**ME 256 Jan. – May 2010, Homework #1 Solution;** **G.K. Ananthasuresh, IISc, Bangalore**

Figure 1 shows the top-view of the accelerometer. We need to determine the values of $w$, $l$, and $s$ such that the specified requirements are satisfied. Let us first mathematically write the requirements as inequalities.

$$\text{Sensitivity} = 10^{-5} - \frac{2nx}{g_0 - x} \leq 0 \tag{1}$$

$$\text{Frequency} = 30\times10^{3} - \frac{1}{2\pi}\sqrt{\frac{k}{m}} \leq 0 \tag{2}$$

$$\text{Size} = 2l + s - 10^{-3} \leq 0 \tag{4}$$

$$\text{Slenderness} = 10w - l \leq 0 \tag{5}$$

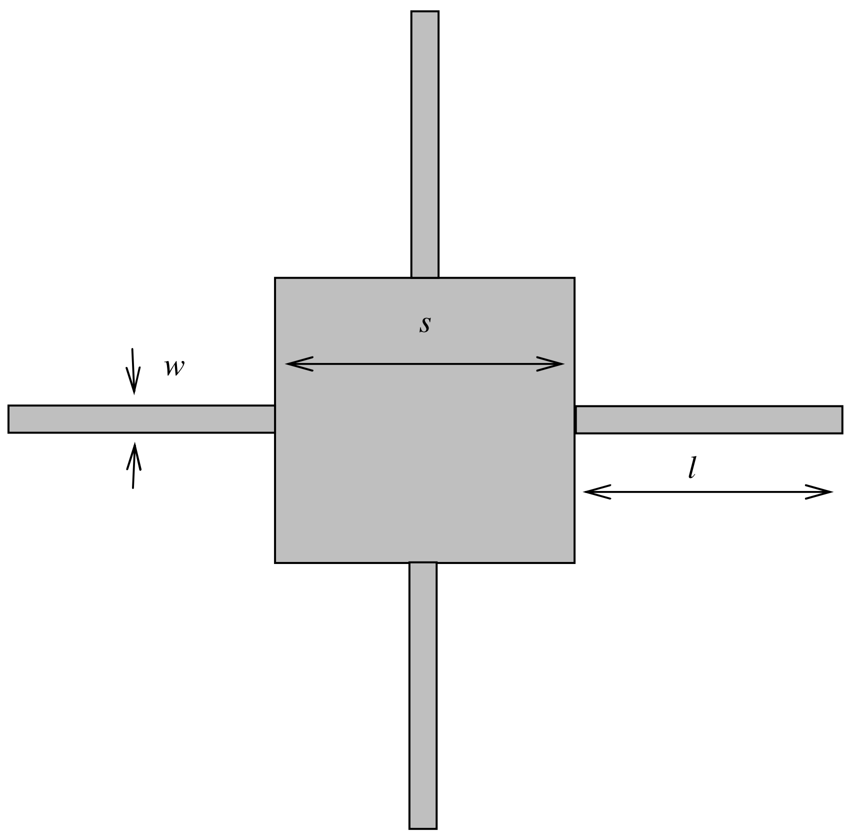


Fig. 1 The top view of the out-of-plane accelerometer with its square proof-mass and four suspension beams.

Equations (1) and (2) can be re-written in terms of the given quantities as follows.

$$\text{Sensitivity} = 10^{-5} - \frac{C_1 s^2 l^3}{C_2 w - C_3 s^2 l^3} \leq 0 \tag{5}$$

$$\text{Frequency} = 30\times 10^3 - C_4\sqrt{\frac{w}{s^2 l^3}} \le 0 \tag{6}$$

where

$$\begin{aligned} C_1 &= 2n\rho t_m a \\ C_2 &= 4g_0 E t_s^3 \\ C_3 &= \rho a t_m \\ C_4 &= \frac{1}{\pi}\sqrt{\frac{E t_s^3}{\rho t_m}} \end{aligned} \tag{6a}$$

with

$n$ = amplification of the displacement (=1.0 until otherwise stated)

$\rho$ = density (mass per unit volume) = 2300 kg/m$^3$

$t_m$ = thickness of the proof-mass = 4.75 μm

$a$ = minimum acceleration to be detected = 10 mg = $10^{-2}\times 9.81$ m/s$^2$

$g_0$ = gap between the proof-mass and the substrate = 2 μm

$E$ = Young's modulus of the material = 150 GPa

$t_s$ = thickness of the suspension beam = 2 μm

The first observation we make in Eqs. (5) and (6) is that $l$ and $s$ occur in the same way (i.e., as $s^2 l^3$) in the sensitivity and frequency requirements. But we also see that the two have opposite relationship with regard to $w$: if we decrease $w$, the sensitivity increases but the frequency decreases.

