Mid-term Examination

Marks: 30 Open notes but closed books **Time: 90 minutes**

Question 1 *(5 marks)*

Consider $J = \int_{x_1}^{x_2} F(x, y, y', y'', \cdots, y^{n^{th}\prime})dx$ and $f(x)$. What possible terms can you add to the integrand of J using $f(x)$ and $y(x)$ and their derivatives so that the new functional and J would have the same Euler-Lagrange extremizing differential equation as the extremizing solution?

Question 2 *(5 marks)*

What functional, when optimized, would lead to the Poisson's equation: $\nabla^2 z(x, y) = g(x, y)$? Please also write the boundary condition that arises when such a functional is extremized.

Question 3 *(3+4+1+2 = 10 marks)*

It is desired that an axially loaded fixed-free bar's cross-section area, $A(x)$, be designed with a given volume of material, V^*, such that the overall stiffness and the displacement at a particular point, $x = \hat{x}$, are both maximized.

(a) Justify that the following problem statement meets the above requirements and explain what the symbols in the problem statement stand for.

$$\underset{A(x)}{\text{Minimize}} \quad J = \frac{\int_0^L \frac{P^2}{2EA} dx}{\int_0^L \frac{Pq}{EA} dx}$$

Subject to

$$\int_0^L A dx - V^* \le 0$$

Data: E = Young's modulus, $f(x)$ = axial load, L = length, V^*

(b) Write down the Euler-Lagrange necessary conditions for this problem (You need not write the boundary conditions).

(c) How do you justify that the volume constraint must be active/inactive?

(d) Determine the expressions for the optimal $A(x)$ and the Lagrange multiplier corresponding to the constraint.

Question 4 (10 marks)

Write down the extremizing differential equation and the boundary conditions for the following functional if the end points of $y(x)$ lie on curves $\phi_1(x)$ and $\phi_2(x)$.

$$J = \left\{ \int_{x_1}^{x_2} F(x, y, y')dx \right\} + \psi(x_1, y_1, x_2, y_2)$$

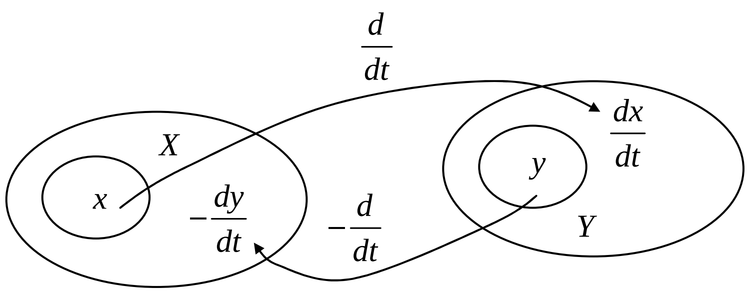
Note that $y_1 = y_{x=x_1}$ and $y_2 = y_{x=x_2}$.

Mid-term Examination

Marks: 30 Open notes and open books **Time: 60 minutes**

Question 1 *(10 marks)*

Consider two inner products defined on vector spaces X and Y as mappings $X \times X$ and $Y \times Y$ respectively into their common scalar field K and denoted as $\left\langle x_1, x_2 \right\rangle_1$, $x_1, x_2 \in X$ and $\left\langle y_1, y_2 \right\rangle_1$, $y_1, y_2 \in Y$. Note also that there are the following relationships between X and Y.



Now, consider $J = \int_a^b x \dfrac{dy}{dt} dt$ with $x(t) \in X$ and $y(t) \in Y$.

If the result of the integration by parts of J can be expressed as $\left\langle x, -\dfrac{dy}{dt} \right\rangle_1 = \left\langle y, \dfrac{dx}{dt} \right\rangle_2$, write down the formulae for the two inner products used here.

- Comment on the validity of your definitions of inner products for extra marks of up to 5.

Question 2 *(10 marks)*

Write the Euler-Lagrange equation of the following problem of minimizing a functional of two independent variables.

