Solution to Mid-term Examination

Marks: 30 Open notes but closed books **Time: 90 minutes**

Question 1 *(5 marks)*

Consider $J = \int_{x_1}^{x_2} F(x, y, y', y'', \cdots, y^{n^{th}\prime})dx$ and $f(x)$. What possible terms can you add to the integrand of J using $f(x)$ and $y(x)$ and their derivatives so that the new functional and J would have the same Euler-Lagrange extremizing differential equation as the extremizing solution?

Clearly, adding $y', y'', y''', \cdots, y^{n^{th}\prime}$ individually or as a product with a constant does not contribute any new terms to the E-L equation of the functional: $J = \int_{x_1}^{x_2} F(x, y, y', y'', y''', \cdots, y^{n^{th}\prime})dx$.

Likewise, $f, f', f'', \cdots, f^{n^{th}\prime}$ behave the same way.

Furthermore, consider $f'y + y'f$. Its terms in the E-L equation will be

$\frac{\partial(f'y + y'f)}{\partial y} - \frac{d}{dx}\left(\frac{\partial(f'y + y'f)}{\partial y'}\right) = f' - f' = 0$. So, it too does not add any new terms to the E-L equation.

Similarly, $f^{n^{th}\prime} y + y^{n^{th}\prime} f$ would also not add any new terms.

Question 2 *(5 marks)*

What functional, when optimized, would lead to the Poisson's equation: $\nabla^2 z(x, y) = g(x, y)$? Please also write the boundary condition that arises when such a functional is extremized.

$$\nabla^2 z(x, y) - g(x, y) = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - g(x, y) = 0$$

Since for a functional of the form $J = \int_{x_1}^{x_2}\int_{y_1}^{y_2} F\left(x, y, z(x, y), z_x, z_y\right)dx\,dy$, the E-L equation is:

$\frac{\partial F}{\partial z} - \frac{\partial}{dx}\left(\frac{\partial z}{\partial z_x}\right) - \frac{\partial}{dy}\left(\frac{\partial z}{\partial z_y}\right) = 0$, it can be seen that the terms in the Poisson's equation arise from

$$F = \frac{1}{2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial z}{\partial y}\right)^2 + g(x, y)z = \frac{1}{2}(z_x)^2 + \frac{1}{2}(z_y)^2 + g(x, y)z.$$

The general form of the boundary condition for $J = \int_{x_1}^{x_2}\int_{y_1}^{y_2} F\left(x, y, z(x, y), z_x, z_y\right)dx\,dy$ is:

$\int_\Gamma \left(\frac{\partial F}{\partial z_x}dy - \frac{\partial F}{\partial z_y}dx\right)\delta z = 0$ where $\Gamma = \text{the boundary of the domain}$. This in this case, gives:

$\int_{\Gamma}\left(z_x dy - z_y dx\right)\delta z = 0$ where $\Gamma =$ the boundary of the domain .

Question 3 *(3+4+1+2 = 10 marks)*

It is desired that an axially loaded fixed-free bar's cross-section area, $A(x)$, be designed with a given volume of material, V^*, such that the overall stiffness and the displacement at a particular point, $x = \hat{x}$, are both maximized.

(a) Justify that the following problem statement meets the above requirements and explain what the symbols in the problem statement stand for.

$$\underset{A(x)}{\text{Minimize}} \quad J = \frac{\int_0^L \frac{P^2}{2EA}dx}{\int_0^L \frac{Pq}{EA}dx}$$

Subject to

$$\int_0^L A dx - V^* \le 0$$

Data: $E =$ Young's modulus, $f(x) =$ axial load, $L =$ length, V^*

(b) Write down the Euler-Lagrange necessary conditions for this problem (You need not write the boundary conditions).

(c) How do you justify that the volume constraint must be active/inactive?

(d) Determine the expressions for the optimal $A(x)$ and the Lagrange multiplier corresponding to the constraint.

