Sample Variational Methods Problem

Joey Peter
SR No: 5510-511-051-03929
Supercomputer Education Research Center

June 29, 2006

1 Introduction

Minimal Time Orbital Transfer
Given a constant thrust (T) rocket engine, find the thrust direction history $\phi(t)$ to transfer the space vehicle from a given initial circular orbit to a final circular orbit.

2 Free Body Diagram

The following figure depicts the problem. The dot represents the space vehicle.

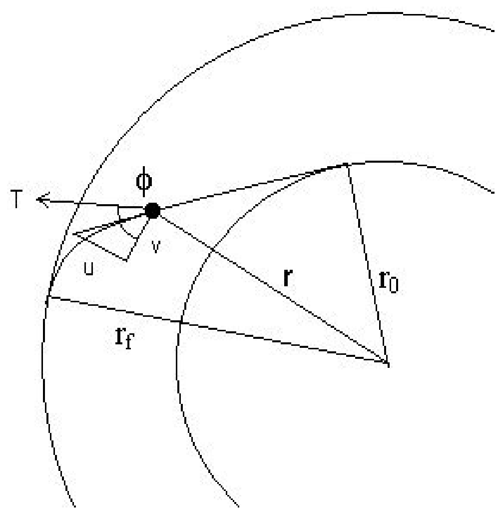

Figure 1: Free body diagram of a space vehicle

3 Symbols

u : Radial component of the velocity
v : Tangential component of the velocity
m : Mass of the vehicle

μ : Gravitational constant of attracting center
r : Radial distance of space vehicle from the attracting center
ϕ : Thrust deflection angle.

4 System Dynamics

$$\frac{dr}{dt} = u \tag{1}$$

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{Tsin\phi}{m} \tag{2}$$

$$\frac{dv}{dt} = -\frac{uv}{r} + \frac{Tcos\phi}{m} \tag{3}$$

The functional to be minimised is:

$$J = \int_{t_0}^{t_f} dt$$

Since the final time is not fixed, the problem needs to be reformulated with 'r' instead to 't' as the independent variable. Then the new functional would be:

$$J = \int_{t_0}^{t_f} \frac{dt}{dr} dr$$

Substituting $\frac{dr}{dt}$ from equation (1), we get:

$$J = \int_{r0}^{r_f} \frac{1}{u} dr$$

Reformulating the differential equations of motion with 'r' as the independent variable, we get:

$$\frac{du}{dr} = \frac{1}{u}(\frac{v^2}{r} - \frac{\mu}{r^2} + \frac{Tsin\phi}{m}) \tag{4}$$

$$\frac{dv}{dr} = \frac{1}{u}(-\frac{uv}{r} + \frac{Tcos\phi}{m}) \tag{5}$$

The augmented functional is given by:

$$J = \int_{r0}^{r_f} (\frac{1}{u} + \lambda_u(\frac{1}{u}(\frac{v^2}{r} - \frac{\mu}{r^2} + \frac{Tsin\phi}{m} - \frac{du}{dr})) + \lambda_v(\frac{1}{u}(-\frac{uv}{r} + \frac{Tcos\phi}{m})) - \frac{dv}{dr})$$

5 Euler Lagrange Equations

$\delta_u = 0 \Rightarrow$

$$1 + \lambda_u(\frac{v^2}{r} - \frac{\mu}{r^2} + \frac{Tsin\phi}{m}) + \lambda_v(\frac{Tcos\phi}{m}) - u^2\frac{d\lambda_u}{dr} = 0 \tag{6}$$

$\delta_v = 0 \Rightarrow$

$$2\lambda_u v - u\lambda_v + ur\frac{d\lambda_v}{dr} = 0 \tag{7}$$

$\delta_\phi = 0 \Rightarrow$

$$tan\phi = \frac{\lambda_u}{\lambda_v} \tag{8}$$

6 Update Formula

1. Assume a control history ϕ.

2. Use governing equations (4) and (5) to solve for u and v.

3. Use relations (6) and (7) to solve for λ_u and λ_v.

4. Use relation (8) to solve for control history.

5. Go to step 2 and repeat till convergence.

Classmates Note: Kindly let me know if you disagree with any of the results shown here.

All the best