ME 256: Variational Methods and Structural Optimization, Indian Institute of Science, Bangalore, India
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Notes #5

1 Size-optimization of an axially loaded bar

We consider the problem of the size optimization of an axially loaded homogeneous bar as shown in Figure 1.

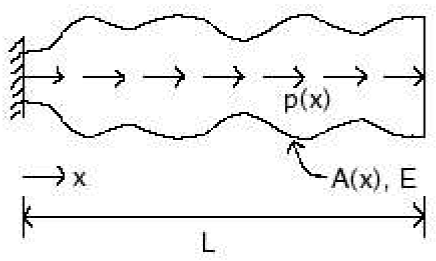

Figure 1: Bar of length L of varying area of cross-section under axial load $p(x)$

Given

Young's modulus of material, E
Length of the bar, L
Axial load, $p(x)$
Prescribed volume, V^*

Wanted Stiffest bar

To be determined $A(x)$ = area of cross-section

1.1 A measure of stiffness

Mean compliance $\int_0^L pu\,\mathrm{d}x$ — the smaller, the stiffer. Here $u(x)$ is the axial deformation in the bar.

1.2 Problem statement

$$\underset{A(x)}{\text{Min}} \int_0^L pu \, \mathrm{d}x \tag{1}$$

subject to

$$\begin{aligned}
\Lambda &\quad : \quad \int\limits_0^L A \, \mathrm{d}x - V^* \leq \; 0 &&\text{Resource constraint} \qquad (2)\\
\lambda &\quad : \quad (EAu')' + p = \; 0 &&\text{Governing equation} \qquad (3)
\end{aligned}$$

We start by defining the Lagrangian, which is to be minimized.

$$\mathcal{L} = \int\limits_0^L pu \, \mathrm{d}x + \Lambda \left\{ \int\limits_0^L A \, \mathrm{d}x - V^* \right\} + \int\limits_0^L \lambda \left\{ (EAu')' + p \right\} \mathrm{d}x \tag{4}$$

Taking variations with respect to each of the dependent variables, we get

$$\begin{aligned}
\underset{A}{\delta\mathcal{L}} = 0 \quad &: \quad \lambda E u'' - (\lambda E u')' + \Lambda = 0 \\
&\Rightarrow \quad \lambda' E u' = \Lambda
\end{aligned} \tag{5}$$

The corresponding boundary condition is given by

$$\lambda E u' \, \delta A\big|_0^L = 0 \tag{6}$$

Similarly, we take variation with respect to u and obtain the corresponding boundary conditions

$$\begin{aligned}
\underset{u}{\delta\mathcal{L}} = 0 \quad &: \quad p - (\lambda E A')' + (\lambda E A)'' = 0 \\
&\Rightarrow \quad p + (\lambda' E A)' = 0
\end{aligned} \tag{7}$$

$$\lambda E A \, \delta u'\big|_0^L = 0 \tag{8}$$

$$\lambda' E A \, \delta u\big|_0^L = 0 \tag{9}$$

Comparing Eqns. 3 and 7, we get

$$\lambda = u \tag{10}$$

Substituting this into Eqn. 5 we can solve for $u(x)$

$$\lambda'^2 E = u'^2 E = \Lambda \quad \Rightarrow \quad u' = \sqrt{\frac{\Lambda}{E}} \tag{11}$$

Putting Eqn. 11 into Eqn. 3, we determine $A(x)$ as

$$\sqrt{\Lambda E} \, A' + p = 0 \quad \Rightarrow \quad A(x) = -\int\limits_0^x \frac{p(\xi)}{\sqrt{\Lambda E}} \, \mathrm{d}\xi + C \tag{12}$$

For the simple case of a constant distributed load, we see that $A(x)$ monotonically decreases as indicated by the negative slope. However, since we require $A(x)$ to be positive everywhere, we have to impose a lower bound at $x = L$. To impose bounds on the range of $A(x)$, we therefore re-state the problem with these two new constraints on the design variable.

1.3 Size optimization with bounds on $A(x)$

$$\underset{A(x)}{\text{Min}} \int_0^L pu \, \mathrm{d}x \tag{13}$$

subject to[1]

$$\begin{aligned}
\Lambda \quad &: \quad \int_0^L A \, \mathrm{d}x - V^* \leq 0 && \text{Resource constraint} && (14)\\
\lambda \quad &: \quad (EAu')' + p = 0 && \text{Governing equation} && (15)\\
\mu_1 \quad &: \quad A_l - A \leq 0 && \text{Lower bound on } A && (16)\\
\mu_2 \quad &: \quad A - A^u \leq 0 && \text{Upper bound on } A && (17)
\end{aligned}$$

where A_l and A^u are the lower and upper bounds of A respectively.