If we decide that we must have a sensitivity of $10^{-5}$ per $a$, then we can pose the following optimization problem to maximize the frequency.

$$\underset{w,l,s}{\text{Minimize}} \; -C_4\sqrt{\frac{w}{s^2 l^3}}$$

Subject to

$$\begin{aligned} &10^{-5} - \frac{C_1 s^2 l^3}{C_2 w - C_3 s^2 l^3} \le 0 \\ &2l + s - 10^{-3} \le 0 \\ &10w - l \le 0 \end{aligned} \tag{7}$$

Since we cannot maximize the frequency and the sensitivity at the same time (because of $w$), we can argue that the sensitivity constraint must be active in Eq. (7). That is, it can only be as high as its lower bound. Thus,

$$10^{-5} = \frac{C_1 s^2 l^3}{C_2 w - C_3 s^2 l^3}$$

$$\Rightarrow \frac{w}{s^2 l^3} = \frac{10^5 C_1 + C_3}{C_2} \tag{8}$$

From the objective function, we can see that its value is immediately fixed because of Eq. (8).

$$\text{Frequency} = C_4 \sqrt{\frac{10^5 C_1 + C_3}{C_2}} \tag{9}$$

So, there is no question of maximizing the frequency. But we are not done; we still need to determine the values of $w$, $l$, and $s$ such that the other two constraints in Eq. (7) are satisfied. For this, we use a graphical portrayal of those constraints as shown in Fig. 2. Here, the red curve is the straight line due to $2l + s - 10^{-3} \le 0$; everything below this curve is feasible. The blue curve in Fig. 2 shows the $10w - l \le 0$ in view of Eq. (8); everything below this curve is feasible. Hence, the shaded region in Fig. 2 is the feasible space of both the constraints. Any value of $l$ and $s$ in this will satisfy the two constraints.

Based on this, if we were to blindly submit the optimization problem in Eq. (7) to an optimization routine such as *fmincon*, we get several solutions depending on the initial guess. But we now know that all the values in the shaded region of Fig. 2 give the same solution. Run hw1prob.m attached at the end to see this for yourself.

But, with

Sensitivity = $10^{-5}$

Frequency = 15.76 kHz

So, the frequency requirement is not met.

Since we have such a large feasible space (shown in Fig. 2), can we try to see if we can narrow down the choice of $l$ and $s$? We can do that by asking for more. We can ask that $w$ be at least 5 μm, a sensible microfabrication limit. We can also impose that $w - s \le 0$ as per Fig. 1 as suggested by Ramnath Babu in the class. This is because the beams have to connect to the proof-mass. When these two constraints are also drawn, we get a new reduced feasible space shown in Fig. 3.

