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CONLIN: an efficient dual optimizer based on convex approximation concepts

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Abstract. The Convex Linearization method (CONLIN) exhibits many interesting features and it is applicable to a broad class of structural optimization problems. The method employs mixed design variables (either direct or reciprocal) in order to get first order, conservative approximations to the objective function and to the constraints. The primary optimization problem is therefore replaced with a sequence of explicit approximate problems having a simple algebraic structure. The explicit subproblems are convex and separable, and they can be solved efficiently by using a dual method approach.

In this paper, a special purpose dual optimizer is proposed to solve the explicit subproblem generated by the CONLIN strategy. The maximum of the dual function is sought in a sequence of dual subspaces of variable dimensionality. The primary dual problem is itself replaced with a sequence of approximate quadratic subproblems with non-negativity constraints on the dual variables. Because each quadratic subproblem is restricted to the current subspace of non zero dual variables, its dimensionality is usually reasonably small. Clearly, the Hessian matrix does not need to be inverted (it can in fact be singular), and no line search process is necessary.

An important advantage of the proposed maximization method lies in the fact that most of the computational effort in the iterative process is performed with reduced sets of primal variables and dual variables. Furthermore, an appropriate active set strategy has been devised, that yields a highly reliable dual optimizer.

1 Introduction

This paper describes an efficient optimization algorithm particularly well adapted to solve many problems arising in structural synthesis. Such design optimization problems consist of minimizing some objective function subject to constraints, insuring the feasibility of structural design. Mathematically, the numerical optimization problem considered herein can be written in the following general form:

$$\min \quad c_0(x)\,, \tag{1}$$

$$\text{subject to} \quad c_j(x) \le 0, \quad (j = 1, \dots, m)\,, \tag{2}$$

$$\underline{x}_i \le x_i \le \overline{x}_i \quad (i = 1, \dots, n)\,. \tag{3}$$

The functions $c_j(x)$ (for $j = 0, \dots, m$) are linear or nonlinear functions of the design variables x_i. The objective function (1) usually represents a structural characteristic to be minimized (e.g. the weight). The inequalities (2) are the behavior constraints that impose limitations on structural response quantities (e.g. upper bounds on stresses and displacements under static loading cases). The design variables must also be bounded by the side constraints (3), where $\underline{x}_i$ and $\overline{x}_i$ are lower and upper limits that reflect manufacturing or analysis validity considerations. It should be noted that the side constraints (3) constitute a particular case of the more general constraints (2). However, they are written separately in our optimization problem statement because the dual method approach described later can handle them more efficiently when considered apart from the behavior constraints.

The nonlinear programming problem (1)-(3) can be solved iteratively by using numerical optimization techniques (Morris 1982). Each iteration begins with a complete analysis of the system behavior in order to evaluate the objective function and constraint values along with their sensitivities to changes in the design variables (i.e. first derivatives). Most often, the analysis capability is based on finite element discretization. A design iteration is concluded by employing the results of these behavioral and sensitivity analyses in a minimization algorithm which searches the n-dimensional design space for a new primal point that decreases the objective function value while remaining feasible (i.e. satisfying the constraints).

The essential difficulty in solving the nonlinear programming problem (1)-(3) lies in the implicit character of the constraint functions $c_j(x)$. In other words, for each new design, these functions can only be evaluated numerically through a finite element analysis. The iterative nature of the optimization process implies that many structural reanalyses must usually be accomplished before finding an acceptable solution. Those repeated analyses can lead to prohibitive computational cost when dealing with large scale problems.

One widely used approach to design optimization is to join together a general purpose optimizer and a finite element package having the required sensitivity analysis capabilities. Many numerical experiments conducted for that purpose have demonstrated that direct approaches like gradient projection or feasible direction methods are inadequate for design optimization problems in view of the large number of iterations required for convergence. On the other hand, recursive linear programming techniques, even though they necessitate difficult adjustments of move limits, have proved to be computationally efficient. Following this idea of sequential linearization, various methods have been proposed during the last decade, leading to the now well established "approximation concepts" approach (Schmit and Miura 1976; Schmit and Fleury 1980; Fleury and Schmit 1980).