The Lagrangian for the modified problem now becomes

$$\begin{aligned}
\mathcal{L} = \int_0^L pu \, \mathrm{d}x + \Lambda \left\{ \int_0^L A \, \mathrm{d}x - V^* \right\} + \int_0^L \lambda \left\{ (EAu')' + p \right\} \mathrm{d}x \\
+ \int_0^L \mu_1 (A_l - A) \, \mathrm{d}x + \int_0^L \mu_2 (A - A^u) \, \mathrm{d}x
\end{aligned} \tag{18}$$

Taking variations with respect to each of the dependent variables, we get

$$\begin{aligned}
\underset{A}{\delta\mathcal{L}} = 0 \quad &: \quad \lambda E u'' - (\lambda E u')' + \Lambda + \mu_2 - \mu_1 = 0 \\
&\Rightarrow \quad \lambda' E u' = \Lambda + \mu_2 - \mu_1
\end{aligned} \tag{19}$$

$$\begin{aligned}
\underset{u}{\delta\mathcal{L}} = 0 \quad &: \quad p - (\lambda E A')' + (\lambda E A)'' = 0 \\
&\Rightarrow \quad p + (\lambda' E A)' = 0
\end{aligned} \tag{20}$$

Comparing Eqns. 15 and 20, we get

$$\lambda = u \tag{21}$$

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[1] By always expressing inequality constraints to be less than or equal to 0, we ensure that the associated Lagrange multipliers are always non-negative.

Substituting this into Eqn. 19 we can solve for $u(x)$

$$\lambda'^2 E = u'^2 E = \Lambda + \mu_2 - \mu_1 \quad \Rightarrow \quad u' = \sqrt{\frac{\Lambda + \mu_2 - \mu_1}{E}} \tag{22}$$

From the Karush-Kuhn-Tucker(KKT) complementarity conditions, we get

$$\mu_1 \left(A_l - A\right) = 0 \tag{23}$$

$$\mu_2 \left(A - A^u\right) = 0 \tag{24}$$

$$\Lambda \left\{ \int_0^L A \, \mathrm{d}x - V^* \right\} = 0 \tag{25}$$

$$\mu_1,\, \mu_2,\, \Lambda \geq 0 \tag{26}$$

Due to the constraints imposed on A, there are three cases possible depending on whether A equals either of the bounds or is in between them. Due to Eqns. 23, 24 and 26 the Lagrange multipliers μ_1 and μ_2 are non-zero only in regions where the corresponding bounding constraint on A is active. We also note that when $A_l < A < A^u$, both μ_1 and μ_2 are zero. This situation becomes identical to the unbounded optimization problem stated in Section 1.2. In this case, we know from our earlier solution for A (Eqn. 12) that it decreases linearly. Hence we conclude that the only possible form of A is as shown in Fig. 2, where x_1 and x_2 are such that $0 \leq x_1 < x_2 \leq L$.

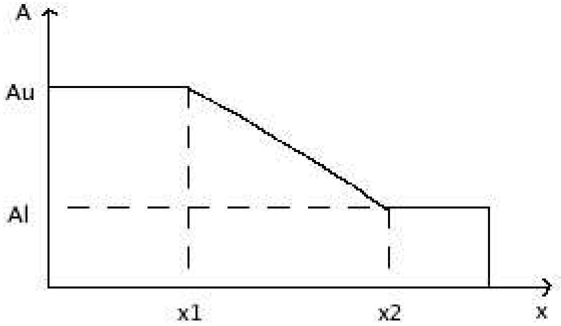

Figure 2: Form of $A(x)$

1.4 Solving the bounded optimization problem

We treat each of the three regions separately and use the boundary conditions as well as continuity at the interfaces in order to evaluate the constants in the solution. For simplicity, we solve the optimization problem for the special case of a constant load distribution $p(x) = p_0$.

Case I $0 \le x \le x_1 \quad A^{\mathrm{I}} = A^u \quad \Rightarrow \mu_1 = 0 \quad \& \quad \mu_2 > 0$
From Eqn. 15 we get

$$p_0 + EA^u u'' = 0 \quad \Rightarrow \quad u^{\mathrm{I}} = -\frac{p_0}{2EA^u}x^2 + C_1 x + C_2 \tag{27}$$

Case II $x_1 \le x \le x_2 \quad A^u > A^{\mathrm{II}} > A_l \quad \Rightarrow \mu_1 = 0 \quad \& \quad \mu_2 = 0$
From Eqn. 22 we get

$$u^{\mathrm{II}} = \sqrt{\frac{\Lambda}{E}}\, x + C_3 \tag{28}$$

Substituting this into Eqn. 15 we solve for A

$$A^{\mathrm{II}} = -\frac{p_0}{\sqrt{\Lambda E}}\, x + C_4 \tag{29}$$

Case III $x_2 \le x \le L \quad A^{\mathrm{III}} = A_l \quad \Rightarrow \mu_1 > 0 \quad \& \quad \mu_2 = 0$
From Eqn. 15 we get

$$p_0 + EA_l u'' = 0 \quad \Rightarrow \quad u^{\mathrm{III}} = -\frac{p_0}{2EA_l}x^2 + C_5 x + C_6 \tag{30}$$

Boundary conditions To evaluate the unknown constants (C_1-C_6, x_1 and x_2) in the solution, we use the boundary conditions at the two ends of the bar and the continuity of u, u' and A at x_1 & x_2. We also use the original volume constraint equation in order to evaluate the Lagrange multiplier Λ. The boundary conditions used correspond to those for a normal fixed-free bar.