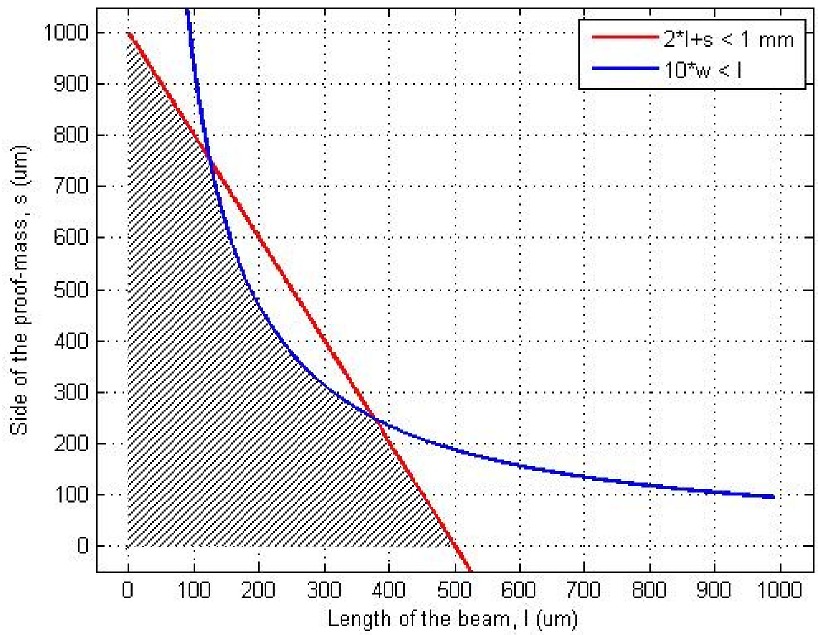


**Figure 2.** Feasible space of *l* and *s* after the sensitivity requirement is met and frequency is at 15.76 kHz.

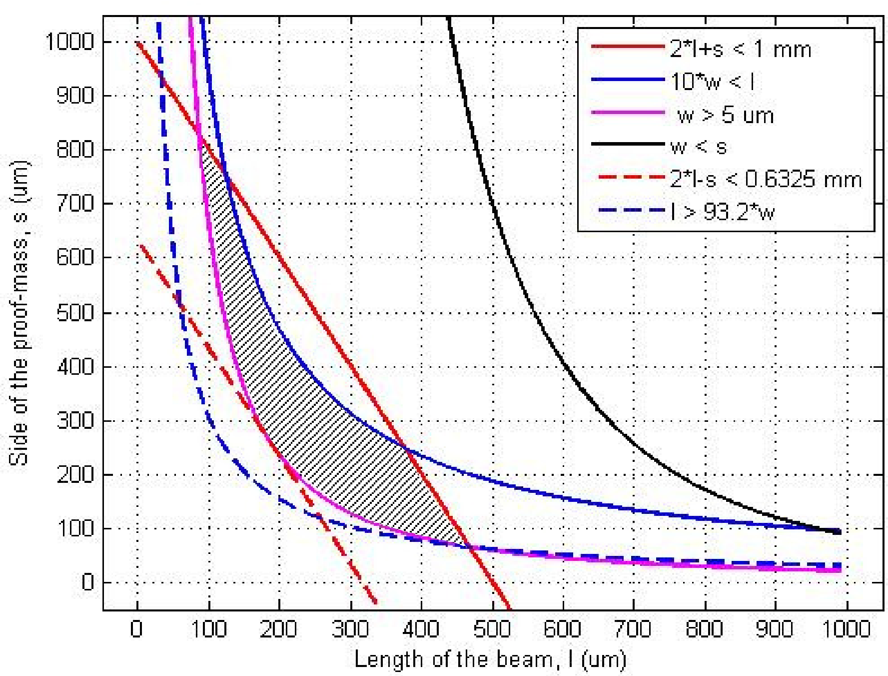


**Figure 3.** Feasible space of *l* and *s* after the sensitivity requirement is met and frequency is at 15.76 kHz with two more constraints added.

In Fig. 3, the magenta line corresponds to the constraint that $w$ > 5 μm; everything above this curve is allowed by this constraint. Therefore, our feasible space decreases as shown by the shaded region in Fig. 3 as compared with that in Fig. 2. The black curve corresponding to $w - s \le 0$ is inactive in the sense that it is automatically satisfied by the feasible region in Fig. 2; everything below this black curve is feasible and that is satisfied by the feasible space of Fig. 2.

Using Fig. 3, we can also answer two more questions: (i) what is the smallest size possible within this feasible space? That is, what is the smallest $(2l + s)$? (ii) what is the largest slenderness ratio possible? That is, what is the largest value of $(l/w)$?

To answer the first question, we simply translate the red line until the feasible space becomes zero. See the dashed red line in Fig. 3 where it is tangential to the magenta curve. By using this requirement, we can conclude that

$$s = \sqrt{\frac{5}{10^6 C_5 l^3}} = 253.01 \text{ μm},\ l = \left( \frac{3}{4} \sqrt{\frac{5}{10^6 C_5}} \right)^{2/5} = 189.76 \text{ μm}$$

and $2l + s = 632.52$ μm.