(a) Recall that the minimizing strain energy is equivalent to maximizing the stiffness of an elastic structure. For an axially loaded bar, the strain energy is given by $\int_0^L \left(\frac{1}{2}EAu'^2\right)dx$. This can be re-written as $\int_0^L \left(\frac{1}{2}EAu'^2\right)dx = \int_0^L \frac{(EAu')^2}{2AE}dx = \int_0^L \frac{P^2}{2AE}dx$ where $P(x)$ is the internal force due to the applied load $f(x)$. Since it is a fixed free-bar, it is statically determinate. In a statically determinate structure $P(x)$ can be determined without knowing the area of cross-section. Furthermore, it is independent of $A(x)$. Thus, the numerator in the functional to be minimized serves the purpose of maximizing stiffness.

Next, note that the mutual strain energy, $\int_0^L (EAu'v')dx = \int_0^L \frac{(EAu')(EAv')}{AE}dx = \int_0^L \frac{Pq}{AE}dx$ is the displacement at $x = \hat{x}$ under $f(x)$ where $q(x)$ is the internal force and $v(x)$ the deformation when a unit load is applied at $x = \hat{x}$. Since this term is in the denominator, minimizing the functional implies maximizing the displacement at $x = \hat{x}$.

The volume constraint is there as per the statement of the problem. Note that this being a statically determinate structure, governing differential equation is absent in the constraint.

(b) First, we write the Lagrangian for the problem as $L = \dfrac{\int_0^L \frac{P^2}{2EA} dx}{\int_0^L \frac{Pq}{EA} dx} + \Lambda \left\{ \int_0^L A dx - V^* \right\}$. Note that the ratio of two integrals is dealt with by using the rules of ordinary calculus while taking the variations.

By taking the variation with respect to $A(x)$, we get:

$$\delta L = 0 \Rightarrow \frac{-\frac{P^2}{2EA^2}}{\left(\int_0^L \frac{Pq}{EA} dx\right)} - \frac{\left(\int_0^L \frac{P^2}{EA} dx\right)\left(-\frac{Pq}{EA^2}\right)}{\left(\int_0^L \frac{Pq}{EA} dx\right)^2} + \Lambda = 0 \Rightarrow -\left(\int_0^L \frac{Pq}{EA} dx\right)\frac{P^2}{2EA^2} + \left(\int_0^L \frac{P^2}{EA} dx\right)\frac{Pq}{EA^2} = \Lambda \left(\int_0^L \frac{Pq}{EA} dx\right)^2$$

(c) If $\Lambda = 0$, from the last equation it implies that

$\delta L = 0 \Rightarrow P\left\{ -\left(\int_0^L \frac{Pq}{EA} dx\right)\frac{P}{2EA^2} + \left(\int_0^L \frac{P^2}{EA} dx\right)\frac{q}{EA^2} \right\} = 0$. Since P cannot be zero in the entire span of the bar for any given load $f(x)$, we note that

$$-\left(\int_0^L \frac{Pq}{EA} dx\right)\frac{P}{2EA^2} + \left(\int_0^L \frac{P^2}{EA} dx\right)\frac{q}{EA^2} = 0 \Rightarrow \left(\int_0^L \frac{Pq}{EA} dx\right)\frac{P}{2} = \left(\int_0^L \frac{P^2}{EA} dx\right) q$$

$$\Rightarrow \frac{\left(\int_0^L \frac{P^2}{EA} dx\right)}{\left(\int_0^L \frac{Pq}{EA} dx\right)} = \frac{P}{2q}$$

The last result in the above equation, i.e., $\frac{P}{2q}$, is determined once $f(x)$ and $\hat{x}$ are given. So, this means that the functional to be minimized is already determined! Clearly, that value is not the mimimum. So, we argue that $\Lambda \neq 0$. So, the constraint must be active.