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\begin{aligned}
\textbf{I} \quad & u^{\mathrm{I}}(0) = 0 \quad \Rightarrow \quad C_2 = 0 \\
\textbf{II} \quad & u^{\mathrm{III}\prime}(L) = 0 \quad \Rightarrow \quad C_5 = \frac{p_0\, L}{E\, A_l} \\
\textbf{III} \quad & u^{\mathrm{II}\prime}(x2) = u^{\mathrm{III}\prime}(x2) \quad \Rightarrow \quad \sqrt{\frac{\Lambda}{E}} = \frac{p_0\,(L - x_2)}{E\, A_l} \\
& \Rightarrow \quad x_2 = L - \frac{A_l\sqrt{\Lambda E}}{p_0} \\
\textbf{IV} \quad & A^{\mathrm{II}}(x1) = A^u \quad \Rightarrow \quad C_4 = A^u + \frac{p_0\, x_1}{\sqrt{\Lambda E}} \\
\textbf{V} \quad & A^{\mathrm{II}}(x2) = A_l \quad \Rightarrow \quad A^u + \frac{p_0(x_1 - x_2)}{\sqrt{\Lambda E}} = A_l \\
& \Rightarrow \quad x_1 = L - \frac{A^u\sqrt{\Lambda E}}{p_0} \\
\textbf{VI} \quad & u^{\mathrm{I}\prime}(x1) = u^{\mathrm{II}\prime}(x1) \quad \Rightarrow \quad C_1 = \frac{p_0 x_1}{EA^u} + \sqrt{\frac{\Lambda}{E}}
\end{aligned}
$$

$$\textbf{VII} \quad u^{\mathrm{I}}(x1) = u^{\mathrm{II}}(x1) \quad \Rightarrow \quad C_3 = C_1 x_1 - \frac{p_0 x_1^2}{2EA^u} - x_1\sqrt{\frac{\Lambda}{E}} = \frac{p_0 x_1^2}{2EA^u}$$

$$\textbf{VIII} \quad u^{\mathrm{II}}(x2) = u^{\mathrm{III}}(x2)$$

$$\Rightarrow \quad C_6 = x_2\sqrt{\frac{\Lambda}{E}} + C_3 + \frac{p_0 x_2^2}{2EA_l} = x_2\sqrt{\frac{\Lambda}{E}} + \frac{p_0}{2E}\left(\frac{x_1^2}{A^u} + \frac{x_2^2}{A_l}\right)$$

Thus the solution of the optimization problem is given by

$$u(x) = \begin{cases} \frac{p_0}{EA^u}\left(x_1\, x - \frac{x^2}{2}\right) + \sqrt{\frac{\Lambda}{E}}\, x & 0 \le x \le x_1 \\ \sqrt{\frac{\Lambda}{E}} + \frac{p_0 x_1^2}{2EA^u} & x_1 \le x \le x_2 \\ x_2\sqrt{\frac{\Lambda}{E}} + \frac{p_0}{2E}\left\{\frac{x_1^2}{A^u} + \frac{(x_2^2 - x^2)}{A_l}\right\} & x_2 \le x \le L \end{cases} \tag{31}$$

$$A(x) = \begin{cases} A^u & 0 \le x \le x_1 \\ A^u + \frac{p_0(x_1 - x)}{\sqrt{\Lambda E}} & x_1 \le x \le x_2 \\ A_l & x_2 \le x \le L \end{cases} \tag{32}$$

where Λ is evaluated by substituting $A(x)$ in the resource constraint[2] Eqn. 14 as follows

$$\begin{aligned} V^* &= A^u\, x_1 + \int\limits_{x_1}^{x_2} \left\{A^u + \frac{p_0(x_1 - x)}{\sqrt{\Lambda E}}\right\} \mathrm{d}x + A_l(L - x_2) \\ &= (A^u - A_l)x_2 + A_l\, L + \frac{p_0}{\sqrt{\Lambda E}}\left\{x_1\, x_2 - (\frac{x_2^2}{2} + \frac{x_1^2}{2})\right\} \\ &= (A^u - A_l)x_2 + A_l\, L - \frac{p_0}{2\sqrt{\Lambda E}}(x_2 - x_1)^2 \\ &= A^u\, L - \frac{\sqrt{\Lambda E}}{2p_0}\left({A^u}^2 - {A_l}^2\right) \end{aligned}$$

$$\Rightarrow \quad \Lambda = \frac{1}{E}\left[\frac{2p_0(A^u\, L - V^*)}{{A^u}^2 - {A_l}^2}\right]^2 \tag{33}$$

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[2] Note that the resource constraint inequality now becomes an equation. Since we are minimizing mean compliance, the stiffest bar will have the maximum possible volume. Hence the volume constraint will always be active. Also note that from Eqn. 25, $\Lambda > 0$.