In the approximation concepts approach, the primary optimization problem is replaced with a sequence of explicit subproblems having a simple algebraic structure. Each subproblem is generated through Taylor series expansion of the objective function and constraints in terms of intermediate linearization variables. For example, linearization of the constraints with respect to reciprocal variables is a well recognized technique to solve optimal sizing problems. There is an intuitive explanation for the success of this technique, in that stresses and displacements are exact linear functions of the reciprocal sizing variables in the case of a statically determinate structure. For shape optimal design problems, there is no such physical guideline for the selection of intermediate linearization variables. Nevertheless, this change of variables continues to have a highly beneficial effect on the convergence properties of the shape optimization process (Braibant and Fleury 1985; Fleury 1986).

A very attractive feature of the approximation concepts approach is that it replaces the primary optimization problem with a sequence of separable subproblems which can be efficiently solved by a dual method formulation. In the dual approach, the constrained primal minimization problem is replaced by maximizing a quasi-unconstrained dual function depending only on the Lagrangian multipliers associated with the linearized constraints. These multipliers are the dual variables subject to simple nonnegativity constraints. The efficiency of the dual formulation is due to the fact that maximization is performed in the dual space, whose dimensionality is relatively low and depends on the number of active constraints at design iteration.

The convex linearization method (CONLIN) (Fleury and Braibant 1986) was initially conceived as an extension to the approximation concepts approach. The key idea in the CONLIN method is to perform the linearization process with respect to mixed variables, either direct or reciprocal, independently for each function involved in the optimization problem. At each successive iteration point, the CONLIN method only requires evaluation of the objective and constraint functions and their first derivatives with respect to the design variables. The optimizer will then select by itself an appropriate approximation scheme on the basis of the signs of the derivatives. This constitutes a major improvement with respect to the regular approximation concept approach, where it is usually assumed that the objective function is linear in the direct variables (e.g. structural weight) and that the constraints can be accurately approximated as linear functions of the reciprocal variables (e.g. stresses and displacements). Furthermore, the CONLIN optimizer has an inherent tendency to generate a sequence of steadily improving feasible designs, in contrast with the previously developed approximation concept approach using dual methods (Schmit and Fleury 1980; Fleury and Schmit 1980). Finally, it is relatively straightforward to equip CONLIN with a built-in strategy for dealing with highly infeasible starting points, by uniformly relaxing the violated behavior constraints.

The CONLIN method proceeds by linearizing each function defining the optimum design problem with respect to a properly selected mix of direct and reciprocal variables, so that a convex and separable subproblem is generated. The selection of the "intermediate" linearization variables is made on the basis of the signs of the first partial derivatives. It is easily proven that, considering any differentiable function $c(x)$, the following linearization scheme yields a convex approximation (hence the term "convex linearization", Fleury and Braibant 1986):

$$c(x) = c(x^0) + \sum_{+} c_i^0 (x_i - x_i^0) - \sum_{-} (x_i^0)^2 c_i^0 \left(\frac{1}{x_i} - \frac{1}{x_i^0} \right), \quad (4)$$

where c_i denote the first derivatives of $c(x)$ with respect to the design variables x_i. The symbol $\sum_{+}$ ($\sum_{-}$) means "summation over the terms for which c_i is positive (negative)". One of the most interesting features of the convex linearization scheme is that it also leads to the most conservative approximation amongst all the possible combinations of mixed direct/reciprocal variables. This property was initially demonstrated by Starnes and Haftka (1979), who employed conservative approximation to handle difficult buckling constraints.