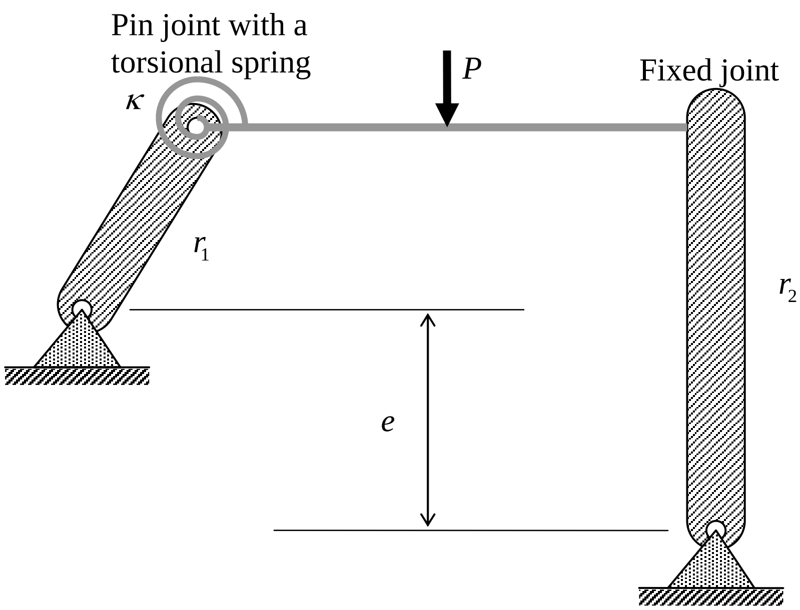
$$\underset{u(x,y)}{\text{Minimize}} \quad J = \iint_R \left[D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)^2 - 2D(1-\nu)\left\{ \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 \right\} - f(x,y)u \right] dxdy$$

Simplify the Euler-Lagrange equation to the extent possible.

- Extra points will be given for writing the boundary conditions. (*Up to 10 extra marks.*)

Question 3 *(10 marks)*

The figure below shows a horizontal beam of length L, Young's modulus E, moment of inertia I, and loaded with a force P at its mid point and constrained as follows. The left end is attached to a rigid crank of radius r_1 with a pin joint and a torsional spring of spring constant κ. The right end is fixed to another rigid crank of radius r_2. The vertical separation between the grounded pivots of the two cranks is shown to be e. Obtain a functional the minimization of which gives the static equilibrium equation to solve for the deformed configuration of this problem. And then, write the Euler-Lagrange equations along with the boundary conditions in terms of the symbols noted here.

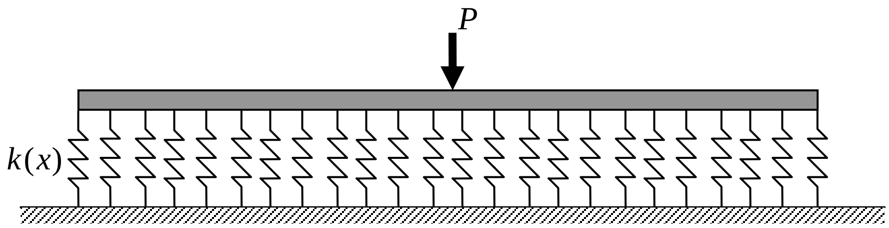


Mid-term Examination

Marks: 30 Open notes and open book **Time: 60 minutes**

Question 1 *(10 marks)*

The figure below shows a beam of elasticity modulus, $EI(x)$, and length, L, lying on an elastic foundation of linear stiffness, $k(x)$. A force, P, is applied at the mid point as shown. Using the principle of minimum potential energy, derive the governing equation for the elasto-static equilibrium of the beam along with the boundary conditions.

