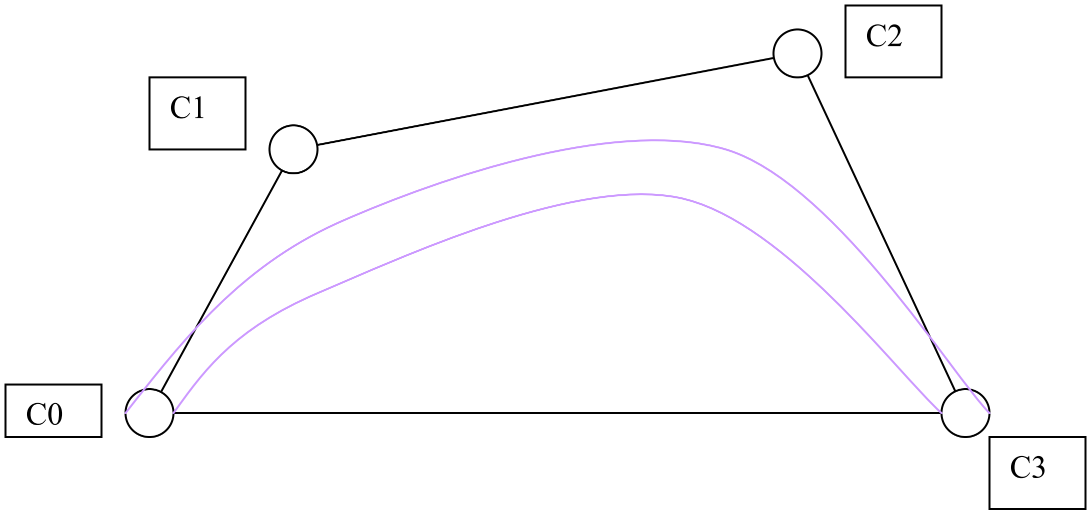
$$=> \alpha = \sqrt{\frac{\sqrt{1+C} - 1}{72R^2}}$$

$$A = \sqrt{\frac{72R^2}{\sqrt{1+C} - 1}}$$

From this Λ can be solved by putting back in the constraint equation. However, by this analysis Area comes out to be uniform.

Numerical approach:

Before talking about the constraints, design and state variables I would like to present the formulae used in wide curve theory.



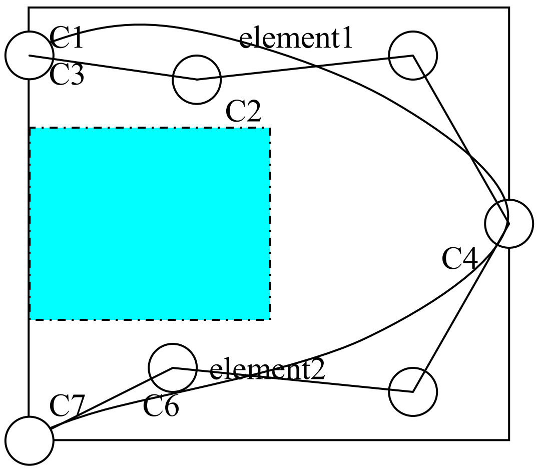
A cubic wide Bezier curve

$$x_c(t) = \sum_{i=0}^{3} C_{ix} B_m^i(t)$$
$$y_c(t) = \sum_{i=0}^{3} C_{iy} B_m^i(t)$$

$$w_c(t) = \sum_{i=0}^{3} C_{id} B_m^i(t)$$
$$B_m^i(t) = C_m^i t^i (1-t)^{m-i}$$

Here $x_c(t)$, $y_c(t)$ is the centre line of the wide Bezier curve which is the regular Bezier curve with control points $\{(C_{ix}, C_{iy})$, i = 0,1,..m$\}$. parameter t lies between 0 & 1.The wide Bezier curve is fully controlled by the set of circles $\{C_0, C_1..C_m\}$ located at centers (C_{ix}, C_{iy}) with diameters C_{id} .

In this problem I am taking 2 Bezier curves as shown below.



None of the curves should enter the blue shaded region (which is the nook of the hook). i.e.

$C_{ix} + C_{id}/2 > x_1$---------------(1)
$C_{iy} + C_{id}/2 < y_1$---------------(2)
$C_{iy} + C_{id}/2 > y_2$---------------(3)
$C_{ix} + C_{id}/2 < L$---------------(4)
$C_{ix} - C_{id}/2 > x_1$---------------(5)
$C_{iy} + C_{id}/2 < L$---------------(6)
$C_{iy} - C_{id}/2 > 0$---------------(7)

Constraints on the wide curve:
A wide Bezier curve may be considered as the trace of moving a variable circle along the center Bezier curve. When the radius of curvature of the center Bezier curve is smaller than the radius of the moving circle, self-intersection of the wide curve occurs. This can be avoided by imposing additional constraint as follows:
GCC = Max [0.5 $w_c(t)$ - $\rho_c(t)$]
where,

$$\rho_c(t) = \frac{\left[\dot{x}_c(t) + \dot{y}_c(t)\right]}{[\dot{x}_c(t)\ddot{y}_c(t) - \ddot{x}_c(t)\dot{y}_c(t)]}$$ = radius of curvature of the center Bezier curve.

However, self intersection can still occur if the center Bezier curve crosses itself and creates a loop. This kind of self-intersection of wide curves can be avoided by the following inequality condition:

GCL = (m – 4/3)(n – 4/3) - 4/9 < 0 where m and n are given by

$$\sqrt{(C_{1x}-C_{0x})^2+(C_{1y}-C_{0y})^2}=m\sqrt{(Q_x-C_{0x})^2+(Q_y-C_{0y})^2}$$

$$\sqrt{(C_{3x}-C_{2x})^2+(C_{3y}-C_{2y})^2}=n\sqrt{(Q_x-C_{3x})^2+(Q_y-C_{3y})^2}$$

where, C_1, C_2, C_3, C_4, are the centers of the circles and Q is the point of self-intersection of the wide curve.

The above 2 types of constraints when considered in my case will introduce 4 new constraints namely,

$GCC_{element1} < 0$---------------(8)

$GCC_{element2} < 0$---------------(9)

$GCL_{element1} < 0$---------------(10)

$GCL_{element2} < 0$---------------(11)

1 additional constraint need to be specified so that the curves don't cross each other.

$$Max\{(C_{3y}-C_{4y})/(C_{3x}-C_{4x}),(C_{2y}-C_{4y})/(C_{2x}-C_{4x})\}-Max\{(C_{7y}-C_{4y})/(C_{7x}-C_{4x}),(C_{6y}-C_{4y})/(C_{6x}-C_{4x})\}$$

---------------(12)

The volume constraint is given by:

$$b\int_0^1 w_c\sqrt{\dot{x}_c+\dot{y}_c}\,dt-V^*\leq 0$$ ---------------(13)

where b = depth of the beam (I will assume it to be unity)

Equations (1) to (13) give the set of constraints for this problem.

C_1 is fixed. The set of design variables are

X = [C_{1d} C_{2d} C_{2x} C_{2y} C_{3d} C_{3x} C_{3y} C_{4d} C_{4x} C_{4y} C_{5d} C_{5x} C_{5y} C_{6d} C_{6x} C_{6y} C_{7d} C_{7x} C_{7y}]

Using suitable shape functions The Tangent stiffness matrix $[K]^T$ is formed. (This I will develop) Then

$$[K]^T\{u\} = \{F\}$$

are solved to get {u} (the displacements).

Now,

$$SE = \{u\}^T[K]^T\{u\}/2$$

This is used as the minimizing function in fmincon optimization program of MATLAB.