To answer the second question, we vary the slenderness ratio limit $(l/w)$ from 10 to large values until the feasible space becomes zero. This happens for $(l/w) = 93.2$ as shown by the dashed blue curve. Find this value using the equations.

If we were to modify the optimization problem in Eq. (7) to minimize the size or maximize the slenderness ratio, we would have arrived at the same results. But we don't have to do that if we use graphical portrayal of the inequalities or by analyzing them first.

But our frequency limit is not achieved yet. So, we turn to increasing the value of $n$, which until now was kept at 1. From Eqs. (5) and (8), we can calculate the value of $n$ that gives the sensitivity of $10^{-5}$ and frequency of 30KHz.

Since $\frac{w}{s^2 l^3} = \frac{10^5 C_1 + C_3}{C_2}$,

$$f = C_4 \sqrt{\frac{w}{s^2 l^3}} = C_4 \sqrt{\frac{10^5 C_1 + C_3}{C_2}} = 30 \text{ kHz},\ n = 3.62.$$

But if the designer does not have the ability to achieve this amplification, he/she may be ready to compromise on the sensitivity to get more frequency than 15.76 kHz. In order to see this tradeoff, we plot both the sensitivity and the frequency in the same plot using *plotyy* command in Matlab, as shown in Fig. 4. Initially, we assume that the size and slenderness ratio constraints are both active and use the width as the free variable.

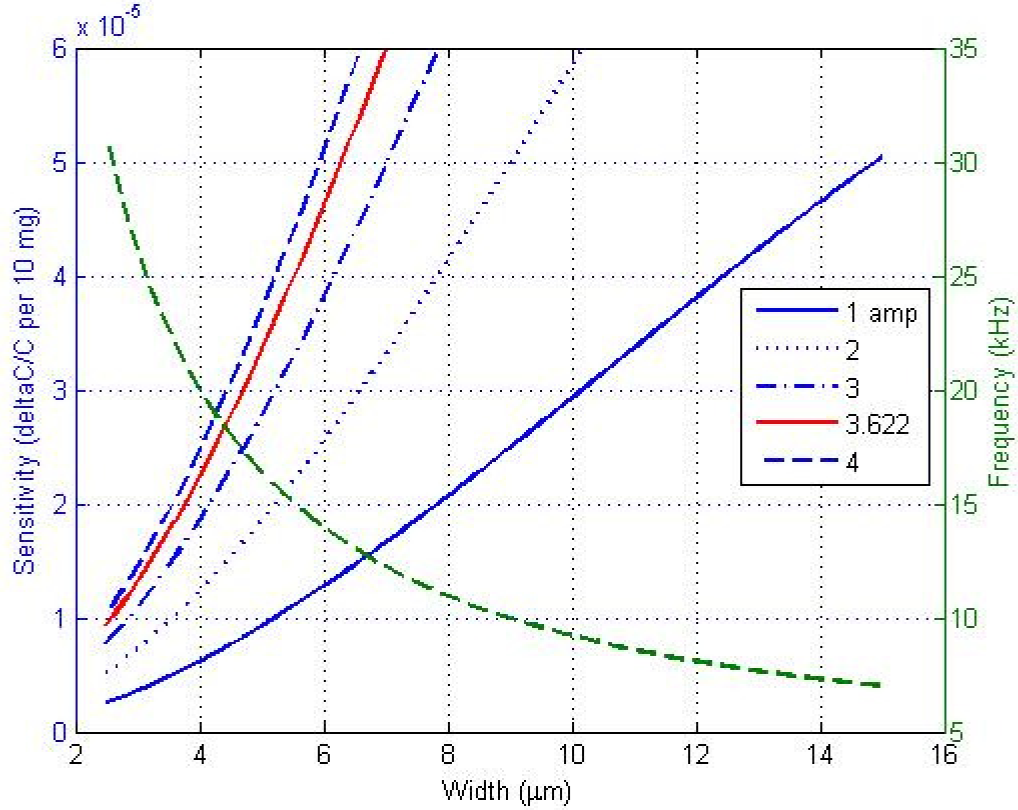


**Figure 4.** A plot that helps one see what tradeoff is needed in relaxing the sensitivity or frequency requirements. We can also see how different amplifications might help.