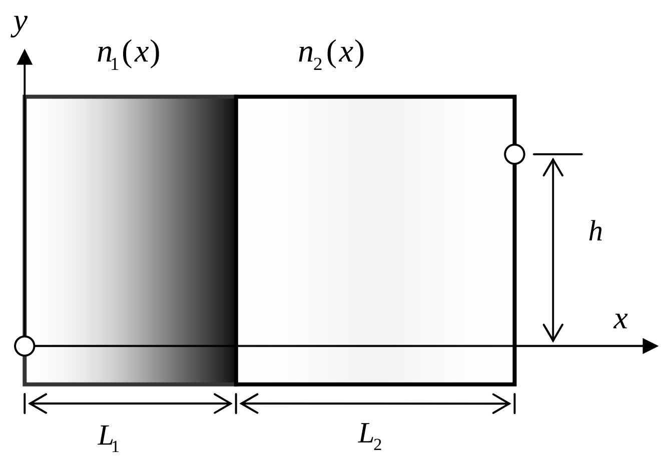


Question 2 *(10 marks)*

For a system of N particles, the time-dependent functions of the generalized coordinates $q_i(t)$ where $(i=1\cdots N)$ can be found by using the Hamilton's principle. This principle states that the Hamiltonian, H, defined as $H = KE - PE = \left\{\sum_{i=1}^{N} 0.5 m_i \left(\frac{dq_i}{dt}\right)^2\right\} - PE(q_1, q_2, \cdots, q_N)$ is stationary with respect to $q_i (i=1\cdots N)$. Note that KE is the kinetic energy and PE is the potential energy. First, write down the Euler-Lagrange necessary conditions for the stationarity of the Hamiltonian. Next, show that conservation of the energy follows from this principle. That is, show that $(KE+PE)$ is equal to a constant for $q_i(t)$s that satisfy the stationarity equations of $q_i (i=1\cdots N)$.

Question 3 *(10 marks)*

Two inhomogeneous transparent slabs are connected as shown in the figure below. The refractive indices, $n_1(x)$ and $n_2(x)$, of the two slabs vary as indicated in the figure. Note that $n_1 \neq n_2$ at the interface, $x = L_1$. First, Write down the calculus of variations problem to minimize the transit time for a light ray to travel from A to B following a path, $y(x)$. Next, obtain the differential equation helps in solving for $y(x)$. Comment on the nature of the solution, write down the boundary conditions and sketch the solution of the path taken by the light ray.



ME 256 Variational Methods and Structural Optimization Jan.-May, 2008

Mid-term Examination

Marks: 30 Open notes and open book **Time: 90 minutes**

Question 1 *(14 marks)*

A chain that has a uniform mass per unit length of γ and length L is tied to two points on a disk as shown in the figures below. In Fig. A, both ends are fixed to the disk as shown. In Fig. B, one end is fixed while the other end is movable in a circumferential slot. The disk is oriented horizontally and is rotating about the vertical axis at a constant angular velocity, ω. Note that a centripetal force of $mr\omega^2$ acts on a mass m located at a distance r away from the centre and rotating at an angular velocity ω.

(a) First, for the chain attachment as shown in Fig. A, formulate a calculus of variations problem to find the shape of the chain as the disk rotates. Obtain the governing differential equation (no need to simplify) and the boundary conditions.

(b) What in your formulation changes when one end is constrained to move in a circumferential slot as shown in Fig. B?

(c) (For *5 extra marks*) Comment on the nature of the solutions. Using your intuition and/or careful study of your governing equations, sketch approximate optimum chain shapes in both cases.

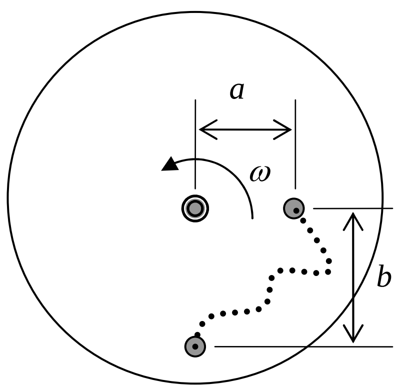


Fig. A

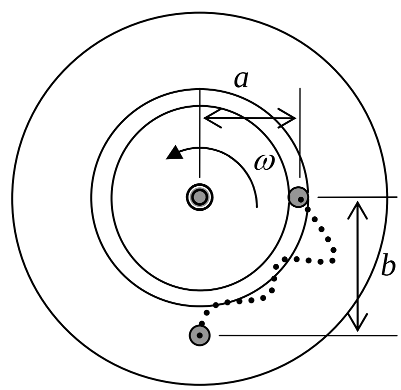


Fig. B

Question 2 *(8 marks)*

Obtain a *nontrivial* functional, which when extremized, gives the following differential equation as a necessary condition.

$$y''(x) + c_1 y'(x) + c_2 y(x) = c_3$$

where $c_{i=1,2,3}$ are constants. Write the Euler-Lagrange equations for your functional and verify that you get the above differential equation as part of the necessary conditions. Do not worry about the boundary conditions.

Question 3 *(8 marks)*

A functional, $\int_0^1 \left\{ (y')^3 + 2yy'' \right\} dx$, is to be minimized with respect to an unknown function, $y(x)$. A solution has not been found yet. Someone started with a function, $y(x) = x^3 - x$ that does not minimize the given functional, but wants to know the following: at which value of $x = \hat{x}$ the functional most is sensitive for a small change to $y(x)$ at $x = \hat{x}$? Please determine $\hat{x}$ accordingly. Show all your work with details.