Fundamental lemmas of calculus of variations

We are now familiar with the notions of a functional, vector spaces (of which function spaces are one type), Gâteaux variation and Fréchet differential. We also know operationally useful definition of Gâteaux variation of a functional. We did all this because we want to derive the necessary conditions for a minimum of the given functional. But then, Gâteaux variation depends on an arbitrary function h. In contrast, the gradient (i.e., the derivative) of an ordinary function does not have such an arbitrary entity. Of course, we noted that h exists in the definition of Gâteaux variation just as a direction is there in the definition of a directional derivative of an ordinary function. In any case, h is there and we have to deal with it. At this point, h is there in between us and the necessary conditions for a minimum of a functional. This is where the fundamental lemma of calculus of variations helps us. So, let us look at it.

Lemma 1

If $F(x)$ is continuous in $[a,b]$ and if $\int_a^b F(x)h(x)\,dx = 0$ for every function $h(x \in)c^0(a,b)$ such that $h(a)=h(b)=0$, then $F(x)=0$ for all $x\in[a,b]$.

It is a simple but profound statement. It is simple in that one can easily see why this is true. It is profound because many results of calculus of variations rest on this. It is interesting to note that its proof was attempted in 1854 by Stegmann before Du Bois-Raymond proved it in 1879. So, we can perhaps assume that Euler, Lagrange and others who dealt with necessary and sufficient conditions for a minimum of a functional tacitly assumed that it is true. For the same of completeness, let us look at a proof of this lemma. It will be proved by contradiction—a legitimate method of proving things although it is simply a process of verifying what you know as truth.

Proof of lemma 1 by contradiction

Let us say that $F(x)$ is not zero over its entire domain. Let us assume that it is positive for some interval $[x_1, x_2]$ contained within $[a,b]$. Let $h(x)=(x-x_1)(x_2-x)$ for $x\in[x_1,x_2]$ and zero outside of $[x_1,x_2]$. Note that $(x-x_1)(x_2-x)$ is positive for $x\in[x_1,x_2]$. Now, consider:

$$\int_a^b F(x)h(x)\,dx = \int_a^{x_1} F(x)h(x)\,dx + \int_{x_1}^{x_2} F(x)h(x)\,dx + \int_{x_2}^{b} F(x)h(x)\,dx$$

$$= 0 + \int_{x_1}^{x_2} F(x)h(x)dx + 0$$

$$= \int_{x_1}^{x_2} F(x)(x - x_1)(x_2 - x)\,dx > 0$$

Thus, we get a contradiction to what is given the lemma. So, we can conclude that $F(x)$ cannot be zero anywhere in the domain $[a,b]$. This proves the lemma.

Lemma 2

If $F(x)$ is continuous in $[a,b]$ and if $\int_a^b F(x)h'(x)\,dx = 0$ for every $h(x) \in c^1(a,b)$ such that $h(a) = h(b) = 0$, then $F(x)$ = constant for all $x \in [a,b]$.

This can also be proved by contradiction. Towards that, let c be defined as in $\int_a^b (F(x) - c)dx = 0$ and let $h(x) = \int_a^x (F(\xi) - c)d\xi$ so that $h(x)$ satisfies the conditions laid out in the statement of the lemma. Now, consider:

$\int_a^b (F(x) - c)h'(x)dx = \int_a^b F(x)h'(x)dx - c\{h(b) - h(a)\} = 0$ (Why is this true? The reason lies in the statement of the lemma.)

and

$\int_a^b (F(x) - c)h'(x)dx = \int_a^b (F(x) - c)^2\,dx$ (Why is this true? The reason lies in our assume $h(x)$.)

Therefore, $\int_a^b (F(x) - c)h'(x)dx = \int_a^b (F(x) - c)^2\,dx = 0 \Rightarrow F(x) - c = 0 \Rightarrow F(x) = c$ for all $x \in [a,b]$. This proved this second lemma.

In calculus of variations, two more lemmas are also stated.

Lemma 3

If $F(x)$ is continuous in $[a,b]$ and if $\int_a^b F(x)h''(x)dx = 0$ for every $h(x) \in c^2(a,b)$ such that $h(a) = h(b) = 0$ and $h'(a) = h'(b) = 0$, then $F(x) = c_0 + c_1 x$ for all $x \in [a,b]$ where c_0 and c_1 are constants.

Lemma 4

If $F_1(x)$ and $F_2(x)$ are continuous in $[a,b]$ and if $\int_a^b \left[F_1(x)h(x) + F_2(x)h'(x)\right]dx = 0$ for every $h(x) \in c^1(a,b)$ such that $h(a) = h(b) = 0$, then $F_2(x)$ is differentiable and $F_2'(x) = F_1(x)$ for all $x \in [a,b]$.

Lemmas 3 and 4 can also be proved by contradiction in the same way as the first two by assuming certain functions for $h(x)$. It is also interesting that lemmas 2-4 can also be derived from lemma 1 using simple rules of integration by parts. In fact, as we will see the rule of integration by parts is an essential tool of calculus of variations. We must also recall that Green's theorem and divergence theorem are essentially integration by parts in higher dimensions.