The CONLIN algorithm applies this convex linearization scheme to the objective function and to all the constraint functions defining the optimization problem (1)-(3). It is convenient to normalize the design variables so that they become equal to unity at the current point x^0 where the problem is linearized in the form

$$x_i' = \frac{x_i}{x_i^0} \implies c_i' = c_i^0 x_i^0 .$$

The factor $(x_i^0)^2$ disappears from (4), which then becomes:

$$c(x') = c(x^0) + \sum_{+} c_i'(x_i' - 1) - \sum_{-} c_i'\left(\frac{1}{x_i'} - 1\right). \tag{4'}$$

Applying this linearization technique to each function $c_j\ (x)$, and dropping the superscript $'$, the following explicit subproblem is generated:

$$\min \sum_{+} c_{i0} x_i - \sum_{-} \frac{c_{i0}}{x_i} - \bar{c}_0 ,$$

$$\text{subject to} \quad \sum_{+} c_{ij} x_i - \sum_{-} \frac{c_{ij}}{x_i} \le \bar{c}_j \quad (j = 1, \dots, m),$$

$$\underline{x}_i \le x_i \le \overline{x}_i , \tag{5}$$

where c_{ij} denote the first derivatives of the objective and constraint functions evaluated at the current point x^0. Note that the constants $\bar{c}_j$ contain the zero order contributions in the Taylor series expansion in the form

$$\bar{c}_j = \sum_i \mid c_{ij} \mid x_i^0 - c_j(x^0) \quad (j = 0, \dots, m). \tag{6}$$

2 Dual method approach

In the convex linearization method, the initial problem is transformed into a sequence of explicit subproblems having a simple algebraic structure. Furthermore each subproblem is convex and separable. These properties make it attractive to solve the subproblem by using dual algorithms. The dual method approach is well-known and quite respected in the mathematical programming community (Lasdon 1970, pp. 396-459; Lootsma 1989). In the context of structural optimization problems, it was initially introduced by Fleury (1979), and it subsequently led to a reconciliation of optimality criteria techniques and mathematical programming methods (Fleury 1982). In this section, the principles of the dual formulation are applied to the explicit subproblem generated by the CONLIN strategy. The solution of the primal problem (5) can be obtained by the following "Max-Min" two-phase procedure:

$$\max l(r) ,$$

$$\text{subject to} \quad r_j \ge 0 , \tag{7}$$

where the dual function $l(r)$ results from minimizing the Lagrangian function

$$L(x, r) = \sum_{j=0}^{m} r_j \left(\sum_{+} c_{ij} x_i - \sum_{-} \frac{c_{ij}}{x_i} - \bar{c}_j \right), \tag{8}$$

over the acceptable primal variables

$$l(r) = \min_{\underline{x}_i \le x_i \le \overline{x}_i} L(x, r) . \tag{9}$$

The separability of the primal problem implies that the Lagrangian function (8) can be written as the sum of n individual functions $L_i(x_i)$, and therefore, the n-dimensional minimum problem (9) can be split into n single variable minimization problems:

$$\min L_i(x_i) = a_i x_i + \frac{b_i}{x_i} ,$$

$$\text{subject to} \quad \underline{x}_i \le x_i \le \overline{x}_i , \tag{10}$$

where the coefficients

$$a_i = \sum_{+} c_{ij} r_j \ge 0 ,$$

and

$$b_i = -\sum_{-} c_{ij} r_j \ge 0 , \tag{11}$$

depend only on the dual variables r_j. These coefficients remain always non-negative in the feasible region of the dual space (i.e. $r_j \ge 0$). Therefore, the Lagrangian problem (10) has necessarily a unique solution, obtained by stating that the first derivative of $L_i(x_i)$ must vanish

$$L_i'(x_i) \equiv a_i - \frac{b_i}{x_i^2} = 0 .$$

Since the side constraints (3) must be satisfied, it comes

$$x_i = \left(\frac{b_i}{a_i}\right)^{\frac{1}{2}} \quad \text{if} \quad \underline{x}_i^2 \le \frac{b_i}{a_i} \le \overline{x}_i^2 , \tag{12}$$

$$x_i = \underline{x}_i \quad \text{if} \quad \frac{b_i}{a_i} \le \underline{x}_i^2 , \tag{13}$$

$$x_i = \overline{x}_i \quad \text{if} \quad \overline{x}_i^2 \leq \frac{b_i}{a_i} . \tag{14}$$

Remembering that a_i and b_i depend on the dual variables [see (11)], we have obtained fully explicit primal-dual relationships. It can be seen that, at each point in the dual space, the primal variables are subdivided into free and fixed variables. It is convenient to introduce the set of indices

$$I = \{i : \underline{x}_i < x_i < \overline{x}_i\} . \tag{15}$$

The free variables ($i \in I$) are given by (12), while the remaining fixed variables are given by (13) or (14).