The green curve in Fig. 4 shows the frequency with its value shown on the vertical axis on the right-hand side. The blue curves and the red curve are for the frequency with its vertical axis shown on the left hand side. We can now see what frequency we get if we settle for less than $10^{-5}$ sensitivity. We can also see, through non-solid blue curves and the red curve, how amplification factor $n$ can also be used in this decision. Figure 5 shows the same data as in Fig. 4 but with the size limit changed to 0.9 mm. Likewise, Fig. 6 shows it for the slenderness ratio changed to 15 instead of 10.

The Matlab scripts used to plot the figures shown in this solution document are also attached. Here is the list and what they do. The comments inside the scripts are self-explanatory.

| | |
|---|---|
| hw1prob_optim.m, obj.m, con.m | Solving the problem in Eq. (7). |
| hw1prob_check | Checking a solution. |
| Hwprob_size.m | Feasible space and extra constraints. |
| hw1prob_tradeoff.m | To graphically see the tradeoff. |

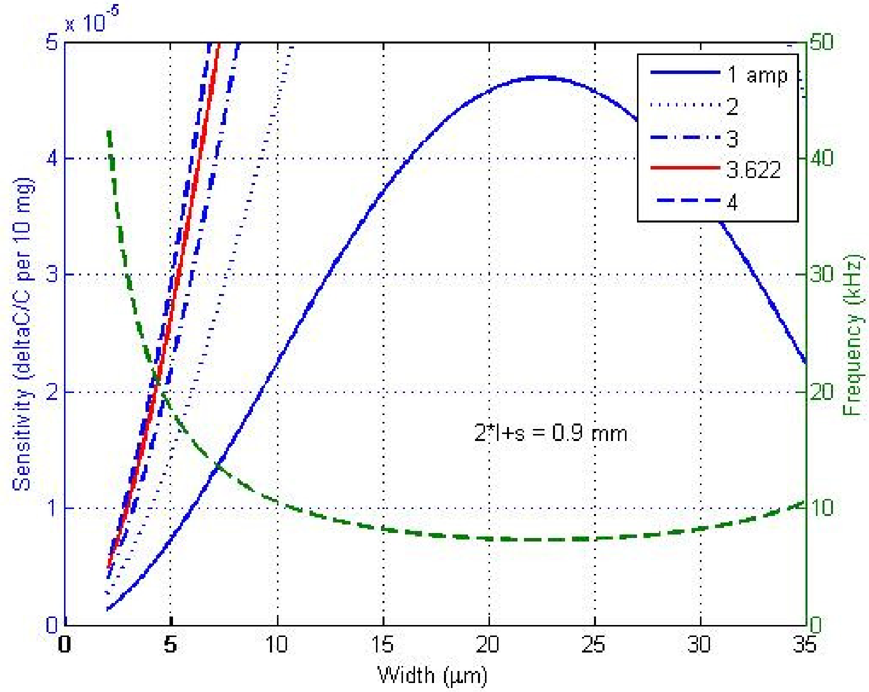


**Figure 5.** Tradeoff between sensitivity and frequency with overall size limit decreased to 0.9 mm.

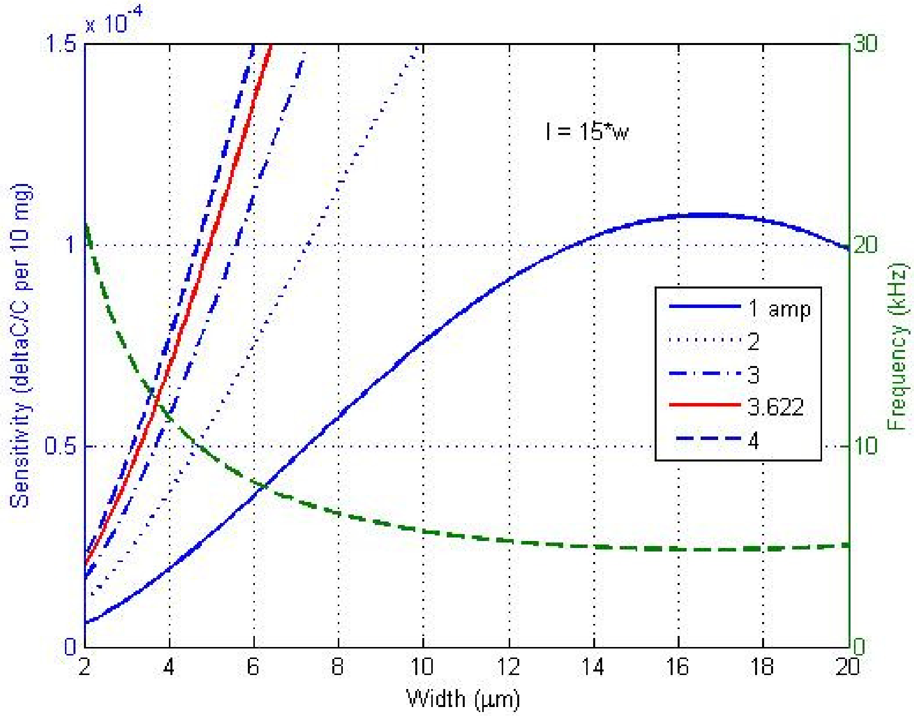


**Figure 6.** Tradeoff between sensitivity and frequency with slenderness ratio limit increased to 15 from 10.

# hw1prob_tradeoff.m

```
% Homework #1, problem 1, ME 256 Jan.-May, 2010
