Mid-term Examination

Marks: 30 Open notes but closed books **Time: 90 minutes**

Question 1 *(5 marks)*

Consider $J = \int_{x_1}^{x_2} F(x, y, y', y'', \cdots, y^{n^{th}\prime})dx$ and $f(x)$. What possible terms can you add to the integrand of J using $f(x)$ and $y(x)$ and their derivatives so that the new functional and $f(x)$ have the same Euler-Lagrange extremizing differential equation as the extremizing solution?

Question 2 *(5 marks)*

What functional, when optimized, would lead to the Poisson's equation: $\nabla^2 z(x, y) = g(x, y)$? Please also write the boundary condition that arises when such a functional is extremized.

Question 3 *(3+4+1+2 = 10 marks)*

It is desired that an axially loaded fixed-free bar's cross-section area, $A(x)$, be designed with a given volume of material, V^*, such that the overall stiffness and the displacement at a particular point, $x = \hat{x}$, are both maximized.

$$\underset{A(x)}{\text{Minimize}} \quad J = \frac{\int_0^L \frac{P^2}{2EA} dx}{\int_0^L \frac{Pq}{EA} dx}$$

Subject to

$$\int_0^L A dx - V^* \leq 0$$

Data: E = Young's modulus, $f(x)$ = axial load, L = length, V^*

(a) Justify that the following problem meets the above requirement and explain what the symbols in the problem statement stand for.
(b) Write down the Euler-Lagrange necessary conditions for this problem (You need not write the boundary conditions).
(c) How do you justify that the volume constraint must be active/inactive?
(d) Determine the expressions for the optimal $A(x)$ and the Lagrange multiplier corresponding to the constraint.

Question 4 (**10 marks**)

Write down the extremizing differential equation and the boundary conditions for the following functional if the end points of $y(x)$ lie on curves $\phi_1(x)$ and $\phi_2(x)$.

$$J = \left\{ \int_{x_1}^{x_2} F(x, y, y')dx \right\} + \psi(x_1, y_1, x_2, y_2)$$

Note that $y_1 = y_{x=x_1}$ and $y_2 = y_{x=x_2}$.