(d) From the E-L equation in (b), the expression for $A(x)$ can be obtained as
$$A(x) = \frac{\sqrt{\left(\int_0^L \frac{Pq}{EA} dx\right)\frac{P^2}{2E} + \frac{Pq}{E}}}{\sqrt{\Lambda}\left(\int_0^L \frac{Pq}{EA} dx\right)}.$$

The expression for the Lagrange multiplier Λ can be determined by substituting the above expression for $A(x)$ into the active volume constraint.

The expressions for $A(x)$ and Λ need to be simultaneously solved here by using a numerical technique.

Question 4 (10 marks)

Write down the extremizing differential equation and the boundary conditions for the following functional if the end points of $y(x)$ lie on curves $\phi_1(x)$ and $\phi_2(x)$.

$$J = \left\{ \int_{x_1}^{x_2} F(x, y, y')dx \right\} + \psi(x_1, y_1, x_2, y_2)$$

Note that $y_1 = y_{x=x_1}$ and $y_2 = y_{x=x_2}$.

By taking variation with respect to $y(x)$, we get:

$$\delta J = 0 \Rightarrow \left\{ \int_{x_1}^{x_2} \left(F_x \delta x + F_y \delta y + F_{y'} \delta y' \right) dx \right\} + \frac{\partial \psi}{\partial x_1} \delta x_1 + \frac{\partial \psi}{\partial y_1} \delta y_1 + \frac{\partial \psi}{\partial x_2} \delta x_2 + \frac{\partial \psi}{\partial y_2} \delta y_2 = 0$$

The first term in the integrand comes out of the integral to join the other (boundary) terms and integration by parts is needed to get rid of $\delta y'$. This gives:

$$\left\{ \int_{x_1}^{x_2} \left\{ F_y - \left(F_{y'} \right)' \right\} \delta y \, dx \right\} + \left(F_{y'} \delta y \right)\Big|_{x_1}^{x_2} + F \delta x_2 - F \delta x_1 + \frac{\partial \psi}{\partial x_1} \delta x_1 + \frac{\partial \psi}{\partial y_1} \delta y_1 + \frac{\partial \psi}{\partial x_2} \delta x_2 + \frac{\partial \psi}{\partial y_2} \delta y_2 = 0$$

Note that $\delta y\big|_{x_1} = \delta y_1 - y'\big|_{x_1} \delta x_1$ and $\delta y\big|_{x_2} = \delta y_2 - y'\big|_{x_2} \delta x_2$. Since the end points lie on give curves $y_1 = \phi_1(x)$ and $y_2 = \phi_2(x)$, we can write $\delta y_1 = \phi_1' \delta x_1$ and $\delta y_2 = \phi_2' \delta x_2$. By substituting these relationships into the long equation preceding this paragraph, we get:

$$\left\{ \int_{x_1}^{x_2} \left\{ F_y - \left(F_{y'} \right)' \right\} \delta y \, dx \right\} - \left[\left\{ F + F_{y'}(\phi_1' - y') - \frac{\partial \psi}{\partial x_1} - \frac{\partial \psi}{\partial y_1} \phi_1' \right\}\Bigg|_{x_1} \right] \delta x_1 + \left[\left\{ F + F_{y'}(\phi_2' - y') + \frac{\partial \psi}{\partial x_2} + \frac{\partial \psi}{\partial y_2} \phi_2' \right\}\Bigg|_{x_2} \right] \delta x_2 = 0$$

Since δy, δx_1 and δx_2 are arbitrary, by fundamental lemma of calculus of variations, we write

$$F_y - \left(F_{y'} \right)' = 0$$

$$\left\{ F + F_{y'}(\phi_1' - y') - \frac{\partial \psi}{\partial x_1} - \frac{\partial \psi}{\partial y_1} \phi_1' \right\}\Bigg|_{x_1} = 0$$

$$\left\{ F + F_{y'}(\phi_2' - y') + \frac{\partial \psi}{\partial x_2} + \frac{\partial \psi}{\partial y_2} \phi_2' \right\}\Bigg|_{x_2} = 0$$

The first in the above set is the E-L differential equation and the other two are boundary conditions.