% Exlporing the tradeoff between the sensitivity and the frequency. This is
% the same as hw1prob1_2.m except that it uses constants C1..C4 as per the
% solution.
clear all
clc
% Fixed data
E = 150E9;
rho = 2300;
tm = 4.75e-6;
ts = 2.5E-6;
g0 = 2e-6;
a = 9.81E-2;
% Data that can be varied
n = 1;
% Derived data
C1 = 2*n*rho*tm*a;
C2 = 4*g0*E*ts^3;
C3 = rho*tm*a;
C4 = sqrt(E*ts^3/rho/tm)/pi;
% Variables
w = 1E-6*(3:0.1:20);
l = 10*w;
s = 1E-3 - 2*l;
A = s.*s;
% Derived quantities
deltaCbyC = C1*s.^2.*l.^3 ./ (C2*w - C3*s.^2.*l.^3 );
f = C4 * sqrt(w ./ (s.^2.*l.^3) );
% Plotting
figure(1)
clf
[AX, H1, H2] = plotyy(w*1E6, deltaCbyC, w*1E6, f/1E3);
hold on
xlabel('Width (\mum)');
ylabel(AX(2),'Frequency (kHz)');
ylabel(AX(1),'Sensitivity (deltaC/C per 10 mg)');
set(H1,'LineStyle','-','LineWidth',2);
set(H2,'LineStyle','--','LineWidth',2);
plot(w*1E6, 2*deltaCbyC, ':','LineWidth',2);
plot(w*1E6, 3*deltaCbyC,'-.','LineWidth',2);
plot(w*1E6, 3.622*deltaCbyC, 'r-','LineWidth',2);
plot(w*1E6, 4*deltaCbyC,'--','LineWidth',2);
legend('1 amp','2','3','3.622','4');
grid
```

# hw1prob_check.m

```
function [] = check(X)
% Check the solution of Hw #1 Prob #1 ME 256 Jan.-May 2010
E = 150E9;
rho = 2300;
tm = 4.75e-6;
ts = 2.5E-6;
g0 = 2e-6;
```

```
a = 9.81E-2;
w = X(1);
l = X(2);
s = X(3);
m = rho*s^2*tm;
k = 4*E*w*ts^3 / l^3;
x = m*a/k;
deltaCbyC = 2*x/(g0-x)
f = sqrt(k/m)/2/pi
con1 = 2*l + s
con2 = 10*w - l
```