From the foregoing developments, it appears that the dual problem (7) can be expressed in closed form

$$\max l(r) = \sum_{j=0}^{m} r_j \left[\sum_{+} c_{ij} x_i(r) - \sum_{-} \frac{c_{ij}}{x_i(r)} - \overline{c}_j \right] , \tag{16}$$

$$\text{s.t.} \quad r_j \geq 0 ,$$

where the primal variables x_i are known explicitly in terms of the dual variables r_j via (12)-(14). In order for the dual function to be bounded, the dual variables r_j must be linked by a linear equality constraint. Following the usual practice, we shall simply assume that the Lagrangian multiplier r_0 associated with the objective function is fixed to a unit value.

A fundamental property of the dual function is that its first derivatives are simply given by the primal constraint values

$$g_j \equiv \frac{\mathrm{d}l}{\mathrm{d}r_j} = \sum_{+} c_{ij} x_i(r) - \sum_{-} \frac{c_{ij}}{x_i(r)} - \overline{c}_j . \tag{17}$$

In addition, because the dual problem is fully explicit and because the corresponding primal problem exhibits a relatively simple algebraic form, the second derivatives of the dual function

$$H_{jk} \equiv \frac{\mathrm{d}^2 l}{\mathrm{d}r_j \mathrm{d}r_k}$$

can be written in closed form. From (17) it comes

$$H_{jk} \equiv \frac{\mathrm{d}g_j}{\mathrm{d}r_k} = \sum_{+} c_{ij} \frac{\mathrm{d}x_i}{\mathrm{d}r_k} + \sum_{-} \left(\frac{c_{ij}}{x_i^2} \right) \frac{\mathrm{d}x_i}{\mathrm{d}r_k} .$$

Differentiating the primal-dual relationship (12), it follows that

$$\frac{\mathrm{d}x_i}{\mathrm{d}r_k} = \frac{\left(a_i \frac{\mathrm{d}b_i}{\mathrm{d}r_k} - b_i \frac{\mathrm{d}a_i}{\mathrm{d}r_k} \right)}{(2 x_i a_i^3)} , \quad (i \in I)$$

for the free primal variables. For the fixed variables, these derivatives are obviously zero.

Using the definition (11) of a_i and b_i, it can be seen that

$$\frac{\mathrm{d}a_i}{\mathrm{d}r_k} = +c_{ik} \quad \text{if} \quad c_{ik} > 0 , \qquad \frac{\mathrm{d}a_i}{\mathrm{d}r_k} = 0 \quad \text{if} \quad c_{ik} \leq 0 ,$$

$$\frac{\mathrm{d}b_i}{\mathrm{d}r_k} = -c_{ik} \quad \text{if} \quad c_{ik} < 0 , \qquad \frac{\mathrm{d}b_i}{\mathrm{d}r_k} = 0 \quad \text{if} \quad c_{ik} \geq 0 .$$

Therefore

$$\frac{\mathrm{d}x_i}{\mathrm{d}r_k} = -\frac{c_{ik}}{2 x_i a_i^2} \quad \text{if} \quad c_{ik} > 0 ,$$

$$\frac{\mathrm{d}x_i}{\mathrm{d}r_k} = -\frac{c_{ik} x_i}{2 a_i^2} \quad \text{if} \quad c_{ik} < 0 . \tag{18}$$

Finally, by regrouping the foregoing results, an explicit form is obtained for the elements of the dual Hessian matrix:

$$H_{jk} = -\frac{1}{2} \sum_{i \in I} n_{ij} n_{jk} \frac{x_i}{a_i} , \tag{19}$$

where

$$n_{ij} = c_{ij} \quad \text{if} \quad c_{ij} > 0 ,$$

$$n_{ij} = \frac{c_{ij}}{x_i^2} \quad \text{if} \quad c_{ij} < 0 ,$$

and the n_{ik} coefficients obey the same rule. It is important to emphasize that the summation in (19) is restricted to the free primal variables, i.e. the variables x_i which do not reach their lower or upper bound [see (12)]. This means that the second derivatives of the dual function are discontinuous whenever a free primal variable becomes fixed, or conversely.

The fundamental difficulty in using Newton type methods for solving the dual problem resides in these inherent discontinuities of the Hessian matrix. Fortunately, the topology of the dual space can be described in an exact mathematical way via the concept of second order discontinuity planes (see e.g. Fleury and Schmit 1980; Fleury 1979; Fleury 1982). Based on this concept, a very reliable sequential quadratic programming method has been devised to solve the dual problem.

3 Sequential quadratic programming in dual space

The initial implementation of the CONLIN optimizer was based on the algorithm DUAL-2 available in ACCESS-3 (Fleury and Schmit 1980). This Newton type dual algorithm had proven to be highly efficient from the computer time viewpoint, when compared to the conventional general purpose optimizers available in ACCESS-3 (NEWSUMT, CONMIN). However, DUAL-2 suffers from some lack of reliability because it breaks down when linearly dependent constraints lead to a singular Hessian matrix. When employing a second order algorithm, the main difficulty was that the second derivatives of the dual function are discontinuous along some planes in the dual space. As shown above, this feature remains true when the CONLIN approximation scheme is used. To cope with this difficulty, a specially devised line search technique was used in DUAL-2. The new optimizer, which is still more efficient and much more reliable, abandons the artifice of a line search procedure. Rather a more elaborate strategy is employed, which permits controlling the second order discontinuities as well as the possible singularity of the Hessian matrix. The new method replaces the basic generalized Newton iteration that was used in DUAL-2, with the solution of an equivalent quadratic subproblem.

The key idea of the DUAL-2 optimizer has been maintained: the maximum of the dual function is sought in a sequence of dual subspaces that only include the dual variables associated with potentially active primal constraints. Such a subspace will be referred to by the set

$$M = \{j : c_j \text{ is potentially active}\}.$$

Of course, the dual subspace M needs to be periodically updated according to some rules which will be discussed later. For the moment, let us assume that we are just interested in finding the maximum value of the dual function in a given supspace M. It should be clearly understood that ultimately, when the maximum of the dual function will be found, the subspace M will only contain positive dual variables. During the iterative optimization process, the set M is mostly made up of positive dual variables, however, a few zero dual variables are allowed to enter the set M occasionally.

In the old version of the DUAL-2 optimizer, the dual function was maximized by using a generalized Newton method, according to the iterative process

$$r_j^+ = r_j + \alpha s_j \qquad (j \in M),$$

where

$$s = -[H]^{-1} g \tag{20}$$

represents a Newton search direction, evaluated from the gradient vector g and the Hessian matrix H of the dual function [see (17) and (19)]. The step length α was chosen to increase $l(r)$ along s without violating the non-negativity constraints on the dual variables belonging to the current subspace M. The line search technique needed to compute an adequate step length α was simple, but it was sometimes computationally expensive. In addition, the Hessian matrix H could happen to be singular, causing a breakdown in the optimization process.

In the new version of the second order dual optimizer, no line search is required, and the Hessian matrix H is allowed to be occasionally singular. Instead of using the Newton method, the dual problem is transformed into a sequence of quadratic subproblems. Indeed, it can be observed that the search direction s given by (20) is also the solution of the optimization problem

$$\max \left(\frac{1}{2} s^T H s + s^T g \right).$$

By selecting a unit step length $\alpha = 1$, it comes $s = r - r^0$, hence the following quadratic problem in the current subspace M:

$$\max q(r) = \frac{1}{2} r^T H r - r^T b,$$

$$\text{subject to} \quad r_j \geq 0 \qquad (j \in M), \tag{21}$$

where

$$b = H r_0 - g.$$

In summary, the new optimizer replaces the primary dual problem with a sequence of approximate quadratic subproblems. Each quadratic subproblem can be readily solved by using, for example, a simple conjugate gradient method with non-negativity constraints on the dual variables. Because each quadratic subproblem is restricted to the current dual subspace, its dimensionality is usually small, and the maximization process is quite fast. Clearly, in this approach the Hessian matrix no longer needs to be inverted (it can, in fact, be occasionally singular), and no line search process is necessary. The main change is therefore to replace the basic generalized Newton iteration in DUAL-2 with the solution of the equivalent quadratic subproblem. However, in order to prevent the instability of convergence that might occur because of the second order discontinuities, additional modifications were found to be necessary.