## hw1prob_optim.m

```
% Homework #1, problem 1, ME 256 Jan.-May, 2010
% We maximize the frequency subject to three inequality constraints,
% namely, on the sensitivity, the overall size, and the slenderness
% requirement. The latter two are linear inequality constraints.
clear all
clc
global C1 C2 C3 C4
% Fixed data
E = 150E9;
rho = 2300;
tm = 4.75e-6;
ts = 2.5E-6;
g0 = 2e-6;
a = 9.81E-2;
% Data that can be varied
n = 1;
% Derived data
C1 = 2*n*rho*tm*a;
C2 = 4*g0*E*ts^3;
C3 = rho*tm*a;
C4 = sqrt(E*ts^3/rho/tm)/pi;
% Intial guess for the three variables
w0 = 30E-6;
l0 = 10*w0;
s0 = 1E-3 - 2*l0;
X0 = [w0; l0; s0];
% Lower and upper bounds for the three variables
lb = [2E-6; 20E-6; 20E-6];
ub = [100E-6; 1E-3; 1E-3];
% Linear inequality constraints
Aineq = [0 2 1; 10 -1 0];
Bineq = [1.0E-3; 0];
% Calling the fmincon function
[X,FVAL,EXITFLAG,OUTPUT,LAMBDA] =
fmincon('obj',X0,Aineq,Bineq,[],[],lb,ub,'confn');
% Checking the solution
w = X(1);
l = X(2);
s = X(3);
m = rho*s^2*tm;
k = 4*E*w*ts^3 / l^3;
x = m*a/k;
deltaCbyC = 2*n*x/(g0-x)
```

```
f = sqrt(k/m)/2/pi
con1 = 2*l + s - 1E-3
con2 = 10*w - l
% You would notice that there are a number of feasible solutions to this
% problem but not really an optimal solution to maximize the frequency.
% Frequency is fixed once the sensitivity constraint is satisfied. Hence,
% we cannot really maximize it; we can only identify a variety of
% solutions. Unless, we impose other constraints, there is no optimization
% here.
```

```
function objfn = obj(X)
global C1 C2 C3 C4
objfn = -C4 * sqrt( X(1) / X(3)^2 / X(2)^3);
```

```
function [confn ceq] = confn(X)
global C1 C2 C3 C4
confn = 1E-5 - C1*X(3)^2*X(2)^3 / (C2*X(1) - C3*X(3)^2*X(2)^3 );
ceq = [];
```

## hw1prob_size.m

```
% Homework #1, problem 1, ME 256 Jan.-May, 2010
% To see what size and slender beam requirements meet the specifications on
% the sensitivity and frequency.
clear all
clc
% Fixed data
E = 150E9;
rho = 2300;
tm = 4.75e-6;
ts = 2.5E-6;
g0 = 2e-6;
a = 9.81E-2;
% Data that can be varied
n = 1;
% Derived data
C1 = 2*n*rho*tm*a;
C2 = 4*g0*E*ts^3;
C3 = rho*tm*a;
C4 = sqrt(E*ts^3/rho/tm)/pi;
C5 = (1E5*C1 + C3) / C2;
% Constraints
l = 1E-6*(0:1:990);
s1 = 1.0*1E-3 - 2*l;
s2 = sqrt(1./(l.^2)) / sqrt(10*C5);
s3 = sqrt(5/1E6/C5./(l.^3)); % w cannot be less than 5 um.
s4 = 1./(C5*l.^3); % w should be less than s.
s1p = 0.6325*1E-3 - 2*l; % Smallest size possible
s2p = sqrt(1./(l.^2)) / sqrt(93.2*C5); % Best slenderness ration possible
% Plotting the constraints
figure(1)
clf
plot(l*1E6, s1*1E6, '-r', 'LineWidth', 2);
hold on
plot(l*1E6, s2*1E6, '-b', 'LineWidth', 2);
plot(l*1E6, s3*1E6, '-m', 'LineWidth', 2);
plot(l*1E6, s4*1E6, '-k', 'LineWidth', 2);
```

```
plot(l*1E6, s1p*1E6, '--r', 'LineWidth', 2);
plot(l*1E6, s2p*1E6, '--b', 'LineWidth', 2);
xlabel('Length of the beam, l (um)');
ylabel('Side of the proof-mass, s (um)');
legend('2*l+s < 1 mm', '10*w < l', ' w > 5 um', 'w < s', '2*l-s < 0.6325 mm',
'l > 93.2*w');
axis([-50 1050 -50 1050]);
grid
%
```