4 Treatment of side constraints

In order to control the discontinuities in the dual Hessian matrix (19), which are due to status change of primal variables from free to fixed, or conversely, each quadratic subproblem is restricted to a given primal subspace where the design variables are no longer subjected to the side constraints. Such a subspace will be referred to by the set

$$N = \{i : \ x_i \text{ is imposed to be free}\} .$$

The remaining variables are frozen to their current value (lower or upper bound), i.e. they are momentarily removed from the primal problem statement.

The variables in set N will be considered as free variables, even if the inequalities (12) are not satisfied at the current dual point. This does not introduce any difficulty, because even in the absence of side constraints, the primal variables, when evaluated at a given feasible point in the dual space, always take on real non-negative values. On the other hand, for the variables which do not belong to the set N, no computation is performed. Rather, these variables remain fixed to their lower or upper bound, even though at the current dual point, the inequalities (13) and (14) are violated.

Therefore, the primal-dual relationships (12)-(14) are replaced with

$$x_i = (b_i/a_i)^{1/2} \qquad (\text{if } i \in N) ,$$

$$x_i \text{ unchanged} \qquad (\text{if } i \notin N) .$$

In order to properly select and update the subspaces M (dual variables) and N (primal variables), it is necessary to devise a reliable active set strategy. The key problem is to correctly identify the sets of free and fixed primal variables corresponding to a given dual subspace M of non-zero dual variables.

Let us assume that the maximum of the dual function has been obtained, together with the associated sets of free and fixed variables, at some stage of the dual maximization process. At this stage, the maximization algorithm will introduce one or more non zero dual variables, corresponding to violated primal constraints. The dimensionality of the current dual subspace M will therefore be increased by a least one dual variable. In this expanded dual space, the maximum of the dual function does not necessarily correspond to the same set of free variables as before. In other words, a scheme must be devised to determine the updated set of free primal variables.

At the current dual point, the free and fixed primal variables have been correctly identified according to (12)-(14). However, because the dimensionality of the dual space has been increased, the new Hessian matrix can be singular. To prevent this singularity from occuring, some of the fixed variables must be artificially freed. The initial set N will be made up of the true set I of free variables, augmented by these freed variables. To determine which new variables should be added in the set N, the derivatives of the primal variables with respect to the dual variables are examined. From (18), it can be seen that, if a dual variable r_k has entered the set M, then a primal variable x_i is likely to increase if $c_{ik} < 0$, and to decrease if $c_{ik} > 0$ (note that r_k is initially zero, and subsequently, it can only take on a positive value). Hence, for each r_k added to M, only the x_i's satisfying

$$x_i = \underline{x}_i \quad \text{and} \quad c_{ik} < 0 ,$$

or

$$x_i = \overline{x}_i \quad \text{and} \quad c_{ik} > 0$$

should be added to N.

Once the updated sets M and N have been properly initialized, the sequential quadratic approach can be started. Only the variables in set N are used to build up the Hessian matrix of the dual function appearing in the quadratic subproblem (21). This means that (19) must be used with the set N replacing the set I.

Whenever a maximum of the dual function has been obtained in a given primal subspace N, the corresponding design variables are adjusted with respect to their lower and upper bounds, by using the exact primal-dual relationships (12)-(14). Briefly stated, the set N is reset to the set I of free primal variables at the current dual point. The iterative maximization process is then restarted with the updated set N of free primal variables. This scheme is repeated until no further change occurs in the definition of the set N. At this stage of the iterative process, the correct primal subspace I has been found, i.e.

$$N \equiv I .$$

The correct sets of free and fixed primal variables have been obtained and, therefore, the maximum of the dual function has been achieved in the current dual subspace.

At this point, the dual subspace of non-zero dual variables M is modified, if required. First, the zero dual variables (if any) are dropped from the current dual subspace M. Second, one or more dual variables are added to the current subspace M, by selecting the most violated primal constraints. This scheme is repeated until no modification occurs in the set M. At this moment, the following relationships are satisfied:

$$g_j = 0 \quad \text{if} \quad r_j > 0 ,$$

$$g_j < 0 \quad \text{if} \quad r_j = 0\,,$$

which means that the maximum of the dual function has been obtained.

An important advantage of this new maximization method lies in the fact that most of the computational effort in the iterative process is performed with reduced sets of primal and dual variables. This innovative active set strategy yields a highly reliable dual optimizer, without any possible breakdown due to singularities in the Hessian matrix. Therefore, the new dual algorithm coupled to the CONLIN formulation, constitutes a robust and very efficient general purpose optimizer.

5 Constraint relaxation

Because of the conservative character of the convex linearization approach, it can happen that the approximate feasible domain be empty, especially when the initial starting point is seriously infeasible. To cope with this difficulty, a built-in constraint relaxation capability was introduced into the CONLIN optimizer.

Introducing an additional variable δ, the following relaxed problem is substituted to the explicit subproblem (5):

$$\min \sum_{+} c_{i0} x_i - \sum_{-} \frac{c_{i0}}{x_i} + z_0 w \delta\,,$$

$$\text{subject to} \quad \sum_{+} c_{ij} x_i - \sum_{-} \frac{c_{ij}}{x_i} \le \bar{c}_j + z_j \left(1 - \frac{1}{\delta}\right),$$

$$\underline{x}_i \le x_i \le \overline{x}_i \qquad 1 \le \delta\,, \tag{22}$$

where w is a user-supplied weighting factor, and

$$z_j = \sum_{i} |c_{ij}| x_i^0 \qquad (j = 0, \ldots, m)$$

represent increments to the functions $c_j(x)$, opening up the feasible domain in the design space if necessary.

Clearly, if the "relaxation" variable δ hits its lower bound ($\delta = 1$), nothing is changed in the problem statement, which will usually happen when the starting point is feasible or nearly feasible. On the other hand, if the starting point is seriously infeasible, the algorithm will find a value of δ greater than unity, which means that the approximate feasible domain is artificially enlarged.

From the modified primal problem (22), it is easily seen that the Lagrangian problem related to the relaxation variable δ has the form

$$\min_{\delta \ge 1} \quad z_0 w \delta - (1 - 1/\delta) \sum r_j z_j\,.$$

From this minimum condition, δ is given in terms of the dual variables by the relations

$$\delta = \left(\sum r_j z_j / w z_0\right)^{1/2} \quad \text{if} \sum r_j z_j > w z_0\,,$$

$$\delta = 1 \qquad \text{if} \sum r_j z_j \le w z_0\,. \tag{23}$$

The second order dual optimizer needs only little modification to take care of the additional relaxation variable. Whenever a new dual point has been obtained in the iterative process, the relaxation variable δ is computed from (23). As long as δ remains fixed to 1, nothing is changed in the optimization process. If the relaxation has to be activated (i.e. $\delta > 1$), then it is necessary to modify the definitions of the dual gradient vector (17) and Hessian matrix (19)

$$g_j = \sum_{+} c_{ij} x_i(r) - \sum_{-} c_{ij}/x_i(r) - \bar{c}_j - z_j(1 - 1/\delta)\,, \tag{24}$$

$$H_{jk} = -1/2 \sum_{i \in I} n_{ij} n_{ik} x_i / a_i - \frac{1}{2} z_j z_k / w \delta^3\,. \tag{25}$$

The dual quadratic subproblem (21) keeps the same form and same dimensionality as before.

6 Optimization algorithm

(i) *initialization* — Equation
define $N = \{i\colon x_i \text{ is imposed to be free}\}$
$M = \{j\colon c_j \text{ is potentially active}\}$

(ii) compute the free primal variables in subspace N

$x_i(r);\ i \in N$ (12)
compute the relaxation variable δ (23)

(iii) evaluate the dual gradient vector in subspace M
$g_j(x);\ j \in M$ (24)

(iv) check optimality in subspace N

if $g_j = 0$ for $r_j > 0$

and $g_j < 0$ for $r_j = 0$

then go to (vii)

(v) evaluate the dual Hessian matrix in subspace M

$$H_{jk}(x);\ j,k \in M \tag{25}$$

compute $b = Hr - g$

(vi) solve the quadratic subproblem and return to (ii)(21)

(vii) *update of set N*

for $i \in N$ if $x_i < \underline{x}_i$ or $x_i > \overline{x}_i$ then remove i from N

for $i \notin N$ evaluate $x_i = (b_i/a_i)^{1/2}$

if $x_i > \underline{x}_i$ and $x_i < \overline{x}_i$ then add i to N

if set N has been modified, go back to (ii)

(viii) *update of set M*

for $j \in M$ if $r_j = 0$ then remove j from M

for $j \notin M$ evaluate the primal constraint values g_j (17)

if $g_j \leq 0$ for all $j \in M$, go to (ix)

otherwise add one or more active constraints to the set M and go back to (ii)

(ix) the maximum of the dual function has been obtained

r_j^*; $j \in M$ (other r_j are zero)

x_i^*; $i \in N$ (other x_i are fixed)

Note. The initialization phase (i) is normally based on the input data provided by the user (x_i and r_j), for example, by employing the results from a previous call to the CONLIN optimizer. However, a built-in starting procedure is also implemented, by assuming that the initial subspace M contains only one active constraint (the most critical one at the given design point).

7 Concluding remarks

The convex linearization method has proven to be a highly efficient and reliable optimization tool. The specially devised second order dual optimizer described in this paper offers many attractive features that make it ideal for most of our research projects. CONLIN is especially adapted to structural synthesis problems and it is envisioned that it will soon have the ability to solve fairly large scale optimization problems (hundreds of design variables and constraints), at the expense of a moderate computational time. It has a built-in constraint relaxation capability that allows the user to start from any infeasible initial design, and even to find a solution to infeasible problems (in the form of minimal relaxation).

At each successive iteration point, the CONLIN procedure only requires evaluation of the objective and constraint functions and their first derivatives with respect to the design variables. This information is provided by the finite element analysis and sensitivity analysis results. The CONLIN optimizer then selects by itself an appropriate approximation scheme on the basis of the sign of the derivatives.

Because this paper is mostly focused on the dual algorithm implemented in the CONLIN optimizer, rather than the convex approximation strategy itself, no examples of application were presented. Indeed, there exists no basis for fair comparison with other numerical optimization methods (feasible directions, projected or reduced gradient, SLP, SQP, etc.). Rather, we shall close this paper by providing a summary of the many various problems solved by CONLIN up to now.

The CONLIN optimizer can, of course, solve all the optimal sizing problems previously treated by DUAL-2 (Fleury and Schmit 1980). It was successfully used by industry to solve real life, large scale problems (Fleury and Braibant 1986). CONLIN has also demonstrated its ability to solve efficiently shape optimal design problems involving two-dimensional structures in plane stress or plane strain. In all the numerical examples reported by Braibant and Fleury (1985); Fleury (1986) and Liefooghe, Shyy and Fleury (1989), convergence has been achieved within ten finite element analyses. More recently, CONLIN has been successfully applied to configuration optimization of trusses, exhibiting similar convergence properties (Kuritz and Fleury 1989). A sampling of typical structural optimization problems can be found in a paper by Fleury (1987). Other applications include: optimization of path robot planning (Braibant and Geradin 1985); optimization of composite structures using NASTRAN (Nagrendra and Fleury 1989); bound and minimax formulation problems (Olhoff 1988). Every optimization problem mentioned in this paragraph was solved by using the dual algorithm presented in this paper.

In addition, the convex linearization method was the subject of a fundamental mathematical study, in which many numerical experiments were carried out on analytical problems (Nguyen *et al.* 1987). The method was found to perform well for a broader class of problems than the one for which convergence can be proven. It was suggested that this feature is due to the conservative nature of the convex linearization scheme. Finally, the CONLIN strategy was the basis for further mathematical developments, leading to the promising "Method for Moving Asymptotes" (Svanberg 1987).

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